Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 5.1

15. Note that the problem is asking you to find a basis of the space of column vectors $\vec{x}$ such that $\vec{v}\cdot\vec{x} = 0$, that is $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$. This is one equation in four unknowns, so the space in question is $\ker(\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix})$, for which a basis can be calculated in the usual way.

22. (i) Suppose that $\vec{x} \in \mathbb{R}^n$ is orthogonal to V. We will show that $\vec{x}$ is orthogonal to the basis vectors $\vec{v_1}, \dots, \vec{v_n}$ of V.

    By definition, $\vec{x}$ is orthogonal to every vector in V. Note $\vec{v_1}, \dots, \vec{v_n} \in V$, so in particular $\vec{x}$ is orthogonal to $\vec{v_1}, \dots, \vec{v_n}$.

    (ii) Suppose that $\vec{x} \in \mathbb{R}^n$ is orthogonal to $\vec{v_1}, \dots, \vec{v_n}$; that is, $\vec{x}\cdot\vec{v_i} = 0$ for $i = 1, \dots, n$. We will show that $\vec{x}$ is orthogonal to V.

    Let $\vec{v} \in V$ be an arbitrary element. Then, it is enough to show that $\vec{x}\cdot\vec{v} = 0$. Since $\vec{v_1}, \dots, \vec{v_n}$ form a basis of V, there exist scalars $c_1, \dots, c_n$ such that $\vec{v} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$. Then, by the linearity of the dot product, $\vec{x}\cdot\vec{v} = \vec{x}\cdot(c_1\vec{v_1} + \cdots + c_n\vec{v_n}) = c_1(\vec{x}\cdot\vec{v_1}) + \cdots + c_n(\vec{x}\cdot\vec{v_n}) = c_1(0) + \cdots + c_n(0) = 0$ as claimed. Thus, $\vec{x}$ is orthogonal to V.

28. Note that the given basis vectors are not orthonormal. However, they are orthogonal, so we can make an orthonormal basis by dividing each vector by its length: $\vec{u_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u_2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, and $\vec{u_3} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Then we can calculate the projection using Theorem 5.1.5. We find $\text{proj}_V(\vec{x}) = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

30. There are a few ways to do this:

    (a) Let $\vec{u_1}, \dots, \vec{u_m}$ be an orthonormal basis of V. Then, by Theorem 5.1.5, $\vec{y} = (\vec{u_1}\cdot\vec{x})\vec{u_1} + \cdots + (\vec{u_m}\cdot\vec{x})\vec{u_m}$, so that $\|\vec{y}\|^2 = (\vec{u_1}\cdot\vec{x})^2 + \cdots + (\vec{u_n}\cdot\vec{x})^2$ (either by the Pythagorean Theorem, or by direct calculation). Also, $\vec{y}\cdot\vec{x} = (\vec{u_1}\cdot\vec{x})^2 + \cdots + (\vec{u_n}\cdot\vec{x})^2$ by direct calculation, so $\|\vec{y}\|^2 = \vec{y}\cdot\vec{x}$.

    (b) For the angle θ defined in Definition 5.1.12, note that $\vec{y}\cdot\vec{x} = \|\vec{y}\|\|\vec{x}\|\cos\theta$. For $\vec{y} = \text{proj}_V(\vec{x})$, geometry shows that $\|\vec{y}\| = \|\vec{x}^{\parallel}\| = \|\vec{x}\|\cos\theta$ (draw a triangle with sides $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$ to see this), so that $\vec{y}\cdot\vec{x} = \|\vec{y}\|(\|\vec{x}\|\cos\theta) = \|\vec{y}\|\|\vec{y}\| = \|\vec{y}\|^2$.

(c) Write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$. Since $\vec{y} = \vec{x}^{\parallel}$, we know that $\vec{y} \cdot \vec{x}^{\perp} = 0$, so that $\vec{y} \cdot \vec{x} = \vec{y} \cdot (\vec{x} - \vec{x}^{\perp}) = \vec{y} \cdot (\vec{x}^{\parallel}) = \vec{y} \cdot \vec{y} = \|\vec{y}\|^2$.

31. We calculate that $p = \|\operatorname{proj}_V \vec{x}\|^2$ (either directly, or using the Pythagorean Theorem), so Theorem 5.1.10 tells us that $p \leq \|\vec{x}\|^2$ and that these two quantities are equal if and only if $\vec{x} \in V$.

Section 5.2

32. First, let's find *any* basis of the plane $x_1 + x_2 + x_3 = 0$, and then we can find an orthonormal basis by applying Gram-Schmidt. Note that this plane is $\ker(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})$, so finding a basis of this kernel is one way to proceed. Alternatively, we know that a plane has dimension 2, so that any two linearly independent vectors form a basis. Finding two linearly independent vectors on this plane is easy (pick values for x_1 and x_2 and then let $x_3 = -x_1 - x_2$), for example, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ form a basis. Taking Gram-Schmidt of this basis, we end up with the basis $\{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3/2}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\}$.
One trick we can use to make this slightly easier is to try to start with orthogonal vectors, so that we just need to normalize them. Can we find a vector in the plane which is orthogonal to (for example) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$? One way to do this is to use the zero in the second component to our advantage: any vector whose first and third components are equal will be orthogonal to this vector, and we can then find something to put in the second component to force the vector to lie in the plane, for example, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Another way (for those of you who have studied the cross product, in Calc 3 for example) is to note that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the normal vector to the plane, so that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lies in the plane and is orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

33. First find a basis for the kernel: $\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\}$. Then use Gram-Schmidt to make it orthonormal. In this case, the given vectors are already orthogonal, so we just need to normalize them, giving us the basis $\{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\}$.

35. First, find a basis for the image: $\{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}\}$. Then use Gram-Schmidt to make it orthonormal. In this case, the given vectors are already orthogonal, so we just need to normalize them, giving us the basis $\{\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix}\}$.

39. Let $\vec{v_1} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$. Then we need to find a third vector, $\vec{v_3}$, such that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ form a basis of $\mathbb{R}^3$ and then perform Gram-Schmidt on this basis. We can pick any $\vec{v_3}$ which is not in the span of $\vec{v_1}$ and $\vec{v_2}$. One way to do this is to find a linear combination of $\vec{v_1}$ and $\vec{v_2}$ and then "jostle" the third component a bit to take it out of the span. For example, $\vec{v_1} + \vec{v_2} = \begin{bmatrix}2\\3\\2\end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix}2\\3\\1\end{bmatrix}$ is linearly independent from $\vec{v_1}, \vec{v_2}$ (since there's a unique way to get the first two components to be 2 and 3, and we saw that that way results in the third component being 2). A more elegant choice, however, is to take $\vec{v_3} = \vec{v_1} \times \vec{v_2}$, as then we'll get orthogonality for free, which makes the subsequent computations a bit easier. In the end, you'll end up with $\{\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix}5\\-4\\1\end{bmatrix}\}$, possibly with some or all of these vectors replaced with its opposite (i.e., multiplied by -1).