Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 4.2

6. T is linear ($T(M+N) = (M+N)\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} = M\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} + N\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} = T(M)+T(N)$ and $T(kM) = kM\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} = k(M\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}) = kT(M)$), but is not an isomorphism since $\ker(T) \neq \{0\}$ (see #52 for the computation of the kernel).

20. T is linear ($T(x+\imath y+a+\imath b) = T((x+a)+\imath(y+b)) = x+a-\imath(y+b) = x-\imath y+a-\imath b = T(x+\imath y)+T(a+\imath b)$ and $T(k(x+\imath y)) = T(kx+\imath ky) = kx-\imath ky = k(x-\imath y) = kT(x+\imath y)$) and is an isomorphism, since it has inverse $T^{-1}(x+\imath y) = x-\imath y$ (that is, T is its own inverse, since applying T twice takes a vector back to itself). Alternatively, compute $\ker(T) = \{0\}$ by noting that $x-\imath y = 0$ forces $x=0$ and $y=0$ since $1, \imath$ are linearly independent.

30. T is linear ($T(f+g) = t(f+g)'(t) = t(f'(t)+g'(t)) = tf'(t)+tg'(t) = T(f)+T(g)$ and $T(kf) = t(kf)'(t) = ktf'(t) = kT(f)$), but is not an isomorphism since $\ker(T) \neq \{0\}$ (see #56 for the computation of the kernel).

52. Write $T(\begin{bmatrix}a & b\\ c & d\end{bmatrix}) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. Performing the matrix multiplication and equating coefficients gives $a+3b=0$ and $c+3d=0$, so that matrices in the kernel are of the form $\begin{bmatrix}-3b & b\\ -3d & d\end{bmatrix}$ and so $\ker(T) = \operatorname{span}(\begin{bmatrix}-3 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ -3 & 1\end{bmatrix})$. The two vectors in this span are linearly independent (zero trick), so $\operatorname{nullity}(T) = 2$.

56. Note $T(a+bt+ct^2) = bt+2ct^2$, so $\operatorname{im}(T) = \operatorname{span}(t, t^2)$. These are linearly independent, so $\operatorname{rank}(T) = 2$ Setting $bt+2ct^2 = 0$ and noting that t, t^2 are linearly independent, we see $b=0$ and $2c=0$ (so $c=0$). Thus, all vectors in the kernel are of the form $f(t) = a$ for some scalar a. Since $a = a1$, we see that $\ker(T) = \operatorname{span}(1)$. Thus, $\operatorname{nullity}(T) = 1$. (Note that $\operatorname{rank}(T)+\operatorname{nullity}(T) = 3$ as required by Rank-Nullity.)

64. One possibility is $T(a+bt+ct^2+dt^3) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. You can either verify directly that this is an isomorphism, or note that $T = L_{\mathfrak{U}}^{-1} \circ L_{\mathfrak{B}}$ where $\mathfrak{B} = \{1, t, t^2, t^3\}$ and $\mathfrak{U} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Another option is to pick four distinct scalars $a, b, c, d \in \mathbb{R}$ and let $T(f(t)) = \begin{bmatrix}f(a) & f(b)\\ f(c) & f(d)\end{bmatrix}$. To show this is an isomorphism, note that the kernel consists of all polynomials such that $f(a) = f(b) = f(c) = f(d) = 0$ and that nonzero polynomials in $\mathcal{P}_3$ can have at most three roots, so that $f(x) = 0$. Most generally, any transformation of the form $T = L_{\mathfrak{U}}^{-1}(BL_{\mathfrak{B}})$ where B is an invertible 4×4 matrix will work (and, moreover, every possible solution can be expressed in this form).

Section 4.3

2. Let $\mathfrak{U}$ be the standard basis for $\mathbb{R}^{2\times 2}$ and transfer the problem to $\mathbb{R}^4$ by applying $L_{\mathfrak{U}}$ to each vector. The resulting vectors in $\mathbb{R}^4$ are not linearly independent (if you put them as the columns of the matrix A, then $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$). As $\operatorname{rref}(A) \neq I_4$, we know they are not linearly independent, and by pulling a redundancy relation among the columns of $\operatorname{rref}(A)$ back to $\mathbb{R}^{2\times 2}$ with $L_{\mathfrak{U}}^{-1}$, we see that $-\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}$.

6. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. B is invertible, so T is too.

16. $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. B is invertible, so T is too.

22. $B = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$. B is not invertible, so T is not an isomorphism. By inspection $\ker(B) = \operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$ and $\operatorname{im}(B) = \operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix})$ (I've simplified $\operatorname{im}(B)$ slightly by multiplying each basis vector by as scalar). Pulling back to $\mathcal{P}_3$ using $L_{\mathcal{U}}^{-1}$, we see that a basis of $\ker(T)$ is $\{1\}$ and a basis of $\operatorname{im}(T)$ is $\{1, 1+4t\}$ (or, you can simplify the image further and see that $\{1, t\}$ is a basis for $\operatorname{im}(T)$). Thus, $\operatorname{rank}(T) = 2$ and $\operatorname{nullity}(T) = 1$.

45. (a) $S_{\mathfrak{B}\to\mathfrak{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

    (b) Note that $SB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} = AS$.

    (c) $S_{\mathfrak{U}\to\mathfrak{B}} = (S_{\mathfrak{B}\to\mathfrak{U}})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

50. $B = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$. Note that $\det(B) = (b-1)^2 + a^2 \neq 0$ unless $a = 0$ and $b = 1$. So, T is invertible unless $(a, b) = (0, 1)$ and is not invertible if $(a, b) = (0, 1)$.

    Aside: Suppose we wanted to solve the differential equation $f'' + af' + bf = c_0 \cos t + c_1 \sin t$ in C^∞. Finding all solutions is tricky based on what you know now, but finding all solutions in $V = \operatorname{span}(\cos t, \sin t)$ is much easier since V is finite dimensional. Note that if we write $\mathfrak{B} =$

$\{\cos t, \sin t\}$, then $[c_0 \cos t + c_1 \sin t]_{\mathfrak{B}} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$, so the solutions will be all $g(t) = x_0 \cos t + x_1 \sin t$ such that $B \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$. If $(a, b) \neq (0, 1)$, then B is invertible and so the differential equation has a unique solution in V. Recall that we've seen that if $A\vec{v} = \vec{b}$, then all solutions to $A\vec{x} = \vec{b}$ are of the form $\vec{x} = \vec{v} + \vec{w}$ where $\vec{w} \in \ker(A)$. The same principle applies here: every solution to $f'' + af' + bf = c_0 \cos t + c_1 \sin t$ in C^∞ (and not just in V) can be written in the form $f = g + h$ where $g(t) = x_0 \cos t + x_1 \sin t$ is the solution we found above and $h(t) \in \ker(T)$ (where we now view T as a linear transformation from C^∞ to C^∞ instead of from V to V). Note that $\ker(T)$ is the set of solutions to $f'' + af' + bf = 0$, which you saw how to solve in Calculus II. Combining your knowledge from Calc II and from Linear Algebra, you should now be able to find all solutions in C^∞ to the differential equation $f'' + af' + bf = c_0 \cos t + c_1 \sin t$ (except in the case $(a, b) = (0, 1)$) For example, $f'' - 2f' + f = 0$ has the two linearly independent solutions e^t and te^t and $f'' - 2f' + f = 10 \cos t - 20 \sin t$ has the solution $-\cos t + 7 \sin t$ in V, so all C^∞ solutions to $f'' - 2f' + f = 10 \cos t - 20 \sin t$ can be expressed as $-\cos t + 7 \sin t + c_1 e^t + c_2 te^t$ for some scalars c_1 and c_2.

51. Write $\mathfrak{B} = \{\cos t, \sin t\}$. We could do this directly, but for fun let's look at the more general problem $T(f(t)) = f(t - \theta)$ where θ is an arbitrary real number instead and then plug in $\theta = \frac{\pi}{2}$ at the end. We'll need to use the addition theorems for sine and cosine to do this. If you don't remember them, note that from what we know about rotation matrices that

$$\begin{bmatrix} \cos(t + \theta) \\ \sin(t + \theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix},$$

so that

$$\cos(t + \theta) = \cos\theta \cos t - \sin\theta \sin t$$

and

$$\sin(t + \theta) = \sin\theta \cos t + \cos\theta \sin t.$$

Thus,

$$[T(\cos t)]_{\mathfrak{B}} = [\cos(t - \theta)]_{\mathfrak{B}} = [\cos t \cos\theta + \sin t \sin\theta]_{\mathfrak{B}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and

$$[T(\sin t)]_{\mathfrak{B}} = [\sin(t - \theta)]_{\mathfrak{B}} = [-\sin\theta \cos t + \cos\theta \sin t]_{\mathfrak{B}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

so that $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. (Note that B is a rotation matrix: coincidence?)

Letting $\theta = \frac{\pi}{2}$, we see $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.