The exam will cover Section 2.3-Section 3.4 and will include computational questions, conceptual questions, and true/false questions. Problems will not be specifically designed to test material from earlier sections, but this doesn't mean that you don't need to know the earlier sections, as the new material builds on older material. In addition to the below, I recommend reviewing the assigned homework problems and the true/false questions at the end of each chapter in the book. Make sure that you know all of the definitions introduced in the text, can state all of the theorems we've covered, understand the basic methods of proof that we've seen, and know how to check your work to ensure that you didn't make any computational mistakes.

You do not need to show any work on the true/false questions. By "true," I will mean "always true." Thus, if something is sometimes true and sometimes false, you should answer false.

1. Find all matrices that commute with $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

2. Let $A = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$. Compute A^{1001} by finding a pattern or by using geometry.

3. Let $A = \frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$. Compute A^{1001} by finding a pattern or by using geometry.

4. Let A be an $n \times n$ diagonal matrix with diagonal entries $a_{11}, \dots, a_{nn}$. Find a formula for A^t in terms of $a_{11}, \dots, a_{nn}$, where t is any positive integer.

5. Are the following matrices invertible? If so, find their inverses. If not, explain how you know that they aren't.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

6. For which values of k are the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ k^2 \end{bmatrix}$ linearly dependent?

7. Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -6 & 1 & 8 \\ 0 & 2 & -4 & 1 & 5 \\ 0 & 7 & -14 & 3 & 18 \end{bmatrix} \text{ with } \operatorname{rref}(A) = \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Find a basis for $\operatorname{im}(A)$.

   (b) Find a basis for $\ker(A)$.

8. Suppose that the vectors $\vec{v_1}, \dots, \vec{v_n}$ form a basis of $\mathbb{R}^n$. Let A be the matrix with column vectors $\vec{v_1}, \dots, \vec{v_n}$. Find $\operatorname{im}(A)$ and $\ker(A)$.

9. Can you find a 2×2 matrix A such that $\operatorname{im}(A) = \ker(A)$. What about a 3×3 matrix? 4×4? $n \times n$?

10. Let $a, b, c \in \mathbb{R}$ where $a \neq 0$. Find a basis for the subspace V of $\mathbb{R}^3$ defined by

$$V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 | ax + by + cz = 0\}.$$

11. Let A be an $n \times p$ matrix such that $\ker(A) = \{\vec{0}\}$ and B be a $p \times m$ matrix. How does $\ker(AB)$ relate to $\ker(B)$?

12. Let V and W be subspaces of $\mathbb{R}^n$ and define $V - W = \{\vec{v} - \vec{w} \in \mathbb{R}^n | \vec{v} \in V, \vec{w} \in W\}$. Show that $V - W$ is a subspace of $\mathbb{R}^n$.

13. Let V be a subspace of $\mathbb{R}^n$ and define $V^\perp = \{\vec{x} \in \mathbb{R}^n | \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}$. Show that $V^\perp$ is a subspace of $\mathbb{R}^n$.

14. Let $\vec{v_1}, \dots, \vec{v_m} \in \mathbb{R}^n$. Let $T(\vec{x})$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$. If $\vec{v_1}, \dots, \vec{v_m}$ are linearly dependent, are $T(\vec{v_1}), \dots, T(\vec{v_m})$ necessarily linearly dependent? If $\vec{v_1}, \dots, \vec{v_m}$ are linearly independent, are $T(\vec{v_1}), \dots, T(\vec{v_m})$ necessarily linearly independent?

15. Consider two 3×3 matrices A and B and a vector $\vec{v} \in \mathbb{R}^3$ such that $AB\vec{v} = \vec{0}$, $BA\vec{v} = \vec{0}$, and $A^2\vec{v} = \vec{0}$, but $A\vec{v} \neq \vec{0}$ and $B^2\vec{v} \neq \vec{0}$.

    (a) Show that the vectors $\mathfrak{B} = \{A\vec{v}, B\vec{v}, \vec{v}\}$ form a basis of $\mathbb{R}^3$.

    (b) Find the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$.

16. We say that a set of vectors $\vec{v_1}, \dots, \vec{v_m} \in \mathbb{R}^n$ are *orthonormal* if they are perpendicular unit vectors. That is, if

$$\vec{v_i} \cdot \vec{v_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

    Show that if $\vec{u_1}, \dots, \vec{u_n} \in \mathbb{R}^n$ are orthonormal, then $\mathfrak{B} = \{\vec{u_1}, \dots, \vec{u_n}\}$ form a basis of $\mathbb{R}^n$. (Hint: Let $c_1\vec{u_1} + \cdots + c_n\vec{u_n} = \vec{0}$ and take the dot product of both sides with $\vec{u_j}$ for $j = 1, \dots, n$.)

17. Let $A = \begin{bmatrix} 12 & 3 & 3 \\ 11 & 2 & 2 \\ 7 & 1 & 1 \end{bmatrix}$ and let $\mathfrak{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ be a basis of $\mathbb{R}^3$. Find the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$.

18. Let A be a matrix of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2 + b^2 = 1$ and $a \neq 1$, and let $\mathfrak{B} = \{ \begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix} \}$ be a basis of $\mathbb{R}^2$. Find the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$. Interpret your answer geometrically.

19. Let A be a matrix of the form $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ where $u_1^2 + u_2^2 = 1$, and let $\mathfrak{B} = \{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \}$ be a basis of $\mathbb{R}^2$. Find the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$. Interpret your answer geometrically.

20. True or False:

(a) Let $\mathfrak{B} = \{\vec{e_n}, \dots, \vec{e_1}\}$ (the standard basis vectors in the opposite order) and let $A = \begin{bmatrix} \vec{v_1} & \dots & \vec{v_n} \end{bmatrix}$. Then the $\mathfrak{B}$-matrix of $T(\vec{x}) = A\vec{x}$ is the matrix $\begin{bmatrix} \vec{v_n} & \dots \vec{v_1} \end{bmatrix}$ (the column vectors in the opposite order).

(b) Let A be an $n \times n$ matrix. If $A^2 = A$, then $A = I_n$.

(c) Let A be an $n \times m$ matrix. Then $\dim(\text{im}(A)) + \dim(\ker(A)) = n$.

(d) If A is a 2×2 matrix representing an orthogonal projection, then A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(e) Let A and B be $n \times n$ matrices. If A is similar to B, then A^t is similar to B^t for every positive integer t.

(f) Let A and B be $n \times n$ matrices such that B is similar to A. If A is invertible, then B is invertible.

(g) If A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(h) Let V be the set of all unit vectors in $\mathbb{R}^n$ (that is, the set of all vectors $\vec{x}$ such that $\vec{x} \cdot \vec{x} = 1$). Then V is a subspace of $\mathbb{R}^n$.

(i) If V is a subspace of $\mathbb{R}^n$, then $\dim(V) \leq n$.

(j) If the vectors $\vec{v_1}, \dots, \vec{v_m} \in \mathbb{R}^n$ are linearly independent and the vector $\vec{v} \in \mathbb{R}^n$ is not in $\text{span}(\vec{v_1}, \dots, \vec{v_m})$, then the vectors $\vec{v_1}, \dots, \vec{v_m}, \vec{v}$ are linearly independent.

(k) If the ith column of an $n \times m$ matrix A is equal to the jth column of A for some $i \neq j$, then the vector $\vec{e_i} - \vec{e_j}$ is in the kernel of A.

(l) Suppose the vectors $\vec{v_1}, \dots, \vec{v_m} \in \mathbb{R}^n$ are linearly independent. Define the vectors

$$\vec{w_j} = \sum_{i=1}^{j} \vec{v_i}$$

for $j = 1, \dots, m$ to be the sum of the first j vectors from $\vec{v_1}, \dots, \vec{v_m}$ (so, for example, $\vec{w_1} = \vec{v_1}, \vec{w_2} = \vec{v_1} + \vec{v_2}, \dots, \vec{w_m} = \vec{v_1} + \cdots + \vec{v_m}$). Then the vectors $\vec{w_1}, \dots, \vec{w_m}$ are linearly independent.

(m) Let $\vec{v_1}, \dots, \vec{v_m}, \vec{v}, \vec{w} \in \mathbb{R}^n$. If $\vec{v}$ is not a linear combination of $\vec{v_1}, \dots, \vec{v_m}$ and $\vec{w}$ is not a linear combination of $\vec{v_1}, \dots, \vec{v_m}$, then $(\vec{v} + \vec{w})$ is not a linear combination of $\vec{v_1}, \dots, \vec{v_m}$.