Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 2.4

9. $\text{rref}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$, so the matrix is not invertible.

19. If we write $\vec{y} = T(\vec{x}) = A\vec{x}$, then $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 2 & 7 & 12 \end{bmatrix}$ and we calculate that $A^{-1} = \begin{bmatrix} 3 & -2.5 & 0.5 \\ -3 & 4 & -1 \\ 1 & -1.5 & 0.5 \end{bmatrix}$, so T is invertible and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

29. In the process of row-reducing this matrix, you'll need to divide by $k^2 - 3k + 2 = (k-1)(k-2)$ at one step in order to continue reducing to I_3. Thus, the matrix has rref I_3 unless $k = 1$ or $k = 2$, in which case it is not invertible.

34. The diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$ is invertible if and only if the components on the diagonal are all nonzero, in which case the inverse is the diagonal matrix with diagonal entries $\frac{1}{a_{11}}, \ldots, \frac{1}{a_{nn}}$.

40. By Summary 3.3.10, there are a variety of ways to do this (some of which we'd seen by 2.4 and some of which we hadn't). Let's look at a few different options. Suppose that the ith and jth columns are equal where $i \neq j$. Then:

    - The linear system $A\vec{x} = \vec{0}$ has the solutions $\vec{x} = \vec{0}$ and $\vec{x} = \vec{e_i} - \vec{e_j}$, so the solution is not unique.
    - Elementary row operations act the same on all columns of a row, so the ith and jth columns of the matrix will still be equal to each other in rref. In particular, this shows us that $\text{rref}(A) \neq I_n$ since no two columns of I_n are equal.
    - Similar to the above, note that we can't have two leading 1's in the same row, so we can't have a leading 1 in both the ith and jth columns, so $\text{rank}(A) < n$.
    - Note $(\vec{e_i} - \vec{e_j}) \in \ker(A)$, so $\ker(A) \neq \{\vec{0}\}$.
    - The ith and jth column of A are equal, so the jth column is redundant. Thus, the column vectors of A are not linearly independent and are not a basis for $\mathbb{R}^n$.

    Any one of these will work, since they are all equivalent to the matrix not being invertible.

42. Yes, all permutation matrices are invertible and their inverses are also permutation matrices (since it can be row-reduced solely by swapping rows.) As an aside, to see why this is a permutation matrix, look at what happens when you multiply the vector $\vec{x}$ by a permutation matrix: you'll end up with a new vector with the same components, but in a different order (that is, they've been "permuted").

67. False. Only true if $AB = BA$.

69. False. In fact, if (for example) $A = -B$, then $(A + B)$ won't even be invertible.

75. True. Theorem 2.4.7 plus the fact that $(A^{-1})^{-1} = A$.

76. Either note that $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is invertible and multiply both sides of the hint by its inverse, or write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and perform the multiplication to set up a system of four equations in four unknowns. Either way, you'll find that the only possibility is $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$.

Section 3.1

2. $\ker(A) = \text{span}(\begin{bmatrix} 3/2 \\ -1 \end{bmatrix}) = \text{span}(\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}) = \text{span}(\begin{bmatrix} 3 \\ -2 \end{bmatrix})$ are some possible options. More generally, the span of any nonzero scalar multiple of $\begin{bmatrix} 3/2 \\ -1 \end{bmatrix}$ will work.

6. One possibility is $\ker(A) = \text{span}(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix})$.

30. Any matrix such that all columns are scalar multiples of $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and at least one column is a nonzero scalar multiple of $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ will work. The simplest solution is the 2×1 matrix $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Another option is $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 5 & 10 & 0 & 15 \end{bmatrix}$.

39. (a) Let $\vec{x} \in \ker(B)$. Then $AB\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$, so $\vec{x} \in \ker(AB)$ and so $\ker(B) \subseteq \ker(AB)$. They need not be equal. For example, let A be the $n \times n$ zero matrix and $B = I_n$.

    (b) Let $\vec{y} \in \text{im}(AB)$. Then there is a vector $\vec{x}$ such that $\vec{y} = AB\vec{x} = A(B\vec{x})$, so $\vec{y} \in \text{im}(A)$. Thus, $\text{im}(AB) \subseteq \text{im}(A)$. Again, they need not be equal. For example, let $A = I_n$ and B be the $n \times n$ zero matrix.

44. (a) Yes. The most straight-forward way to see this is to note that $\text{rref}(\begin{bmatrix} A & \vec{0} \end{bmatrix}) = \begin{bmatrix} \text{rref}(A) & \vec{0} \end{bmatrix}$, so that $A\vec{x} = \vec{0}$ and $\text{rref}(A)\vec{x} = \vec{0}$ have the same solution set. Another elegant way is to note that the kernel of a matrix A is the set of all vectors which are perpendicular to the row-vectors of A and use properties of the dot product to show that the set of such vectors is unaltered by performing elementary row operations.

    (b) No. For example, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (or just $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$). Many people mistakenly tried to show that this wasn't true with matrices that row-reduced to the identity matrix, but such examples fail because the columns of the original matrix are in fact just a different basis of $\mathbb{R}^n$, so that $\text{im}(A) = \text{im}(B) = \mathbb{R}^n$. cf. Summary 3.1.8, parts (i) and (v).