The exam will cover Section 1.1-Section 2.2 and will include computational questions, conceptual questions, and true/false questions. In addition to the below, I recommend reviewing the assigned homework problems and the true/false questions at the end of each chapter in the book. Make sure that you know all of the definitions introduced in the text, can state all of the theorems we've covered, understand the basic methods of proof that we've seen, and know how to check your work to ensure that you didn't make any computational mistakes.

You do not need to show any work on the true/false questions. By "true," I will mean "always true." Thus, if something is sometimes true and sometimes false, you should answer false.

1. Use Gauss-Jordan elimination to find all solutions of the system:

   (a)
$$\begin{array}{rcrcrcrcr} 3x & + & 2y & - & 2z & - & w & = & 3 \\ x & + & y & + & z & + & 2w & = & 5 \\ & & 3y & - & 3z & - & 3w & = & 0. \end{array}$$

   (b)
$$\begin{array}{rcrcrcr} 2x & + & y & - & z & = & 0 \\ x & + & 2y & + & 4z & = & 3 \\ & & 2y & + & 6z & = & 4. \end{array}$$

   (c)
$$\begin{array}{rcrcrcr} 5x & + & 6y & + & 2z & = & 28 \\ 4x & + & 4y & + & z & = & 20 \\ 2x & + & 3y & + & z & = & 13 \end{array}$$

   (d)
$$\begin{array}{rcrcrcrcr} 3x_1 & + & 6x_2 & + & x_3 & - & 2x_4 & = & 9 \\ 2x_1 & + & 4x_2 & + & x_3 & - & x_4 & = & 6 \end{array}$$

2. Determine the values of $k$ for which each system has: i) no solution, ii) a unique solution, iii) infinitely many solutions:

   (a)
$$\begin{array}{rcrcrcr} 3x & - & y & + & 5z & = & 2 \\ 2x & + & 4y & + & 6z & = & 8 \\ 5x & + & 3y & - & 11z & = & k+6 \end{array}$$

   (b)
$$\begin{array}{rcrcr} x & + & 2y & = & 2 \\ 2x & + & (k^2-5)y & = & k+1 \end{array}$$

3. Let
$$A = \begin{bmatrix} 1 & 3 & 7 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -6 & 24 \\ 1 & -2 & 6 \\ -1 & 2 & -4 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 7 & 5 \\ -2 & 1 & 0 & -3 \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

(a) Find rref($A$).
(b) Find rank($A$).
(c) Is $A$ invertible? If so, find $A^{-1}$. If not, explain how you know that it isn't.
(d) Compute $A(2\vec{e_1} + 3\vec{e_3})$.
(e) Same questions for $B$, $C$, and $D$.

4. In each case below, express the vector $\vec{w}$ as a linear combination of $\vec{v_1}, \ldots, \vec{v_m}$ or explain why you can't.

(a) $\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \vec{v_1} = \vec{e_1}, \vec{v_2} = \vec{e_2}$

(b) $\vec{w} = \begin{bmatrix} -1 \\ 13 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\vec{w} = \begin{bmatrix} 7 \\ 2 \\ 6 \\ 3 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}$

(d) $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

5. For scalars $a$ and $b$ with $a^2 + b^2 = 1$, consider the matrix

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

(a) What is the geometrical effect of multiplying $\vec{x}$ by $A$?
(b) Compute $A^{-1}$ when it exists. For what values of $a$ and $b$ does $A^{-1}$ not exist?
(c) Use geometry to explain your result in part (b).

6. Let $T$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ with $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Compute $T(3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \end{bmatrix})$.

7. A linear system of the form $A\vec{x} = \vec{0}$ is called **homogeneous**. Justify the following facts:

(a) If $\vec{v_1}$ and $\vec{v_2}$ are solutions of $A\vec{x} = 0$, then $(\vec{v_1} + \vec{v_2})$ is a solution as well.
(b) If $\vec{v}$ is a solution of $A\vec{x} = 0$ and $k$ is a scalar, then $k\vec{v}$ is a solution as well.

8. Classify each of the following matrices as either a scaling, an orthogonal projection, a shear, a reflection, or a rotation. (Use each option once.)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

9. Calculate the matrix for a rotation of $\theta$ in the counterclockwise direction around the $y$-axis in $\mathbb{R}^3$. (You may assume that this transformation is linear.)

10. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Define the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}^1$.

    (a) Show that $T$ is a linear transformation by finding its matrix.

    (b) Using part (a), show that $\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$ and that $\vec{v} \cdot (k\vec{w}) = k(\vec{v} \cdot \vec{w})$ for all vectors $\vec{u}, \vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ and for all scalars $k$.

11. True or false:

    (a) If a system of equations has fewer equations than unknowns, then it has infinitely many solutions.

    (b) If $A$ in an $n \times m$ matrix, then $\text{rank}(A) \leq n$.

    (c) If $A$ in an $n \times n$ matrix and $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

    (d) If a square matrix has two equal columns, then it is not invertible.

    (e) If a square matrix has two equal rows, then it is not invertible.

    (f) There exists a $2 \times 2$ matrix $A$ such that $\text{rank}(A) = 0$.

    (g) There exists a $2 \times 2$ matrix $A$ such that $\text{rank}(A) = 4$.

    (h) A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$ must be invertible.

    (i) If $A$ is a $3 \times 4$ matrix, then $A\vec{x} = \vec{0}$ has infinitely many solutions.

    (j) The solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{e_1}$ form a line in $\mathbb{R}^2$.

    (k) If $\vec{v}$ and $\vec{w}$ are two solutions to $A\vec{x} = \vec{b}$, then $(\vec{v} + \vec{w})$ is a solution too.

    (l) If $A$ is an upper-triangular matrix, then $A$ is invertible.

12. Let $A$ be an $n \times m$ matrix and $\vec{v}$ and $\vec{w}$ vectors in $\mathbb{R}^m$. Prove that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.

13. Let $T$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that the matrix of $T$ is

$$A = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}.$$

14. Let $S(\vec{x})$ and $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Define $R(\vec{x}) = S(\vec{x}) + T(\vec{x})$. Prove that $R(\vec{x})$ is a linear transformation.

15. Let $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $k$ be a scalar. Define $R(\vec{x}) = kT(\vec{x})$. Prove that $R(\vec{x})$ is a linear transformation.