This collection of problems is intended as a challenge, as a way of examining some of the more interesting uses of what we've seen, and as an introduction to some generalizations of some of these ideas that you might encounter in later courses. Some are straight-forward, while some are broad and only amorphously defined. Some are easy, while some are quite difficult and some may be impossible (but I don't think so). Don't worry if you can't do all of them: that was my intention. You do not need to attempt these problems if you don't want to. None of these will be collected. These haven't been thoroughly proofread, but I'm pretty sure that all of the statements in the problems are true. If not, correct them before proving them.

1 Chapter 4 Challenge Problems

Book
Section 4.1: 46, 47 (additionally, do #47 for a finite dimensional subspace V of $\mathcal{F}(\mathbb{R}, \mathbb{R})$: how do the bases of the spaces in #47 relate to the basis of V?), 57, 58
Section 4.2: 70, 74, 83
4.3: 68 (can you generalize it?), 69, 71 (why is the hint true? Generalize the hint to other problems.)

More problems

1. Let $V = \{f \text{ in } C^\infty \mid f''(x) - f(x) = 0\}$. Show that $\mathcal{U} = \{\sinh x, \cosh x\}$ and $\mathcal{B} = \{e^x, e^{-x}\}$ are two bases of V. Show that $e^x = \sinh x + \cosh x$ and $e^{-x} = -\sinh x + \cosh x$. Use these facts to find the change of basis matrix $S_{\mathcal{B}\to\mathcal{U}}$ and invert it to find $S_{\mathcal{U}\to\mathcal{B}}$. Use this to verify that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. (We've seen pieces of this problem in class.)

2. Let $V_1 = \{f \text{ in } C^\infty \mid f''(x) - f(x) = 0\}$ and let $V_2 = \{f \text{ in } C^\infty \mid f''(x) + f(x) = 0\}$. Let $V = \{f \text{ in } C^\infty \mid f^{(4)}(x) - f(x) = 0\}$. How do the bases of V_1 and V_2 relate to the basis of V? Then, explain why (nonetheless) the following proposition is false: "If $f^{(4)}(x) = f(x)$, then either $f''(x) = f(x)$ or $f''(x) = -f(x)$." Modify it to make it true.

3. Recall your work on #47 in Section 4.1. Explain why the following proposition is false: "If $f(x)$ is any function, then either $f(-x) = f(x)$ or $f(-x) = -f(x)$." Then, modify it to make it true (as in the previous problem). (*Hint:* You might want to work in an arbitrary finite dimensional subspace instead of the entire space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ so that you have a basis. This will work, so long as you choose your vector space so that it contains $f(-x)$ whenever it contains $f(x)$.)

4. Let $V = \{\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2 \,|\, b > 0\}$ be endowed with the operations

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 b_2 \end{bmatrix},$$

$$k \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ b^k \end{bmatrix}.$$

   Show that V is a vector space. Show that V is isomorphic to $\mathbb{R}^2$ by finding an isomorphism between them. Explain why the condition $b > 0$ was included in the definition of V.

5. Let V_a be the set of Taylor series centered at $x = a$ with infinite radius of convergence. Show that V_a is a vector space for every real number a. Given any real numbers a and b, show that $V_a \cong V_b$ by finding an isomorphism between them.

6. Let S and T be subsets of the real numbers. Let $\mathcal{F}(S,T)$ be the set of functions with domain S and target space T (that is, their range is a subset of T), endowed with the standard operations of function addition and scalar multiplication. (Note that we've seen the space $\mathcal{F}(\mathbb{R},\mathbb{R})$ in class.) What conditions are necessary on S and T for $\mathcal{F}(S,T)$ to be a vector space? (*Hint:* If it is a vector space, it'll be a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$.) Answer the same questions for $\mathcal{D}(S,T)$ (differentiable functions from S to T) and $C(S,T)$ (continuous functions from S to T).

7. Let V and W be arbitrary vector spaces. Show that $\mathcal{L}(V,W)$ (the set of linear transformations from V to W) is a vector space. If V and W are finite dimensional, show that $\mathcal{L}(V,W)$ is isomorphic to $\mathbb{R}^{n\times m}$ for some positive integers n and m. (If you have trouble, start with the special case $V = W$.)

8. Let V be the set of continuous functions f in $\mathcal{F}(\mathbb{R},\mathbb{R})$ which satisfy the property $f(a+b) = f(a) + f(b)$ for all real numbers a and b. Show that V is a vector space. Show that V is isomorphic to $\mathbb{R}$. (This is hard.)

9. Let V be the set of continuous functions f in $\mathcal{F}(\mathbb{R},\mathbb{R})$ which satisfy the property $f(a+b) = f(a)f(b)$ for all real numbers a and b. Show that V is not a vector space when endowed with the addition and scalar multiplication rules of $\mathcal{F}(\mathbb{R},\mathbb{R})$. Can you find different rules for addition and scalar multiplication such that V is a vector space? If so, show that V is isomorphic to $\mathbb{R}$. (This is really hard.)

10. Let V be a finite-dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ with basis $\mathcal{B}$. Let B be the $\mathcal{B}$-matrix of the linear transformation $T(f) = f'$. Describe the $\mathcal{B}$-matrix of the linear transformation $S(f) = a_0 f + a_1 f' + a_2 f'' + \cdots + a_n f^{(n)}$ in terms of the matrix B.

11. Let V be a finite-dimensional vector space and T be a linear transformation from V to W. Let $f_1, \ldots, f_m$ be a basis for $\ker(T)$ and $h_1, \ldots, h_n$ be a basis for $\operatorname{im}(T)$. Pick $g_1, \ldots, g_n$ in V such that $T(g_i) = h_i$ for $i = 1, \ldots, n$. Show that $f_1, \ldots, f_m, g_1, \ldots, g_n$ is a basis for V.

12. Is there a vector space V with the property that for every vector space W, V is isomorphic to a subspace of W?

13. Let a be a real number and define the map L_a from $\mathcal{F}(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$ by $L_a(f) = f(a)$. Show that L_a is a linear transformation. Find $\operatorname{im}(L_a)$ and $\ker(L_a)$. (L_a is sometimes called the *evaluation map*.)

14. Let $V_1 = \operatorname{span}\{\sin x, \cos x\}$ and $V_2 = \operatorname{span}\{\sinh x, \cosh x\}$. Define the transformation $T(f) = f'$ on both of these spaces and find the $\mathcal{B}$-matrix of T where $\mathcal{B} = \{\sin x, \cos x\}$ and $\mathcal{B} = \{\sinh x, \cosh x\}$ respectively. Interpret these matrices geometrically. Draw a coordinate plane with perpendicular axes labeled "$\sin x$" and "$\cos x$" (and, respectively "$\sinh x$" and "$\cosh x$"). Does T act like you'd expect geometrically? Can you find any other vector spaces in which the $\mathcal{B}$-matrix of the linear transformation T has a geometric interpretation? Is there a reason why, or is it just a coincidence?

15. When we defined vector spaces, our definition stated that all scalars came from $\mathbb{R}$. This condition is actually unnecessary, and only defines a special type of vector space called an $\mathbb{R}$-*vector space*. We say that a set F is a *field* if you can add, subtract, multiply, and divide by nonzero elements in F (this is an informal definition). The notion of a vector space can be generalized to a F-vector space, by taking scalars from F instead of $\mathbb{R}$. Some examples of fields are $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers). Show that $\mathbb{R}^n$ is a $\mathbb{Q}$-vector space and an $\mathbb{R}$-vector space, but not a $\mathbb{C}$-vector space. Show that it is infinite dimensional as a $\mathbb{Q}$-vector space (this is hard). Show that $\mathbb{C}^n$ is an $\mathbb{R}$-vector space and a $\mathbb{C}$-vector space. Compute its dimension as an $\mathbb{R}$-vector space and as a $\mathbb{C}$-vector space.

16. $\mathbb{Z}$ (the integers) is not a field since we can't do division in it (e.g., $3 \div 2$ doesn't have an answer in the integers). Why does this matter? What problems would we face if we tried to construct a "$\mathbb{Z}$-vector space?" (*Hint:* Consider the relation between redundant vectors and linear independence.)

17. Let $V = \{f \text{ in } C^\infty(\mathbb{C},\mathbb{C}) \mid f''(x) + f(x) = 0\}$ be a $\mathbb{C}$-vector space. ($C^\infty(\mathbb{C},\mathbb{C})$ is like C^∞, except from functions on $\mathbb{C}$ instead of $\mathbb{R}$.) Assuming that the rules of calculus work over $\mathbb{C}$ the same way that they do over $\mathbb{R}$ (not quite true, but close enough for our purposes here), show that $e^{\imath x}$ is in V and show that $e^{\imath x} = \cos x + \imath \sin x$ by adapting our argument that showed $e^x = \sinh x + \cosh x$. Draw a coordinate plane with perpendicular axes labeled "1" and "$\imath$" as in 14 and give a geometric interpretation of the vector $re^{\imath\theta}$. (*Hint*: polar coordinates.)

18. In class, I mentioned that isomorphism is an equivalence relation. Prove this, and use it to do #70 in Section 4.3.

2 Chapter 5 Challenge Problems

Book

Section 5.1: 31, 32, 39
Section 5.2: 40, 41, 44, 45 (*Hint:* Do Gram-Schmidt, but change the order.)
Section 5.3: 64
Section 5.4: 16, 35 (compare to what we did in Section 5.5), 42
Section 5.5: 31, 32, 33, 34

More problems

1. Recall Section 5.3, #64. The quaternions can also be viewed as a generalization of $\mathbb{C}$ by adding vectors (so, in this case, matrices) j and k with the properties that $i^2 = j^2 = k^2 = ijk = -1$. Find matrices i, j, k in H which satisfy these properties.

2. We saw Gram-Schmidt for $\mathbb{R}^n$. Can you do it in an arbitrary inner product space V? Pick your favorite inner product space and find an orthonormal basis for it.

3. More generally, try redoing anything from chapter 5 in an inner-product space V.

4. Suppose that A is an $n \times n$ matrix with $\text{rank}(A) = n$ (so that we can find a QR factorization). Does knowing the QR factorization of A help you compute least-squares solutions to $A\vec{x} = \vec{b}$. How?

5. Suppose that we have a set of data points and want to fit a polynomial of degree m to them using least-squares. If we do the same thing with a polynomial of degree $n > m$, explain (mathematically) why the error will be smaller. Even though this holds mathematically, will this always be true in the real world? Why not?

6. Let $\langle , \rangle$ be any inner product on $\mathbb{R}^n$. Show that $\langle \vec{v}, \vec{w} \rangle = \vec{w}^{\mathrm{T}} M \vec{v}$ for some matrix M. (The product $\vec{w}^{\mathrm{T}} M \vec{v}$ is called a *sesquilinear form*.) Show that M is symmetric and $\vec{v}^{\mathrm{T}} M \vec{v} > 0$ for all nonzero vectors $\vec{v}$. (A matrix with these properties is called *positive definite*.)

7. Let V be a vector space with the (finite) basis $\mathcal{B} = \{f_1, \ldots, f_n\}$. Find an inner product $\langle f, g \rangle$ for which $\mathcal{B}$ is an orthonormal basis. (*Hint:* Consider problem #17 in Section 5.5 where T is an appropriately chosen isomorphism.) Note that this problem shows that we can put an inner product on any finite dimensional vector space.

8. Let V be a vector space with the (finite) basis $\mathcal{B} = \{f_1, \ldots, f_n\}$. Let A be an $n \times n$ matrix whose diagonal entries are all positive. Let a_{ij} be the ijth entry of this matrix. Find an inner product $\langle f, g \rangle$ for which $\langle f_i, f_j \rangle = a_{ij}$ for all i, j. (This generalizes the previous problem: to get the previous problem, let $A = I_n$.)

9. Let $\langle , \rangle$ be an inner product on $\mathbb{R}^n$. Let A be an $n \times n$ matrix. Suppose $\langle A\vec{v}, A\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w}$ in $\mathbb{R}^n$. Show that A is an orthogonal matrix.

10. Let $\langle , \rangle$ be an inner product on V. Let A be a transformation from V to V. Suppose $\langle A(f), A(g) \rangle = \langle f, g \rangle$ for all f, g in V. Show that A is an orthogonal transformation. (*Note*: We only defined orthogonal transformations on $\mathbb{R}^n$, so to begin you'll have to figure out what the term "orthogonal transformation" should mean in this case.)

11. Let $V = \text{span}\{\sin x, \cos x\}$ and pick an inner product so that $\sin x, \cos x$ is an orthonormal basis. How does this relate to question 14 in Chapter 4 above? What does it tell us about the transformation T? What if we picked an inner product in which $\sin x$ and $\cos x$ are not orthonormal?

12. Adapt our work on Fourier Analysis to find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for any positive integer k. (*Hint:* Start with $f(t) = t^k$.) If this is too difficult in general, try it for a few small values of k. (This is the *Riemann zeta function*: $\zeta(k) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$. You can check your work by entering "zeta(k)" for particular values of k at `www.wolframalpha.com`. While we can of course approximate them numerically, exact values of $\zeta(k)$ are unknown for the odd integers; if you find them, you'll be the mathematics-equivalent of famous.)

13. We've seen in this class that most of our work on $\mathbb{R}^n$ carries over to abstract vector spaces V once we define some basic notions like "linear combination" and

"inner product." You know how to do calculus on $\mathbb{R}^n$. Think about what notions you'd need to carry over in order to do calculus on V. Can you carry them over? (This question is partially answered in graduate-level Real Analysis, as well as courses on Point-Set Topology, Representation Theory, Functional Analysis, and Harmonic Analysis. Thus, you almost certainly won't come up with a complete answer to it. As a start, think about how you'd define limits like $\lim_{f \to f_0} T(f)$ and $\lim_{n \to \infty} f_n$ where f, f_0, f_n are vectors in V. Does $\lim_{f \to \infty} T(f)$ make sense? If not, can you modify it so that it does?)