Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 3.2

2. W is not a subspace of $\mathbb{R}^3$ as it is not closed under scalar multiplication. For example, let $k = -1$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then $\vec{v} \in W$ but $k\vec{v} \notin W$.

3. $W = \text{im}(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix})$, so W is a subspace of $\mathbb{R}^3$.

6. (a) Note $\vec{0} \in V$ and $\vec{0} \in W$, so $\vec{0} \in V \cap W$.
   Suppose that $\vec{v}, \vec{w} \in V \cap W$. Then $\vec{v}, \vec{w} \in V$ and $\vec{v}, \vec{w} \in W$, so $\vec{v} + \vec{w} \in V$ and $\vec{v} + \vec{w} \in W$ since V and W are subspaces of $\mathbb{R}^n$ (and so closed under addition). Thus, $\vec{v}+\vec{w} \in V \cap W$.
   Suppose that $\vec{v} \in V \cap W$ and $k \in \mathbb{R}$. Then $\vec{v} \in V$ and $\vec{v} \in W$, so $k\vec{v} \in V$ and $k\vec{v} \in W$, so $k\vec{v} \in V \cap W$.
   Thus, $V \cap W$ is a subspace of $\mathbb{R}^n$.

   (b) $V \cup W$ is not necessarily a subspace of $\mathbb{R}^n$ since it's not always closed under addition. For example, let $V = \ker(\begin{bmatrix} 1 & -1 \end{bmatrix})$ and $W = \ker(\begin{bmatrix} 1 & 1 \end{bmatrix}$ (that is, V is the line $y = x$ and W is the line $y = -x$). Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in W$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin V \cup W$.

8. One possible answer: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \vec{0}$.

24. The third column is redundant: $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, so $3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0}$, so that $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \ker A$.

36. (also in the review solutions for Exam 2) Let $c_1\vec{v_1} + \cdots + c_m\vec{v_m} = \vec{0}$ be a nontrivial relation among $\vec{v_1}, \ldots, \vec{v_m}$. Applying T to both sides and using the properties of linear transformations, we see that $c_1T(\vec{v_1}) + \cdots + c_mT(\vec{v_m}) = \vec{0}$, so that $T(\vec{v_1}), \ldots, T(\vec{v_m})$ are linear dependent.

37. (also in the review solutions for Exam 2) Not necessarily. For example, let $T(\vec{x}) = \vec{0}$ for all $\vec{x}$.

Section 3.3

24. Note that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, so a basis for $\text{im}(A)$ is $\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix} \}$ and a basis for $\ker(A)$ is $\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \}$.

26. (a) L

    (b) H and X

    (c) L

    Note that there is more than one way to express a space as a span of a set of vectors. For example, $\text{im}(C) = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$ since $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

28. They form a basis for all k such that $k \neq 29$. To see this, either form the 4×4 matrix that has these vectors as columns and try to reduce it to I_4, or look at possible relations between the columns and see for which k these are possible. (In this case, the second method is simpler since the copious zeroes mean that there's at most one way to possibly write the fourth vector as a linear combination of the first three.)

33. Note that $V = \ker(\begin{bmatrix} c_1 & \dots & c_n \end{bmatrix})$. Since at least one of the c_i is nonzero, the rank of this matrix will be 1, so by Rank-Nullity the kernel will be $n-1$ dimensional. A hyperplane in $\mathbb{R}^3$ is a plane. A hyperplane in $\mathbb{R}^2$ is a line.

35. The dimension is $n - 1$ since the set of solutions to $\vec{x} \cdot \vec{v} = 0$ is a hyperplane.

36. (also in the review solutions for Exam 2) This can't be done. If $\text{rank}(A) = \text{nullity}(A)$, then by Rank-Nullity we'd have $\text{rank}(A) = 1.5$, which can't happen since the rank of a matrix must be an integer.

39. Any of the characterizations in Summary 3.3.10 can be used (although some are simpler than others). Conditions (ii) or (vi) are probably the easiest to use in this case. As C is 4×5, we know that $\text{nullity}(C) \geq 1$ (by Rank-Nullity, since $\text{rank}(C) \leq 4$). So, since $\ker(C) \subseteq \ker(A)$, it must be that $\ker(A) \neq \{\vec{0}\}$.

66. Note that $\text{rref}(\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1/4 \\ 0 & 1 & -1 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/4 \end{bmatrix}$, so that the first, second, fourth, and fifth columns are linearly independent (as they have leading 1's). Thus, $\mathfrak{B} =$

$$\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\4\\6\\8\end{bmatrix}, \vec{e_2}, \vec{e_3}\right\}$$ is a basis of $\mathbb{R}^4$ by #65.