1. Let $V = \{f(t) \in \mathcal{P}_3 | f(0) = f(1)\}$. Show that V is a subspace of $\mathcal{P}_3$ and find a basis of V.

   The neutral element of $\mathcal{P}_3$ is the function $n(t) = 0$. In particular, $n(0) = 0 = n(1)$, so $n(t) \in V$.

   Let $f, g \in V$. Then $f(0) = f(1)$ and $g(0) = g(1)$, so $(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1)$, so $(f+g) \in V$.

   Let $f \in V$ and $k \in \mathbb{R}$. Then $f(0) = f(1)$, so $(kf)(0) = kf(0) = kf(1) = (kf)(1)$, so $(kf) \in V$.

   Thus, V is a subspace of $\mathcal{P}_3$.

   A general element of $\mathcal{P}_3$ has the form $f(t) = a + bt + ct^2 + dt^3$ and the condition $f(0) = f(1)$ becomes $a = a + b + c + d$, so $b = -c - d$. Thus, $f(t) = a + c(t^2 - t) + d(t^3 - t)$, so that $V = \text{span}(1, t^2 - t, t^3 - t)$. Since these vectors are linearly independent (for example, because they are polynomials of different degrees), we see that $\mathfrak{B} = \{1, t^2 - t, t^3 - t\}$ is a basis of V. Other answers are possible and can arise, for example, if you solved $b + c + d = 0$ for a different variable. If your answer consists of three linearly independent vectors in V, then it is correct.

2. Let $V = \{A \in \mathbb{R}^{3\times 3} | A^T = A\}$. Show that V is a subspace of $\mathbb{R}^{3\times 3}$ and find a basis of V.

   The neutral element is the 3×3 zero matrix 0. Clearly $0^T = 0$, so $0 \in V$.

   Let $A, B \in V$. Then $(A+B)^T = A^T + B^T = A + B$ (we haven't formally seen the first equality, but if you think about how these operations are defined componentwise, it's easy to see that it's true), so $A + B \in V$.

   Let $A \in V$, $k \in \mathbb{R}$. Then $(kA)^T = kA^T = kA$, so $kA \in V$.

   Thus, V is a subspace of $\mathbb{R}^{3\times 3}$.

   A general element of $\mathbb{R}^{3\times 3}$ has the form $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$ and the condition $A^T = A$ becomes $b = d$, $c = g$, and $f = h$. Thus, $A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & j \end{bmatrix}$, so that

   $$V = \text{span}(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix})$$

   and these matrices are linearly independent (for example, zero trick), so we see that these six matrices form a basis of V. As above, other answers are possible.

3. Let $a, b, c \in \mathbb{R}$ with $a \neq b \neq c \neq a$. Show that $\mathfrak{B} = \{(x-b)(x-c), (x-a)(x-c), (x-a)(x-b)\}$ form a basis of $\mathcal{P}_2$.

   Since we know that $\dim(\mathcal{P}_2) = 3$, it's enough to show that these three vectors are linearly independent. Write $c_1(x-b)(x-c) + c_2(x-a)(x-c) + c_3(x-a)(x-b) = 0$. We want to show that $c_1 = c_2 = c_3 = 0$. Letting $x = a, b, c$ respectively shows that $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, so that the three vectors are indeed linearly independent and hence form a basis of $\mathcal{P}_2$.

4. Let $V = \text{span}(\sinh x, \cosh x)$ and $T(f) = f - f'$ be a transformation from V to V. Show that T is a linear transformation and compute $\text{im}(T), \ker(T), \text{rank}(T)$, and $\text{nullity}(T)$. Is T an

isomorphism? What do your answers above tell you about solutions to the differential equation $f(x) = f'(x)$?

Let $f, g \in V$ and $k \in \mathbb{R}$. Note that $T(f+g) = f+g-(f+g)' = f+g-f'-g' = f-f'+g-g' = T(f)+T(g)$ and $T(kf) = kf-(kf)' = kf-kf' = k(f-f') = kT(f)$, so T is a linear transformation.

Now, $T(a\sinh x + b\cosh x) = a\sinh x + b\cosh x - a\cosh x - b\sinh x = a(\sinh x - \cosh x) + b(\cosh x - \sinh x)$. This shows that $\mathrm{im}(T) = \mathrm{span}(\sinh x - \cosh x, \cosh x - \sinh x)$. The second of these vectors is clearly redundant (as it's the first multiplied by -1), so $\{\sinh x - \cosh x\}$ forms a basis of $\mathrm{im}(T)$ and $\mathrm{rank}(T) = \dim(\mathrm{im}(T)) = 1$.

Now, suppose $T(a\sinh x + b\cosh x) = 0$. That is, $a(\sinh x - \cosh x) + b(\cosh x - \sinh x) = 0$. Here, it's useful to rewrite in terms of the basis elements: $(a-b)\sinh x + (b-a)\cosh x = 0$. Since $\sinh x$ and $\cosh x$ are linearly independent (we've checked this previously; also see #7 below), this implies that $a-b=0$ and $b-a=0$, so that $a=b$. Thus, $\ker(T)$ consists of all vectors of the form $a\sinh x + a\cosh x = a(\sinh x + \cosh x)$, so that $\ker(T) = \mathrm{span}(\sinh x + \cosh x)$, so that $\{\sinh x + \cosh x\}$ forms a basis of $\ker(T)$ and $\mathrm{nullity}(T) = \dim(\ker(T)) = 1$. (Note that $\dim(V) = 2$ so that Rank-Nullity is satisfied.)

The solutions to $f(x) = f'(x)$ in V are precisely the elements of $\ker(T)$. Thus, any scalar multiple of $\sinh x + \cosh x$ is a solution to this differential equation. (In fact, all C^∞ solutions are also of this form, by Theorem 4.1.7. Since e^x is also clearly a solution, this shows us that $e^x = a(\sinh x + \cosh x)$ for some scalar a. Letting $x = 0$, we can find that this scalar is $a = 1$.)

5. Let $T(A) = A + A^T$ be a transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Show that T is a linear transformation and compute $\mathrm{im}(T), \ker(T), \mathrm{rank}(T)$, and $\mathrm{nullity}(T)$. Is T an isomorphism? What do your answers above tell you about skew-symmetric 2×2 matrices?

Let $A, B \in \mathbb{R}^{2\times 2}$ and let $k \in \mathbb{R}$. Then $T(A+B) = A+B+(A+B)^T = A+B+A^T+B^T = A+A^T+B+B^T = T(A)+T(B)$ and $T(kA) = kA+(kA)^T = kA+kA^T = k(A+A^T) = kT(A)$, so T is a linear transformation.

Now, $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = 2a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b+c)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. These three matrices are linearly independent (zero trick), so that $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ forms a basis of $\mathrm{im}(T)$ and $\mathrm{rank}(T) = 3$.

Consider $\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $a = 0, b = -c$, and $d = 0$, so that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$ forms a basis of $\ker(T)$ and $\mathrm{nullity}(T) = 1$.

Note that the image of T consists of precisely the symmetric 2×2 matrices and the kernel of T consists of precisely the skew-symmetric 2×2 matrices.

6. Let V and W be two vector spaces of dimension n. Let $\mathfrak{B} = \{f_1, \ldots, f_n\}$ be a basis of V and let T be an isomorphism from V to W. Show that $\{T(f_1), \ldots T(f_n)\}$ is a basis of W.

Since $\dim(W) = n$, it's enough to show that these vectors are linearly independent. Suppose that $c_1T(f_1) + \cdots + c_nT(f_n) = 0$. Since T is an isomorphism, it has an inverse, T^{-1}. Apply

T^{-1} to both sides of this equation: $T^{-1}(c_1 T(f_1) + \cdots + c_n T(f_n)) = T^{-1}(0)$. Since T^{-1} is a linear transformation, we can split on addition and pull out scalars and $T^{-1}(0) = 0$, so that $c_1 T^{-1}(T(f_1)) + \cdots + c_n T^{-1}(T(f_n)) = 0$. By definition, $T^{-1}(T(f)) = f$ for all $f \in V$, so this simplifies to $c_1 f_1 + \ldots c_n f_n = 0$. Since $f_1, \ldots, f_n$ are linearly independent (as they form a basis of V), this shows us that $c_1 = \cdots = c_n = 0$, which in turn shows us that $T(f_1), \ldots, T(f_n)$ are linearly independent, and so form a basis of W.

Aside: Somewhat related to this, we can show that if $\{f_1, \ldots, f_n\}$ is a basis of V and $\{g_1, \ldots, g_n\}$ is a basis of W, then the transformation $T(c_1 f_1 + \cdots + c_n f_n) = c_1 g_1 + \cdots + c_n g_n$ from V to W is a linear transformation and an isomorphism. Compare this to #64 in Section 4.2.

7. We've seen that $V = \{f \in C^\infty | f''(x) - f(x) = 0\}$ is a vector space of dimension 2.

   (a) Verify that $\mathfrak{B} = \{e^x, e^{-x}\}$ and $\mathfrak{U} = \{\sinh x, \cosh x\}$ are both bases of V.

   (b) Find the change of basis matrix $S_{\mathfrak{B} \to \mathfrak{U}}$

   (c) Find $S_{\mathfrak{U} \to \mathfrak{B}}$ by inverting $S_{\mathfrak{B} \to \mathfrak{U}}$.

   (d) Use $S_{\mathfrak{U} \to \mathfrak{B}}$ to write formulas of the form $\sinh x = ae^x + be^{-x}$ and $\cosh x = ce^x + de^{-x}$ for some scalars $a, b, c, d \in \mathbb{R}$.

   (*Hint:* $\sinh(0) = 0$, $\cosh(0) = 1$, $\sinh(\ln 2) = \frac{3}{4}$, and $\cosh(\ln 2) = \frac{5}{4}$.)

   (a) We know that V has dimension 2, so in both cases it's sufficient to show that the vectors are linearly independent. Write $c_1 e^x + c_2 e^{-x} = 0$. Setting $x = 0, \ln 2$ (and noting that $e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$), we see that $c_1 + c_2 = 0$ and $2c_1 + \frac{1}{2}c_2 = 0$. Solving this system of equations, we see that the unique solution is $c_1 = c_2 = 0$. Similarly, writing $c_1 \sinh x + c_2 \cosh x = 0$ and letting $x = 0, \ln 2$ gives us the system of equations $c_2 = 0$ and $\frac{3}{4}c_1 + \frac{5}{4}c_2 = 0$, which also has the unique solution $c_1 = c_2 = 0$, so that both $\mathfrak{B}$ and $\mathfrak{U}$ are bases of V.

   (b) We can construct $S = S_{\mathfrak{B} \to \mathfrak{U}}$ column by column, by writing the vectors of $\mathfrak{B}$ in terms of the vectors of $\mathfrak{U}$. If we write $e^x = c_1 \sinh x + c_2 \cosh x$ and set $x = 0, \ln 2$, solving the system of equations that results shows us that $e^x = \sinh x + \cosh x$, so that $[e^x]_{\mathfrak{U}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we write $e^{-x} = c_1 \sinh x + c_2 \cosh x$ and again set $x = 0, \ln 2$, we find that $e^{-x} = -\sinh x + \cosh x$, so that $[e^{-x}]_{\mathfrak{U}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus, $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   (c) $S_{\mathfrak{U} \to \mathfrak{B}} = S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

   (d) We know that the first column of $S_{\mathfrak{U} \to \mathfrak{B}}$ is $[\sinh x]_{\mathfrak{B}}$ and the second column is $[\cosh x]_{\mathfrak{B}}$, so we have $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

8. Let $V = \operatorname{span}(\sin x, \cos x, x \sin x, x \cos x)$ and let $\mathfrak{B} = \{\sin x, \cos x, x \sin x, x \cos x\}$ be a basis of V. Let $T(f) = f'$ be a linear transformation from V to V. Find the $\mathfrak{B}$-matrix of T and use it to determine whether T is an isomorphism.

Let's find B column-by-column:

$$[T(\sin x)]_{\mathfrak{B}} = [\cos x]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[T(\cos x)]_{\mathfrak{B}} = [-\sin x]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[T(x\sin x)]_{\mathfrak{B}} = [x\cos x + \sin x]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$[T(x\cos x)]_{\mathfrak{B}} = [-x\sin x + \cos x]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

so that the $\mathfrak{B}$ matrix of T is $B = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. We see that B is invertible (as, for example, the columns are seen to be linearly independent by the zero trick), so that T is an isomorphism.

9. Let A be an invertible $n \times n$ matrix with ijth entry a_{ij}. Let V be a vector space with basis $\mathfrak{U} = \{f_1, \ldots, f_n\}$. Define $g_j = a_{1j}f_1 + \cdots + a_{nj}f_n$ for $1 \leq j \leq n$. Show that $\mathfrak{B} = \{g_1, \ldots, g_n\}$ is a basis of V and find the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{U}}$ in terms of A.

   Write $A = \begin{bmatrix} \vec{v_1} & \ldots & \vec{v_n} \end{bmatrix}$. First, we'll show that $\mathfrak{B}$ is a basis by using the isomorphism $L_{\mathfrak{U}}$ (note that we can use this map since we already know $\mathfrak{U}$ is a basis; however, since we don't know that $\mathfrak{B}$ is a basis, it doesn't make sense to talk about $L_{\mathfrak{B}}$ until we show that it is.) Note that $L_{\mathfrak{U}}(g_i) = \vec{v_i}$ for $i = 1, \ldots, n$. Since $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent (since A is invertible), this shows us that $g_1, \ldots, g_n$ are linearly independent. Then, since $\dim(V) = n$ (since $\mathfrak{U}$ is a basis of V), this shows that $\mathfrak{B}$ is a basis of V also (so, in particular, it makes sense to talk about $S_{\mathfrak{B}\to\mathfrak{U}}$). We compute $S_{\mathfrak{B}\to\mathfrak{U}}$ column by column: $[g_i]_{\mathfrak{U}} = \vec{v_i}$, so that $S_{\mathfrak{B}\to\mathfrak{U}} = \begin{bmatrix} \vec{v_1} & \ldots & \vec{v_n} \end{bmatrix} = A$.

   Aside: This shows us that every invertible matrix appears as a change of basis matrix $S_{\mathfrak{B}\to\mathfrak{U}}$ for some bases $\mathfrak{B}$ and $\mathfrak{U}$. Since we also know that $S_{\mathfrak{B}\to\mathfrak{U}}$ is always an invertible matrix, this tell us that if we have one basis $\{f_1, \ldots, f_n\}$ of a vector space V, then every other basis of V can be expressed in the form of the basis $\{g_1, \ldots, g_n\}$ for some invertible matrix A.

10. Let $f_0, \ldots, f_n \in \mathcal{P}_n$ be polynomials such that $\deg(f_i) = i$. (Note that the degree of the zero polynomial is defined to be $-\infty$, so in particular $f_0 \neq 0$.) Show that $f_0, \ldots, f_n$ form a basis of $\mathcal{P}_n$.

(Note that we've previously seen this example in class.) Write $f_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots + a_{ii}x^i$ where $a_{ii} \neq 0$ for $i = 0, \dots n$. As in the previous problem, the easiest way to show that some vectors form a basis is often to use a map $L_{\mathfrak{B}}$ to translate the problem to $\mathbb{R}^m$ (where in this case $m = \dim(\mathcal{P}_n) = n+1$). Of course, this only works if you already know some other basis $\mathfrak{B}$ of the space. Luckily, we do: let $\mathfrak{B} = \{1, x, \dots, x^n\}$. Then $L_{\mathfrak{B}}(f_i) = \begin{bmatrix} a_{i0} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Note that the $(n+1) \times (n+1)$ matrix $\begin{bmatrix} L_{\mathfrak{B}}(f_0) & \dots L_{\mathfrak{B}}(f_n) \end{bmatrix}$ is upper-triangular and that the diagonal entries are nonzero, so that the matrix is invertible. Thus, $L_{\mathfrak{B}}(f_0), \dots, L_{\mathfrak{B}}(f_n)$ are linearly independent. Since $L_{\mathfrak{B}}$ is an isomorphism, this shows us that $f_0, \dots, f_n$ are linearly independent and (since we know $\dim(\mathcal{P}_n) = n+1$) so are a basis of $\mathcal{P}_n$.

Alternatively, we can let A be the upper-triangular matrix whose ijth entry is $a_{(j-1),(i-1)}$ where a_{ij} are the coefficients of $g_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots + a_{ii}x^i$ for $i \leq j$ and whose ijth entry is 0 if $i > j$. Then, if we let $f_i(x) = x^i$ for $i = 0, \dots, n$, then the previous problem shows that the vectors g_i are linearly independent since the matrix A is invertible.

11. Let $f(x) \in \mathcal{P}_n$ be a polynomial of degree n. Show that $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ form a basis of $\mathcal{P}_n$.

Note that the polynomials $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ are all of different degrees, and so form a basis of $\mathcal{P}_n$ by the previous problem.

Alternatively, write $f(x) = c_0 f(x) + \cdots + c_n f^{(n)}(x) = 0$ and take derivatives n times and set $x = 0$ (for example) in the resulting equations (to compute these, note $f^{(n+k)}(x) = 0$ for all $k > 0$ since f has degree n). Then, solving the system of $n+1$ equations in $n+1$ unknowns (by noting that, after reversing the order of the rows, it's a system $A\vec{x} = \vec{0}$ where A is lower-triangular with nonzero diagonal entries, so that A is invertible and the system has the unique solution $\vec{x} = \vec{0}$) will show you that $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ are linearly independent.

12. Use the Gram-Schmidt process to find an orthonormal basis of $V = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix})$ and in the process find the QR-factorization of the matrix $M = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

Please be sure to check your work on this problem (and ones like it). It's really easy to make a calculation mistake on something like this.

Write $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Then $\vec{u_1} = \frac{1}{\sqrt{6}}\vec{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}$ and $r_{11} = \sqrt{6}$.

Thus, $\vec{v_2}^{\perp} = \vec{v_2} - (\vec{u_1}\cdot\vec{v_2})\vec{u_1} = \frac{1}{3}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}$ and $r_{12} = \sqrt{6}$. (Before going on, you should verify that $\vec{u_1}$ and $\vec{v_2}^{\perp}$ really are orthogonal: if they aren't, you made a computational error and if you don't fix it now, everything that follows will be wrong too.)

Thus, $\vec{u_2} = \frac{1}{\sqrt{3}}\vec{v_2}^{\perp} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}$ and $r_{22} = \sqrt{3}$.

Now, $\vec{v_3}^{\perp} = \vec{v_3} - (\vec{u_1}\cdot\vec{v_3})\vec{u_1} - (\vec{u_2}\cdot\vec{v_3})\vec{u_2} = \frac{1}{2}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}$, and $r_{13} = \frac{3}{\sqrt{6}}, r_{23} = 0$. (Before going on, verify that $\vec{v_3}^{\perp}$ is orthogonal to both $\vec{u_1}$ and $\vec{u_2}$.)

Finally, $\vec{u_3} = \frac{1}{\sqrt{3/2}}\vec{v_3}^{\perp} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}$ and $r_{33} = \sqrt{3/2}$.

Thus, our orthonormal basis for V is $\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}\}$, and the QR factorization is

$$Q = \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & 1/\sqrt{3} & -2/\sqrt{6}\\ 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{6}\\ 2/\sqrt{6} & 0 & 0\end{bmatrix}, R = \begin{bmatrix}\sqrt{6} & \sqrt{6} & 3/\sqrt{6}\\ 0 & \sqrt{3} & 0\\ 0 & 0 & \sqrt{3/2}\end{bmatrix}.$$

(Theoretically, you can check your work one final time by multiplying QR and seeing if you end up with the matrix you started with, but since most errors should be caught by checking orthogonality, this isn't really necessary.)

13. Use the Gram-Schmidt process to find an orthonormal basis of $V = \text{span}(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix})$ and in the process find the QR-factorization of the matrix $M = \begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$.

The basis is $\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}\}$ and the QR-factorization is $Q = \frac{1}{2}\begin{bmatrix}1 & 1 & -1\\1 & 1 & 1\\1 & -1 & 1\\1 & -1 & -1\end{bmatrix}$, $R = \begin{bmatrix}2 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

14. In the previous two problems, verify that $R = Q^T M$. Explain why this happens.

    Note that Q has orthonormal columns and so Q is an orthogonal matrix. Thus, Q is invertible and $Q^{-1} = Q^T$. Thus, $M = QR \Rightarrow Q^{-1}M = R \Rightarrow R = Q^T M$.

    Aside: If you don't like the method of computing R that we saw in class, this gives you another fairly easy way to compute it. (The method we saw in class is, however, slightly more efficient, since you need to do every step of it in the process of finding Gram-Schmidt anyway.)

15. Let V be a subspace of $\mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$. Under what circumstances does $\text{proj}_V \vec{x} = \vec{x}$?

    This happens if and only if $\vec{x} \in V$. This can either be seen geometrically, or checked using Theorem 5.1.10.

16. Let $a_1, \ldots, a_n \in \mathbb{R}$. Find an inequality relating $\sum_{k=1}^{n}(a_k)^2$ and $\left(\sum_{k=1}^{n} a_k\right)^2$.

    (*Hint:* Let $\vec{v} = \begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$ and consider $\vec{v} \cdot \vec{w}$.)

    By the Cauchy-Schwarz Inequality (Theorem 5.1.11), $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$. Squaring both sides and computing these quantities for the vectors $\vec{v}, \vec{w}$ from the hint, we get $\left(\sum_{k=1}^{n} a_k\right)^2 \leq n \sum_{k=1}^{n}(a_k)^2$.

    Aside: What does this say? Let's look at the case $n = 2$: $(a+b)^2 \leq 2(a^2+b^2)$. We know that $(a+b)^2 = a^2 + ab + ba + b^2$, so this tells us that if we don't feel like FOIL'ing, we can at least get a bound on $(a+b)^2$ by noting that $ab+ba \leq a^2+b^2$. There are a few other ways to see that this last inequality is true without using Cauchy-Schwarz. Once clever trick is to note that $(a-b)^2 \geq 0$ can be FOIL'ed and rearranged as $a^2 + b^2 \geq 2ab$ (if you take more math classes in the future, some professor is going to do this at some point; if you want to sound smart, you should respond "can't we just use Cauchy-Schwarz instead?"), but such clever tricks are harder to find for large n, in which case Cauchy-Schwarz is very handy. Since $(a_1 + \cdots + a_n)^2$ has n^2 terms and $n(a_1^2 + \cdots + a_n^2)$ has only n terms, the above inequality is extremely useful if you only care about getting a rough bound on the size of the square of a sum.

17. Find the matrix of the orthogonal projection from $\mathbb{R}^3$ onto the subspace $W = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\}$.

We'd like to use Theorem 5.3.10, but **first we need an orthonormal basis**. Use Gram-Schmidt to find the orthonormal basis $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Thus, $Q = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2}\\ 1/\sqrt{3} & -1/\sqrt{2}\\ 1/\sqrt{3} & 0\end{bmatrix}$ and $QQ^{\mathrm{T}} = \frac{1}{6}\begin{bmatrix}5 & -1 & 2\\ -1 & 5 & 2\\ 2 & 2 & 2\end{bmatrix}$.

18. Find the least-squares solution to the inconsistent system

$$\begin{array}{rcrcr} x & + & 4y & = & -2\\ x & + & 2y & = & 6\\ 2x & + & 3y & = & 1.\end{array}$$

$x = 3, y = -1$.

19. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0, 1)$, $(1, 3)$, $(2, 4)$, $(3, 4)$ using least-squares.

Write $A = \begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\3\\4\\4\end{bmatrix}$. Then we want to solve $A^T A\begin{bmatrix}c_0\\c_1\end{bmatrix} = A^T\vec{b}$. That is, $\begin{bmatrix}4 & 6\\6 & 14\end{bmatrix}\begin{bmatrix}c_0\\c_1\end{bmatrix} = \begin{bmatrix}12\\23\end{bmatrix}$. Solving, we get the unique solution $c_0 = 1.5, c_1 = 1$, so $f(t) = 1.5 + t$.

20. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(-1, 0)$, $(0, -1)$, $(1, 3)$, and $(1, 4)$, using least-squares.

Proceed as above to find $f(t) = 1 + 2t$.

21. Let A be an $n \times m$ matrix whose ijth entry is a_{ij}. Compute $\vec{e_i}^T A\vec{e_j}$.

You can do this directly, but it may be conceptually easier to rewrite $\vec{e_i}^T A\vec{e_j} = \vec{e_i} \cdot A\vec{e_j}$. Then $A\vec{e_j}$ is the jth column of A, and so $\vec{e_i} \cdot A\vec{e_j}$ is the ith component of the jth column of A, so that $\vec{e_i}^T A\vec{e_j} = a_{ij}$.

22. True or False:

(a) Let $V = \{f \text{ in } C^\infty \mid f'(x) \neq 0 \text{ for all } x\}$. Then V is a subspace of C^∞. False; e.g., V does not contain the zero vector $f(x) = 0$.

(b) Let $T(f) = f(0)$ be a linear transformation from $\mathcal{P}_3$ to $\mathbb{R}$. Then T is an isomorphism. False; $\dim(\mathcal{P}_3) = 4$, $\dim(\mathbb{R}) = 1$, so the spaces are not isomorphic.

(c) $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^n$. False; $\dim(\mathcal{P}_n) = n + 1$, $\dim(\mathbb{R}^n) = n$.

(d) $\mathcal{P}_{11}$ is isomorphic to $\mathbb{R}^{6\times 2}$. True; one isomorphism is to arrange the values $f(1), \ldots, f(12)$ as the twelve entries of the matrix. Another is to arrange the coefficients of the terms of f as the entries of the matrix.

(e) There is a basis of $\mathbb{R}^{2\times 2}$ consisting of four diagonal matrices. False; any linear combination of diagonal matrices will be a diagonal matrix, so it's impossible to span $\mathbb{R}^{2\times 2}$ with diagonal matrices.

(f) Let V and W be vector spaces. Let $\mathfrak{B} = \{f_1, \ldots, f_n\}$ be a basis of V and $\mathfrak{U} = \{g_1, \ldots g_n\}$ be a basis of W. Define a linear transformation T from V to W by $T(f) = c_1 g_1 + \cdots + c_n g_n$ where $[f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then T is an isomorphism. True. $\dim(V) = \dim(W)$, so it's enough to show that $\ker(T) = \{0\}$. Since $g_1, \ldots, g_n$ are linearly independent, the only way for $c_1 g_1 + \cdots + c_n g_n = 0$ is if $c_1 = \cdots = c_n = 0$, in which case $[f]_{\mathfrak{B}} = \vec{0}$ and so $f = 0$, so $\ker(T) = \{0\}$. Or, see #6

(g) Let V be a finite dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $T(f) = f'$ from V to V is a linear transformation. Then T is not an isomorphism. False; it depends on the space V. See for example Section 4.3, #48.

(h) Let $V = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} | a + b + c + d = 0\}$. Then V is a subspace of $\mathbb{R}^{2\times 2}$. True. $0 + 0 + 0 + 0 = 0$, so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in V$, then $(a + e) + (b + f) + (c + g) + (d + h) = a + b + c + d + e + f + g + h = 0 + 0 = 0$, so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in V$ and $ka + kb + kc + kd = k(a + b + c + d) = k0 = 0$, so $k \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$.

(i) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional, then W is finite dimensional. False; e.g., $T(f) = f$ from $\mathcal{F}(\mathbb{R}, \mathbb{R})$ to $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

(j) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then V is finite dimensional. True. $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.

(k) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then W is finite dimensional. False; e.g., $T(f) = f$ from $\mathcal{P}_1$ to $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

(l) $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$. True. For example, $T(x + \imath y) = \begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism.

(m) Let T be a linear transformation from a vector space V to a vector space W. If W is finite dimensional, then $\dim(W) = \text{rank}(T) + \ker(T)$. False; this statement doesn't even make sense: you can't add a number ($\text{rank}(T)$) to a vector space ($\ker(T)$). Modifying this, the similar statement $\dim(V) = \text{rank}(T) + \text{nullity}(T)$ is true.

(n) Let $\mathcal{U} = \{\vec{u_1}, \ldots, \vec{u_n}\}$ and $\mathcal{B} = \{\vec{b_1}, \ldots, \vec{b_n}\}$ be two bases of a vector space V. Then the change of basis matrix S from $\mathcal{U}$ to $\mathcal{B}$ is given by

$$S = \begin{bmatrix} [\vec{b_1}]_{\mathcal{U}} & \ldots & [\vec{b_n}]_{\mathcal{U}} \end{bmatrix}.$$

False; this is the change of basis matrix from $\mathcal{B}$ to $\mathcal{U}$.

(o) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $\vec{x} \cdot \vec{y} = 0$. Then $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$. False. By the Pythagorean Theorem, $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$, but the stated relation need not hold (and counterexamples are easy to find by picking a couple of vectors and trying it).

(p) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Then $|\vec{x} \cdot \vec{y}| = \|\vec{x}\|\|\vec{y}\|$ if and only if $\vec{x}$ and $\vec{y}$ are parallel. True; see Theorem 5.1.11 (the Cauchy-Schwarz inequality).

(q) If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for $\mathbb{R}^n$, then for $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} = (\vec{v_1} \cdot \vec{x})\vec{v_1} + \cdots + (\vec{v_n} \cdot \vec{x})\vec{v_n}$. False. This is only true for an *orthonormal* basis.

(r) If A is a symmetric $n \times n$ matrix, then $A^2 = I_n$. False; e.g., $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. However, this is true if A is both symmetric and orthogonal (since then $A = A^{\mathrm{T}} = A^{-1}$).

(s) Let A and B be $n \times n$ matrices and let A be similar to B. Then $T(M) = AM - MB$ from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$ is an isomorphism. False. As A is similar to B, there is a matrix S such that $AS = SB$. Then $T(S) = 0$, so $\ker(T) \neq \{0\}$. Is this an isomorphism if A is not similar to B? Not necessarily: even though A not being similar to B implies that there is no *invertible* matrix S such that $AS = SB$, it's still possible that there is a noninvertible matrix S with this property.