Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 5.3

2. Is orthogonal.

4. Is not orthogonal. The first column is not orthogonal to the third column, and the second column is not orthogonal to the third column.

28. There are a few ways to do this. The simplest is to use #27: $L(\vec{v}) \cdot L(\vec{w}) = (A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot (A^T A\vec{w}) = \vec{v} \cdot \vec{w}$ since $A^T A = I_n$ for symmetric matrices.

    Alternatively, you can rewrite the equation $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$ as $\vec{v} \cdot \vec{w} = \frac{1}{2}(\|\vec{v} + \vec{w}\|^2 - \|\vec{v}\|^2 - \|\vec{w}\|^2)$, so that $L(\vec{v}) \cdot L(\vec{w}) = \frac{1}{2}(\|L(\vec{v}) + L(\vec{w})\|^2 - \|L(\vec{v})\|^2 - \|L(\vec{w})\|^2) = \frac{1}{2}(\|\vec{v} + \vec{w}\|^2 - \|\vec{v}\|^2 - \|\vec{w}\|^2) = \vec{v} \cdot \vec{w}$ since L is a linear transformation and preserves the length of vectors.

    Or, a third method is to write everything in terms of an orthonormal basis and then compute both sides directly. That is, let $\vec{u_1}, \dots, \vec{u_n}$ form an orthonormal basis for $\mathbb{R}^n$ (for example, take $\vec{u_i} = \vec{e_i}$ for $i = 1, \dots, n$) and note that $L(\vec{u_1}), \dots, L(\vec{u_n})$ will also be an orthonormal basis of $\mathbb{R}^n$ since L is orthogonal (see Theorem 5.3.2, or Theorem 5.3.3 for the special case $\vec{u_i} = \vec{e_i}$). Write $\vec{v} = c_1\vec{u_1} + \cdots + c_n\vec{u_n}$ and $\vec{w} = d_1\vec{u_1} + \cdots + d_n\vec{u_n}$. Then, compute $\vec{v} \cdot \vec{w} = c_1 d_1 + \cdots + c_n d_n$ and $L(\vec{v}) \cdot L(\vec{w}) = (c_1 L(\vec{u_1}) + \cdots + c_n L(\vec{u_n})) \cdot (d_1 L(\vec{u_1}) + \cdots + d_n L(\vec{u_n})) = c_1 d_1 + \cdots + c_n d_n$, so that $\vec{v} \cdot \vec{w} = L(\vec{v}) \cdot L(\vec{w})$.

30. L preserves length, so the only solution to $L(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$. Thus, $\ker(L) = \{\vec{0}\}$. By Rank-Nullity, $\dim(\operatorname{im}(L)) = m$. We saw back in chapter 1 that $\operatorname{rank}(L) \leq n$, so $m \leq n$ as $\operatorname{rank}(L) = m$. Following the proofs of Theorems 5.3.2 and 5.3.3 (but dropping the assumption that $m = n$), we see that the columns of A will be orthonormal. Computing $A^T A$ as in Theorem 5.3.7 (again, dropping the assumption $m = n$), we see that $A^T A = I_m$. From Theorem 5.3.10, we see that AA^T is the matrix of the orthogonal projection onto the space V spanned by the columns of A, and that the jth column of A is $\operatorname{proj}_V(\vec{e_j})$.

32. (a) Not necessarily. Counterexamples are easy to find. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will work.

    (b) This is necessarily true, as Summary 5.3.8 tells us that $A^T = A^{-1}$ in this case.

36. Either find a basis of $\mathbb{R}^3$ containing the first two columns of the matrix as basis vectors and use Gram-Schmidt, or note that $A^T A = I_3$ and using the resulting equations to solve for a,b, and c. Either way, you should come up with either $\begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \end{bmatrix}$ or $\begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

42. Geometrically, an orthogonal projection does nothing on the space it projects onto, so that the second projection will do nothing new and $A^2 = A$. Algebraically, write $A = QQ^T$ where Q is orthogonal, so that $A^2 = QQ^TQQ^T = QQ^T = A$ since $Q^TQ = I_n$.

Section 5.4

3. By Theorem 5.1.8(c), we know that $p+q = n$, so it is enough to either show that these vectors are linearly independent or that they span $\mathbb{R}^n$. I've provided arguments for both techniques. If you noted that $p + q = n$, you only need one of them. Otherwise, you need both halves.

   Linear Independence:

   Write $c_1\vec{v_1} + \cdots + c_p\vec{v_p} + d_1\vec{w_1} + \cdots + w_q\vec{w_q} = \vec{0}$. We want to show that $c_1 = \cdots = c_p = d_1 = \cdots = d_q = 0$.

   One strategy is to rewrite this relation as $c_1\vec{v_1}+\cdots+c_p\vec{v_p} = -d_1\vec{w_1}-\cdots-w_q\vec{w_q}$. The left-hand side is in V and the right-hand side is in $V^\perp$, so both sides are in $V \cap V^\perp$. But by Theorem 5.1.8(b), we know that $V \cap V^\perp = \{\vec{0}\}$, so that $c_1\vec{v_1} + \cdots + c_p\vec{v_p} = -d_1\vec{w_1} - \cdots - w_q\vec{w_q} = 0$. Then, the linear independence of $\vec{v_1}, \ldots, \vec{v_p}$ tells us that $c_1 = \cdots = c_p = 0$ and the linear independence of $\vec{w_1}, \ldots, \vec{w_q}$ tells us that $d_1 = \cdots = d_q = 0$. Thus, $\vec{v_1}, \ldots, \vec{v_p}, \vec{w_1}, \ldots, \vec{w_q}$ are linearly independent.

   Another strategy is to take proj_V of both sides of this relation to show that $c_1\vec{v_1}+\cdots+c_p\vec{v_p} = \vec{0}$ and subtract this from the original relation to see that $d_1\vec{w_1} + \cdots + w_q\vec{w_q} = \vec{0}$ as well, and then proceed as above. This is a special case of the solution to #5 on the 2nd midterm exam.

   Spanning:

   Let $\vec{x} \in \mathbb{R}^n$ be an arbitrary vector. We will show that $\vec{x}$ is in the span of $\vec{v_1}, \ldots, \vec{v_p}, \vec{w_1}, \ldots, \vec{w_q}$. We know that we can uniquely write $\vec{x} = \vec{x}^\parallel + \vec{x}^\perp$ where $\vec{x}^\parallel \in V$ and $\vec{x}^\perp \in V^\perp$. As $\vec{x}^\parallel \in V$, we know that $\vec{x}^\parallel = c_1\vec{v_1} + \cdots + c_p\vec{v_p}$ for some choice of scalars $c_1, \ldots, c_p$, since $\vec{v_1}, \ldots, \vec{v_p}$ form a basis of V. Similarly, the fact that $\vec{w_1}, \ldots, \vec{w_q}$ form a basis of $V^\perp$ tells us that we can write $\vec{x}^\perp = d_1\vec{w_1}+\cdots+w_q\vec{w_q}$ for some scalars $d_1, \ldots, d_q$. Then, $\vec{x} = c_1\vec{v_1}+\cdots+c_p\vec{v_p}+d_1\vec{w_1}+\cdots+w_q\vec{w_q}$ so that $\vec{x}$ is indeed in $\text{span}(\vec{v_1}, \ldots, \vec{v_p}, \vec{w_1}, \ldots, \vec{w_q})$, and, as $\vec{x}$ was an arbitrary vector in $\mathbb{R}^n$, we see that $\text{span}(\vec{v_1}, \ldots, \vec{v_p}, \vec{w_1}, \ldots, \vec{w_q}) = \mathbb{R}^n$.

4. Start with $(\text{im}(A))^\perp = \ker(A^T)$. Taking perps of both sides gives $\text{im}(A) = (\ker(A^T))^\perp$. Now, this relation holds for any matrix A, so in particular we can replace the A on both sides by A^T. Then, since $(A^T)^T = A$, we have that $(\ker(A))^\perp = \text{im}(A^T)$.