The final exam is cumulative and will cover Section 1.1-7.4, with about 50% on Sections 1.1-5.4 and about 50% on the new material, Sections 6.1-7.4. This review sheet covers Sections 6.1-7.4; see the old review sheets for the earlier sections.

1. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & 3 & 6 \\ 0 & 9 & 7 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 & 0 \end{bmatrix}$.

2. Find $\det(A)$ where $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 3 & 2 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 0 & 2 & 0 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ 5 & 6 & 7 & \boxed{8} \\ 9 & 10 & \boxed{11} & 12 \\ \boxed{13} & 14 & 15 & 16 \end{bmatrix}$ and let P be the pattern indicated (by the boxed entries). Find $\operatorname{sgn}(P)$ and $\operatorname{prod}(P)$.

4. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 6 \\ 0 & 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

5. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

6. Let A be an $n \times n$ matrix. Find $\det(kA)$ in terms of $\det(A)$.

7. Let A be an orthogonal $n \times n$ matrix. What are the possible values of $\det(A)$?

8. Let A be an $n \times n$ matrix. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Is $\vec{v}$ an eigenvector of $A^2 + 3A$? If so, what is its eigenvalue?

9. Find all 2×2 matrices for which $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

10. Let A be a 2×2 matrix with $\operatorname{tr}(A) = 6$ and $\det(A) = 5$. Find the eigenvalues of A.

11. Let A be the matrix of an orthogonal projection onto a plane V in $\mathbb{R}^3$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

12. Let A be the matrix of a vertical shear in $\mathbb{R}^2$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

13. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It can be shown that A is the matrix of a 90° counterclockwise rotation about the z-axis in $\mathbb{R}^3$ as viewed from the positive z-axis. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

14. Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. It so happens that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Let $a(t+1) = 3a(t) + 4b(t)$ and $b(t+1) = 4a(t) + 3b(t)$ and suppose $a(0) = 6$ and $b(0) = 2$. Find closed formulas for $a(t)$ and $b(t)$.

15. Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.

16. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 6 \\ 0 & 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.

17. Let A be a 2×2 matrix with eigenvalues 1 and 5. Find the characteristic polynomial of A.

18. Let A be a 3×3 matrix with eigenvalue 0 with algebraic multiplicity 3. Find the characteristic polynomial of A.

19. Let A be a 2×2 matrix with $\text{tr}(A) = 5$ and $\det(A) = 11$. Find the characteristic polynomial of A.

20. Find the characteristic polynomial of the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix}.$$

(That is, A is the matrix whose jth column is $\vec{e}_{j+1}$ for $j = 1, \dots, n-1$ and whose last column has the arbitrary entries $a_0, \dots, a_{n-1}$.)

21. Using what you learned in the previous problem, find a 6×6 matrix A such that the characteristic polynomial of A is $f_A(\lambda) = \lambda^6 - \lambda^5 + 3\lambda^2 - 7$.

22. Let $A = \begin{bmatrix} 1 & k \\ k & 2 \end{bmatrix}$. Find all scalars k so that 1 is an eigenvalue of A.

23. Let $A = \begin{bmatrix} 1 & k \\ k & 2 \end{bmatrix}$. Find all scalars k so that 2 is an eigenvalue of A.

24. Let $A = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$. Find all real eigenvalues and eigenvectors of A and find an eigenbasis for A if possible. (If not, explain why not.)

25. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find all real eigenvalues and eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)

26. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find all real eigenvalues and eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)

27. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

28. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

29. For which constants k is the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ diagonalizable?

30. Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$. Find a formula for the entries of A^t where t is a positive integer. Also, find the vector $A^t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

31. Let k be a fixed scalar and consider the linear transformation $T(f(x)) = f(kx)$ from $\mathcal{P}_2$ to $\mathcal{P}_2$. Find all eigenvalues and eigenfunctions of T. Is T diagonalizable?

32. Let $V = \operatorname{span}(\sinh x, \cosh x)$ and consider the linear transformation $T(f) = f'$ from V to V. Find all eigenvalues and eigenfunctions of T. Is T diagonalizable?

33. Consider the linear transformation $T(f(x)) = f(x-1)$ from $\mathcal{P}_2$ to $\mathcal{P}_2$. Find all eigenvalues and eigenfunctions of T. Is T diagonalizable?

34. Let A and B be 2×2 matrices with $\det(A) = \det(B) = -1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)

35. Let A and B be 2×2 matrices with $\det(A) = \det(B) = 1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = -2$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)

36. True or false:

(a) Let A be a 3×3 matrix. Then there is a pattern in A with precisely 2 inversions.

(b) Let A be a 3×3 matrix. Then there is a pattern in A with precisely 3 inversions.

(c) Let A be a 3×3 matrix. Then there is a pattern in A with precisely 4 inversions.

(d) Let A be a 4×4 matrix. Then all patterns of A have at most 5 inversions.

(e) Let A be an $n \times n$ matrix. Then $\det(A^{\mathrm{T}}) = \det(A)$.

(f) Let A be an invertible $n \times n$ matrix. Then $\det(A^{-1}) = \det(A)$.

(g) Let B be an $(n-1) \times (n-1)$ matrix and A be the $n \times n$ block matrix $\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$ (where the 0 entries represent zero matrices of the appropriate size). Then $\det(A) = \det(B)$.

(h) Let A be an $n \times n$ matrix. If $\operatorname{rank}(A) \neq n$, then 0 is an eigenvalue of A.

(i) Let A be the 2×2 matrix of a rotation by angle θ where θ is not a multiple of π radians. Then A has no real eigenvalues.

(j) If a matrix has no real eigenvalues, then it has no real eigenvectors.

(k) Let A be an $n \times n$ matrix. Let $\vec{e_1}$ be an eigenvector of A with eigenvalue 1. Then the first column of A is $\vec{e_1}$.

(l) Let E_2 be an eigenspace of the matrix A. Let $\vec{v}$ be a nonzero vector in E_2. Then $A\vec{v} = 2\vec{v}$.

(m) Let λ be an eigenvalue of the matrix A. Then $\dim(E_\lambda) \geq 1$.

(n) Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 2.

(o) Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 4.

(p) If an $n \times n$ matrix has n distinct real eigenvalues, then it has an eigenbasis.

(q) Let A be an $n \times n$ matrix. If $\operatorname{tr}(A) = \det(A)$, then A is invertible.

(r) Let A be an $n \times n$ matrix. Then the eigenvalues of A are the diagonal entries of A.

(s) Let A be a lower triangular matrix with all entries on the diagonal distinct. Then there is an eigenbasis for A.

(t) Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to algebraic multiplicity). Then $\det(A) = \lambda_1 + \cdots + \lambda_n$.

(u) Let A be an $n \times n$ matrix with n distinct eigenvalues. If the largest of the absolute values of the eigenvalues is 0.95, then $\lim_{t \to \infty} A^t \vec{v} = \vec{0}$ for every vector $\vec{v}$ in $\mathbb{R}^n$.

(v) If A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$ and $\det(A) = \det(B)$.

(w) If $\operatorname{tr}(A) = \operatorname{tr}(B)$ and $\det(A) = \det(B)$, then A is similar to B.