Answers in blue. If you have questions or spot an error, let me know.

1. Use Gauss-Jordan elimination to find all solutions of the system:

   (a)

$$\begin{array}{rcrcrcrcl} 3x & + & 2y & - & 2z & - & w & = & 3 \\ x & + & y & + & z & + & 2w & = & 5 \\ & & 3y & - & 3z & - & 3w & = & 0. \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 - t/3 \\ 2 - t/3 \\ 2 - 4t/3 \\ t \end{bmatrix}$$

   (b)

$$\begin{array}{rcrcrcl} 2x & + & y & - & z & = & 0 \\ x & + & 2y & + & 4z & = & 3 \\ & & 2y & + & 6z & = & 4. \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t - 1 \\ 2 - 3t \\ t \end{bmatrix}$$

   (c)

$$\begin{array}{rcrcrcl} 5x & + & 6y & + & 2z & = & 28 \\ 4x & + & 4y & + & z & = & 20 \\ 2x & + & 3y & + & z & = & 13 \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

   (d)

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 6x_2 & + & x_3 & - & 2x_4 & = & 9 \\ 2x_1 & + & 4x_2 & + & x_3 & - & x_4 & = & 6 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - 2s + t \\ s \\ -t \\ t \end{bmatrix}$$

2. Determine the values of k for which each system has: i) no solution, ii) a unique solution, iii) infinitely many solutions:

   (a)

$$\begin{array}{rcrcrcl} 3x & - & y & + & 5z & = & 2 \\ 2x & + & 4y & + & 6z & = & 8 \\ 5x & + & 3y & - & 11z & = & k + 6 \end{array}$$

   There is a unique solution for all values of k.

(b)

$$\begin{array}{rcrcl} x & + & 2y & = & 2 \\ 2x & + & (k^2-5)y & = & k+1 \end{array}$$

There are infinitely many solutions for $k = 3$, no solution for $k = -3$, and a unique solution for $k \neq \pm 3$.

3. Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -6 & 24 \\ 1 & -2 & 6 \\ -1 & 2 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 7 & 5 \\ -2 & 1 & 0 & -3 \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

(a) Find rref(A).

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{rref}(B) = I_3$,

$\text{rref}(C) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{rref}(D) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) Find rank(A).

Counting leading ones, $\text{rank}(A) = 2$, $\text{rank}(B) = 3$, $\text{rank}(C) = 1$, and $\text{rank}(D) = 2$.

(c) Is A invertible? If so, find A^{-1}. If not, explain how you know that it isn't.

A and C aren't invertible since they don't have rank 3. D isn't invertible since it isn't square.

$B^{-1} = \begin{bmatrix} -1 & 18/5 & 3/5 \\ -1/2 & 8/5 & 3/5 \\ 0 & 1/10 & 1/10 \end{bmatrix}$.

(d) Compute $A(2\vec{e_1} + 3\vec{e_3})$.

$A(2\vec{e_1} + 3\vec{e_3}) = \begin{bmatrix} 23 \\ -4 \\ 11 \end{bmatrix}$, $B(2\vec{e_1} + 3\vec{e_3}) = \begin{bmatrix} 77 \\ 20 \\ -14 \end{bmatrix}$, $C(2\vec{e_1} + 3\vec{e_3}) = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$, $D(2\vec{e_1} + 3\vec{e_3}) = \begin{bmatrix} 23 \\ -4 \\ 11 \end{bmatrix}$.

4. In each case below, express the vector $\vec{w}$ as a linear combination of $\vec{v_1}, \dots, \vec{v_m}$ or explain why you can't.

(a) $\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \vec{v_1} = \vec{e_1}, \vec{v_2} = \vec{e_2}$ $\vec{w} = 2\vec{v_1} + 3\vec{v_2}$

(b) $\vec{w} = \begin{bmatrix} -1 \\ 13 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{w} = 3\vec{v_1} - 7\vec{v_2}$

(c) $\vec{w} = \begin{bmatrix} 7 \\ 2 \\ 6 \\ 3 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}$ $\vec{w} = \vec{v_1} + 3\vec{v_2} + 0\vec{v_3}$

(d) $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{w}$ is not a linear combination of $\vec{v_1}$ and $\vec{v_2}$. If we write $\vec{w} = a\vec{v_1} + b\vec{v_2}$, the corresponding system of equations in the variables a and b is inconsistent.

5. For scalars a and b with $a^2 + b^2 = 1$, consider the matrix

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

(a) What is the geometrical effect of multiplying $\vec{x}$ by A?
A is a reflection matrix.

(b) Compute A^{-1} when it exists. For what values of a and b does A^{-1} not exist?
Using the formula for #13 in Section 2.1 (which is good to know!), $A^{-1} = A$ (because $a^2 + b^2 = 1$). The inverse wouldn't exist if and only if $a = b = 0$, but since we have the restriction $a^2 + b^2 = 1$, it always works for the values of a and b that we're considering.

(c) Use geometry to explain your result in part (b).
The way to "undo" the reflection of a vector across a line is to reflect it back across the same line, which is why A is its own inverse.

6. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ with $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Compute $T(3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \end{bmatrix})$.

$$T(3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 3T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) - 4T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}.$$

7. A linear system of the form $A\vec{x} = \vec{0}$ is called **homogeneous**. Justify the following facts:

(a) If $\vec{v_1}$ and $\vec{v_2}$ are solutions of $A\vec{x} = \vec{0}$, then $(\vec{v_1} + \vec{v_2})$ is a solution as well.
$A(\vec{v_1} + \vec{v_2}) = A\vec{v_1} + A\vec{v_2} = \vec{0} + \vec{0} = \vec{0}$.

(b) If $\vec{v}$ is a solution of $A\vec{x} = \vec{0}$ and k is a scalar, then $k\vec{v}$ is a solution as well.
$A(k\vec{v}) = kA\vec{v} = k\vec{0} = \vec{0}$.

8. Classify each of the following matrices as either a scaling, an orthogonal projection, a shear, a reflection, or a rotation. (Use each option once.)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

A is a rotation, B is an orthogonal projection, C is a scaling, D is a shear, and E is a reflection.

9. Calculate the matrix for a rotation of θ in the counterclockwise direction around the y-axis in $\mathbb{R}^3$. (You may assume that this transformation is linear.)

   Note: This problem doesn't actually have a unique solution as stated, since which direction is "counterclockwise" depends on whether we're looking from the positive y-axis or from the negative y-axis. I meant to say "the counterclockwise direction as viewed from the positive y-axis." Oops. Sorry about that; let's pretend that that's what I said.

   We don't have a formula for this, so let's think it out. The matrix of a linear transformation is $\begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}$, so we just need to figure out what such a rotation will do to the standard vectors. We have $T(\vec{e_1}) = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}$, $T(\vec{e_2}) = \vec{e_2}$, and $T(\vec{e_3}) = \begin{bmatrix} \cos(\theta - \frac{\pi}{2}) \\ 0 \\ \sin(\theta - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix}$, so the matrix is $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}$.
   (Note that this is for a right-handed coordinate system. If you prefer working in a left-handed coordinate system, the answer is $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$. As the use of right-handed coordinate systems is conventional in math, you should explicitly specify if you aren't doing this.)

10. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Define the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}^1$.

    (a) Show that T is a linear transformation by finding its matrix.
    The matrix is $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

    (b) Using part (a), show that $\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$ and that $\vec{v} \cdot (k\vec{w}) = k(\vec{v} \cdot \vec{w})$ for all vectors $\vec{u}, \vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ and for all scalars k.
    Since we know that T is linear, $\vec{v} \cdot (\vec{u} + \vec{w}) = T(\vec{u} + \vec{w}) = T(\vec{u}) + T(\vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$ and $\vec{v} \cdot (k\vec{w}) = T(k\vec{w}) = kT(\vec{w}) = k(\vec{v} \cdot \vec{w})$.

11. True or false:

    (a) If a system of equations has fewer equations than unknowns, then it has infinitely many solutions. False; it could have no solution.

    (b) If A in an $n \times m$ matrix, then $\text{rank}(A) \leq n$. True. See Example 3 in Section 1.3.

    (c) If A in an $n \times n$ matrix and $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$. False; this is only true if $\text{rank}(A) = n$. e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

    (d) If a square matrix has two equal columns, then it is not invertible. True. Its rref will not be I_n.

    (e) If a square matrix has two equal rows, then it is not invertible. True. Its rref will not be I_n.

(f) There exists a 2×2 matrix A such that $\text{rank}(A) = 0$. True. The zero matrix.

(g) There exists a 2×2 matrix A such that $\text{rank}(A) = 4$. False. $\text{rank}(A) \leq 2$.

(h) A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$ must be invertible. True. cf. Section 2.1, #13. Or, note that it's a rotation, so clearly invertible by geometry (as you can invert it by rotating in the opposite direction).

(i) If A is a 3×4 matrix, then $A\vec{x} = \vec{0}$ has infinitely many solutions. True. It has more variables than equations, so it can't have a unique solution, and it's homogeneous, so it can't be inconsistent (as $\vec{x} = \vec{0}$ is one solution.)

(j) The solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{e_1}$ form a line in $\mathbb{R}^2$. False. This system is inconsistent.

(k) If $\vec{v}$ and $\vec{w}$ are two solutions to $A\vec{x} = \vec{b}$, then $(\vec{v} + \vec{w})$ is a solution too. False. $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{b} + \vec{b} = 2\vec{b}$. As $2\vec{b} \neq \vec{b}$ except when $\vec{b} = \vec{0}$, this is not generally true.

(l) If A is an upper-triangular matrix, then A is invertible. False. For example, the zero matrix is upper-triangular, but it isn't invertible.

12. Let A be an $n \times m$ matrix and $\vec{v}$ and $\vec{w}$ vectors in $\mathbb{R}^m$. Prove that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.

    See Theorem 1.3.10 in your book for a proof.

13. Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that the matrix of T is

$$A = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}.$$

    See Theorem 2.1.2 in your book. The book doesn't provide a full proof, but I gave one in class. In case you missed it:

    Let $A = \begin{bmatrix} \vec{v_1} & \dots & \vec{v_m} \end{bmatrix}$ be the matrix of T (that is, $\vec{v_1}, \dots, \vec{v_m}$ are the columns of A.) Then $T(\vec{e_i}) = A\vec{e_i} = \vec{v_i}$, so that $A = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}$.

14. Let $S(\vec{x})$ and $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Define $R(\vec{x}) = S(\vec{x}) + T(\vec{x})$. Prove that $R(\vec{x})$ is a linear transformation.

    We need to show two things: $R(\vec{v} + \vec{w}) = R(\vec{v}) + R(\vec{w})$ and $R(k\vec{v}) = kR(\vec{v})$.
    Let $\vec{v}$, $\vec{w}$ in $\mathbb{R}^m$ and k be a scalar. Then:

$$R(\vec{v}+\vec{w}) = S(\vec{v}+\vec{w})+T(\vec{v}+\vec{w}) = S(\vec{v})+S(\vec{w})+T(\vec{v})+T(\vec{w}) = S(\vec{v})+T(\vec{v})+S(\vec{w})+T(\vec{w}) = R(\vec{v})+R(\vec{w}).$$
$$R(k\vec{v}) = S(k\vec{v}) + T(k\vec{v}) = kS(\vec{v}) + kT(\vec{v}) = k(S(\vec{v}) + T(\vec{v})) = kR(\vec{v}).$$

15. Let $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and k be a scalar. Define $R(\vec{x}) = kT(\vec{x})$. Prove that $R(\vec{x})$ is a linear transformation.

    We need to show two things: $R(\vec{v} + \vec{w}) = R(\vec{v}) + R(\vec{w})$ and $R(k\vec{v}) = kR(\vec{v})$.
    Let $\vec{v}$, $\vec{w}$ in $\mathbb{R}^m$ and c be a scalar (c as k is already in use). Then:

$$R(\vec{v} + \vec{w}) = kT(\vec{v} + \vec{w}) = k(T(\vec{v}) + T(\vec{w})) = kT(\vec{v}) + kT(\vec{w}) = R(\vec{v}) + R(\vec{w}).$$
$$R(c\vec{v}) = kT(c\vec{v}) = kcT(\vec{v}) = ckT(\vec{v}) = cR(\vec{v}).$$