Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 3.4

14. $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

24. $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

56. If we write $\mathfrak{B} = \{\vec{v}, \vec{w}\}$, then the conditions that we want $\mathfrak{B}$ to satisfy are $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\vec{v} + 5\vec{w}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\vec{v} + 3\vec{w}$. If we write $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}$, then these two vector equations give us four equations relating the variables a, b, c and d. Solving (by row-reducing, say), we find that the solution is $\mathfrak{B} = \{\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}\}$.

58. (a) Since we know that $\mathbb{R}^3$ has dimension 3, it's enough to show that these vectors are linearly independent. As is usually the case, the best way to do this is to consider a relation of the form
$$c_1 A^2\vec{v} + c_2 A\vec{v} + c_3\vec{v} = \vec{0}$$
and then calculate that $c_1 = c_2 = c_3 = 0$ to show that the vectors satisfy only the trivial relation. Proceeding as in the hint, we can start by multiplying by A^2. Since $A^3\vec{v} = \vec{0}$ (and so $A^{3+k} = A^k A^3\vec{v} = A^k\vec{0} = \vec{0}$ implies in particular $A^4\vec{v} = \vec{0}$), the relation becomes
$$c_3 A^2\vec{v} = \vec{0}.$$
We know that $A^2\vec{v} \neq \vec{0}$, so this can only happen if $c_3 = 0$, so that our original relation is actually of the form
$$c_1 A^2\vec{v} + c_2 A\vec{v} = \vec{0}$$
once we plug in $c_3 = 0$. Next, multiply both sides of this new relation by A and simplify by again noting that $A^3\vec{v} = \vec{0}$, so that
$$c_2 A^2\vec{v} = \vec{0}$$
so that $c_2 = 0$ as above. This leaves us with $c_1 A^2\vec{v} = \vec{0}$, so that $c_1 = 0$ also. Thus, all of the coefficients are zero, and so the vectors are linearly independent.

(b) Write $\mathfrak{B} = \{A^2\vec{v}, A\vec{v}, \vec{v}\}$. Of our three methods for finding the matrix of the transformation, either working column-by-column or making a commutative diagram will work. (It may be possible to do it using the formula $B = S^{-1}AS$, but I don't see a way since computing S^{-1} is difficult from the information we're provided.) Let's try the column-by-column method:

$$[T(A^2\vec{v})]_{\mathfrak{B}} = [A^3\vec{v}]_{\mathfrak{B}} = [\vec{0}]_{\mathfrak{B}} = \vec{0}$$
$$[T(A\vec{v})]_{\mathfrak{B}} = [A^2\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$[T(\vec{v})]_{\mathfrak{B}} = [A\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Aside: These conditions sound odd, perhaps. Can we want a matrix A and a vector $\vec{v}$ with these properties? One thing you might note is that B^3 is the zero matrix (for B as in part (b) above), so taking $A = B$ and letting $\vec{v}$ be any vector in $\mathbb{R}^3$ such that $A^2\vec{v} \neq \vec{0}$ will work; for example, take $\vec{v} = \vec{e_3}$. Is the fact that $A = B$ worked in this case a coincidence?

Aside 2: Of course, the choice in the above aside is not the only one. In particular, it's possible for $A^3\vec{v} = \vec{0}$ even if A^3 is not the zero matrix. Just find any matrix A such that $\ker(A^3) \neq \ker(A^2)$. Then, since we know that $\ker(A^2) \subseteq \ker(A^3)$, you can find a vector $\vec{v}$ that's in $\ker(A^3)$ but not in $\ker(A^2)$.

Section 4.1

1. Not a subspace, as it does not contain the neutral element (i.e., zero vector), is not closed under addition, and is not closed under scalar multiplication.

2. Let $W = \{p(t) \in \mathcal{P}_2 | p(2) = 0\}$. The zero vector of $\mathcal{P}_2$ is the function $n(t) = 0$ for all t. Note $n(2) = 0$, so, $n(t) \in W$. Let $f(t), g(t) \in W$. That is, $f(2) = g(2) = 0$. Then $(f+g)(2) = f(2) + g(2) = 0 + 0 = 0$, so $(f+g)(t) \in W$. Let k be any real scalar. Then $(kf)(2) = kf(2) = k0 = 0$, so $(kf)(t) \in W$. Thus, W is a subspace of $\mathcal{P}_2$.

   If we write $p(t) = a + bt + ct^2$ (as a general element of $\mathcal{P}_2$), the condition $p(2) = 0$ implies that $a + 2b + 4c = 0$, or $a = -2b - 4c$. Plugging in, $p(t) = -2b - 4c + bt + c^2 = b(t-2) + c(t^2 - 4)$, so that every element of W can be written as a linear combination of $t - 2$ and $t^2 - 4$. Note that these two vectors are linearly independent (for example, we saw in class that polynomials of different degree are always linearly independent; or, just prove linear independence the standard way), so $\{t - 2, t^2 - 4\}$ is a basis of W. As an aside, note that now that we know that the dimension of W is 2, any two linear independent vectors in W will form a basis, so other answers are possible. For example $\{t - 2, (t-2)^2\}$ is also a basis, as is $\{t^2 - 3t + 2, t^2 - t - 2\}$, etc.

   Aside: Based on problem #33 in Section 3.3, would it be reasonable to describe W as a hyperplane? Why or why not?

6. Not a subspace, as it doesn't contain the zero vector (in this case, the 3×3 zero matrix), is not closed under addition (try $A + (-A)$ for A invertible), and is not closed under scalar multiplication (try $k = 0$).

9. Not a subspace since it is not closed under scalar multiplication (try $k = -1$).

10. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and let W be the set under consideration. If N is the zero matrix, note that $N\vec{v} = \vec{0}$, so $N \in W$. Let $A, B \in W$. Then $A\vec{v} = \vec{0}$ and $B\vec{v} = \vec{0}$, so $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$, so $A + B \in W$. Let $k \in \mathbb{R}$. Then $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$, so $kA \in W$. Thus, W is a subspace of $\mathbb{R}^{3\times3}$.

17. If we define E_{ij} to be the $n \times m$ matrix with a 1 in the ijth spot and 0's elsewhere, then the set of all E_{ij} for $1 \leq i \leq n, 1 \leq j \leq m$ form a basis of $\mathbb{R}^{n\times m}$. So, the dimension of this space is nm.

25. Compare your answer in this problem to your answer in #2.

34. Write $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the matrix equation $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} S = S$ gives us four equations relating the variables a, b, c, d (one from each component of the matrices) Solving, we see that $c = -a$ and $d = -b$, so that $S = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$ and so a basis is $\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\}$ (since we see that these two vectors are linearly independent, e.g., by the zero trick).

39. The set of E_{ij} (as defined above in the solution to #17) for which $i \leq j$ form a basis, so we just need to count how many of these there are. There are a couple of nice ways to do this.

    Way 1: Note that if $i = 1$, there are n choices for j. If $i = 2$, there are $n - 1$ choices for j, and so on, until we get to $i = n$ and there is only one choice for j. Thus, the total number of basis vectors is $1 + 2 + \cdots + (n - 1) + n = \sum_{k=1}^{n} k$.

    Way 2: The matrix has n^2 entries, of which n are on the diagonal. Thus, the matrix has $n^2 - n$ entries not on the diagonal. Half of them are above the diagonal, so the matrix has $\frac{n^2-n}{2}$ entries strictly above the diagonal. Thus, there are $n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$ entries on or above the diagonal.

    Aside: Note that we have proven that $\sum_{k=1}^{n} k = \frac{n^2+n}{2}$. This is a result that the mathematician Gauss allegedly discovered while in elementary school (although it was already known before he came up with it). See `http://mathforum.org/library/drmath/view/57919.html` for details of how Gauss allegedly did it.

48. Write $f(t) = a + bt + ct^2 + dt^3 + et^4$ as a general element of $\mathcal{P}_4$. Note that the conditions $f(t) = f(-t)$ and $f(-t) = -f(t)$ both give us relations among the variables a, b, c, d, e (since $1, t, t^2, t^3, t^4$ are linearly independent, we can equate the coefficients of each power of degree in these equations). Solving we get:

    (a) $\{1, t^2, t^4\}$, so dimension 3

    (b) $\{t, t^3\}$, so dimension 2

Aside: Note that the only function which is both even and odd is the zero function, which is the only vector in both of these spaces. Note also that this shows us that every vector in $\mathcal{P}^4$ can be written as a sum of an even function and an odd function. In fact, this is true for all functions from $\mathbb{R}$ to $\mathbb{R}$, since we can write

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2},$$

where the first term on the right-hand side is even and the second term is odd. In fact, this is the only way of writing the function $f(x)$ as the sum of an even and an odd function.