The exam will cover Section 4.1-Section 5.4 and will include computational questions, conceptual questions, and true/false questions. Problems will not be specifically designed to test material from earlier sections, but this doesn't mean that you don't need to know the earlier sections, as the new material builds on older material. In addition to the below, I recommend reviewing the assigned homework problems and the true/false questions at the end of each chapter in the book. Make sure that you know all of the definitions introduced in the text, can state all of the theorems we've covered, understand the basic methods of proof that we've seen, and know how to check your work to ensure that you didn't make any computational mistakes.

You do not need to show any work on the true/false questions. By "true," I will mean "always true." Thus, if something is sometimes true and sometimes false, you should answer false.

1. Let $V = \{f(t) \in \mathcal{P}_3 | f(0) = f(1)\}$. Show that V is a subspace of $\mathcal{P}_3$ and find a basis of V.

2. Let $V = \{A \in \mathbb{R}^{3\times 3} | A^T = A\}$. Show that V is a subspace of $\mathbb{R}^{3\times 3}$ and find a basis of V.

3. Let $a, b, c \in \mathbb{R}$ with $a \neq b \neq c \neq a$. Show that $\mathfrak{B} = \{(x-b)(x-c), (x-a)(x-c), (x-a)(x-b)\}$ form a basis of $\mathcal{P}_2$.

4. Let $V = \text{span}(\sinh x, \cosh x)$ and $T(f) = f - f'$ be a transformation from V to V. Show that T is a linear transformation and compute $\text{im}(T), \ker(T), \text{rank}(T)$, and $\text{nullity}(T)$. Is T an isomorphism? What do your answers above tell you about solutions to the differential equation $f(x) = f'(x)$?

5. Let $T(A) = A + A^T$ be a transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Show that T is a linear transformation and compute $\text{im}(T), \ker(T), \text{rank}(T)$, and $\text{nullity}(T)$. Is T an isomorphism? What do your answers above tell you about skew-symmetric 2×2 matrices?

6. Let V and W be two vector spaces of dimension n. Let $\mathfrak{B} = \{f_1, \ldots, f_n\}$ be a basis of V and let T be an isomorphism from V to W. Show that $\{T(f_1), \ldots T(f_n)\}$ is a basis of W.

7. We've seen that $V = \{f \in C^\infty | f''(x) - f(x) = 0\}$ is a vector space of dimension 2.
   (a) Verify that $\mathfrak{B} = \{e^x, e^{-x}\}$ and $\mathfrak{U} = \{\sinh x, \cosh x\}$ are both bases of V.
   (b) Find the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{U}}$
   (c) Find $S_{\mathfrak{U}\to\mathfrak{B}}$ by inverting $S_{\mathfrak{B}\to\mathfrak{U}}$.
   (d) Use $S_{\mathfrak{U}\to\mathfrak{B}}$ to write formulas of the form $\sinh x = ae^x + be^{-x}$ and $\cosh x = ce^x + de^{-x}$ for some scalars $a, b, c, d \in \mathbb{R}$.

   (*Hint:* $\sinh(0) = 0$, $\cosh(0) = 1$, $\sinh(\ln 2) = \frac{3}{4}$, and $\cosh(\ln 2) = \frac{5}{4}$.)

8. Let $V = \text{span}(\sin x, \cos x, x \sin x, x \cos x)$ and let $\mathfrak{B} = \{\sin x, \cos x, x \sin x, x \cos x\}$ be a basis of V. Let $T(f) = f'$ be a linear transformation from V to V. Find the $\mathfrak{B}$-matrix of T and use it to determine whether T is an isomorphism.

9. Let A be an invertible $n \times n$ matrix with ijth entry a_{ij}. Let V be a vector space with basis $\mathfrak{U} = \{f_1, \ldots, f_n\}$. Define $g_j = a_{1j}f_1 + \cdots + a_{nj}f_n$ for $1 \leq j \leq n$. Show that $\mathfrak{B} = \{g_1, \ldots, g_n\}$ is a basis of V and find the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{U}}$ in terms of A.

10. Let $f_0, \dots, f_n \in \mathcal{P}_n$ be polynomials such that $\deg(f_i) = i$. (Note that the degree of the zero polynomial is defined to be $-\infty$, so in particular $f_0 \neq 0$.) Show that $f_0, \dots, f_n$ form a basis of $\mathcal{P}_n$.

11. Let $f(x) \in \mathcal{P}_n$ be a polynomial of degree n. Show that $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ form a basis of $\mathcal{P}_n$.

12. Use the Gram-Schmidt process to find an orthonormal basis of $V = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right)$ and in the process find the QR-factorization of the matrix $M = \begin{bmatrix}1 & 2 & 1\\0 & 1 & -1\\1 & 0 & 0\\2 & 2 & 1\end{bmatrix}$.

    Please be sure to check your work on this problem (and ones like it). It's really easy to make a calculation mistake on something like this.

13. Use the Gram-Schmidt process to find an orthonormal basis of $V = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)$ and in the process find the QR-factorization of the matrix $M = \begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\1 & 0 & 1\\1 & 0 & 0\end{bmatrix}$.

14. In the previous two problems, verify that $R = Q^T M$. Explain why this happens.

15. Let V be a subspace of $\mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$. Under what circumstances does $\operatorname{proj}_V \vec{x} = \vec{x}$?

16. Let $a_1, \dots, a_n \in \mathbb{R}$. Find an inequality relating $\displaystyle\sum_{k=1}^{n} (a_k)^2$ and $\left(\displaystyle\sum_{k=1}^{n} a_k\right)^2$.

    (*Hint:* Let $\vec{v} = \begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$ and consider $\vec{v} \cdot \vec{w}$.)

17. Find the matrix of the orthogonal projection from $\mathbb{R}^3$ onto the subspace $W = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$.

18. Find the least-squares solution to the inconsistent system

$$\begin{array}{rcrcr} x & + & 4y & = & -2 \\ x & + & 2y & = & 6 \\ 2x & + & 3y & = & 1. \end{array}$$

19. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0, 1)$, $(1, 3)$, $(2, 4)$, $(3, 4)$ using least-squares.

20. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(-1, 0)$, $(0, -1)$, $(1, 3)$, and $(1, 4)$, using least-squares.

21. Let A be an $n \times m$ matrix whose ijth entry is a_{ij}. Compute $\vec{e_i}^T A \vec{e_j}$.

22. True or False:

    (a) Let $V = \{f \text{ in } C^\infty \mid f'(x) \neq 0 \text{ for all } x\}$. Then V is a subspace of C^∞.

    (b) Let $T(f) = f(0)$ be a linear transformation from $\mathcal{P}_3$ to $\mathbb{R}$. Then T is an isomorphism.

    (c) $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^n$.

    (d) $\mathcal{P}_{11}$ is isomorphic to $\mathbb{R}^{6\times 2}$.

    (e) There is a basis of $\mathbb{R}^{2\times 2}$ consisting of four diagonal matrices.

    (f) Let V and W be vector spaces. Let $\mathfrak{B} = \{f_1, \ldots, f_n\}$ be a basis of V and $\mathfrak{U} = \{g_1, \ldots g_n\}$ be a basis of W. Define a linear transformation T from V to W by $T(f) = c_1 g_1 + \cdots + c_n g_n$ where $[f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then T is an isomorphism.

    (g) Let V be a finite dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $T(f) = f'$ from V to V is a linear transformation. Then T is not an isomorphism.

    (h) Let $V = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} | a + b + c + d = 0\}$. Then V is a subspace of $\mathbb{R}^{2\times 2}$.

    (i) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional, then W is finite dimensional.

    (j) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then V is finite dimensional.

    (k) Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then W is finite dimensional.

    (l) $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$.

    (m) Let T be a linear transformation from a vector space V to a vector space W. If W is finite dimensional, then $\dim(W) = \text{rank}(T) + \ker(T)$.

    (n) Let $\mathcal{U} = \{\vec{u_1}, \ldots, \vec{u_n}\}$ and $\mathcal{B} = \{\vec{b_1}, \ldots, \vec{b_n}\}$ be two bases of a vector space V. Then the change of basis matrix S from $\mathcal{U}$ to $\mathcal{B}$ is given by

    $$S = \begin{bmatrix} [\vec{b_1}]_{\mathcal{U}} & \ldots & [\vec{b_n}]_{\mathcal{U}} \end{bmatrix}.$$

    (o) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $\vec{x} \cdot \vec{y} = 0$. Then $||\vec{x} + \vec{y}|| = ||\vec{x}|| + ||\vec{y}||$.

    (p) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Then $|\vec{x} \cdot \vec{y}| = \|\vec{x}\|\|\vec{y}\|$ if and only if $\vec{x}$ and $\vec{y}$ are parallel.

    (q) If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for $\mathbb{R}^n$, then for $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} = (\vec{v_1} \cdot \vec{x})\vec{v_1} + \cdots + (\vec{v_n} \cdot \vec{x})\vec{v_n}$.

    (r) If A is a symmetric $n \times n$ matrix, then $A^2 = I_n$.

    (s) Let A and B be $n \times n$ matrices and let A be similar to B. Then $T(M) = AM - MB$ from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$ is an isomorphism.