Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 2.2

4. T will rotate $\vec{x}$ clockwise by $\frac{\pi}{4}$ and scale by $\sqrt{2}$.

10. $\vec{u} = \frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$, so the matrix is $\frac{1}{25}\begin{bmatrix}16 & 12\\12 & 9\end{bmatrix}$.

25. The inverse is $\begin{bmatrix}1 & -k\\0 & 1\end{bmatrix}$. This is a shear in the opposite direction as $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$.

28. D, E, C, A, F.

38. $\det\begin{bmatrix}u_1^2 & u_1u_2\\u_1u_2 & u_2^2\end{bmatrix} = u_1^2u_2^2 - u_1^2u_2^2 = 0$. So, an orthogonal projection is not invertible. $\det\begin{bmatrix}a & b\\b & -a\end{bmatrix} = -a^2 - b^2 = -1$. $\det\begin{bmatrix}a & -b\\b & a\end{bmatrix} = a^2 + b^2 = 1$. $\det\begin{bmatrix}1 & k\\0 & 1\end{bmatrix} = \det\begin{bmatrix}1 & 0\\k & 1\end{bmatrix} = 1$, so reflections about a line, rotations, and shears are all invertible.

42. Note $\text{proj}_L(\vec{x}) = \vec{x}$ for $\vec{x}$ on line L, so $\text{proj}_L(\text{proj}_L(\vec{x})) = \text{proj}_L(\vec{x})$. In the language of Section 2.3, this says that if A is the matrix of an orthogonal projection, then $A^2 = A$. (You can also easily verify that this is true by actually computing A^2 in this case.)

Section 2.3

21. Let $B = \begin{bmatrix}a & b\\c & d\end{bmatrix}$. Setting $AB = BA$ and equating components, we see that $a + 2c = a + 2b$, $b+2d = 2a-b$, $2a-c = c+2d$, and $2b-d = 2c-d$. Solving this system of four equations in four unknowns, we see that $b = c$ and $d = a - b$, so that all matrices of the form $B = \begin{bmatrix}a & b\\b & a-b\end{bmatrix}$ will compute with matrix A.

27. Show $A(C + D) = AC + AD$:

    Let A be a $n \times p$ matrix. Let C be a $p \times m$ matrix with columns $\vec{v_1}, \ldots, \vec{v_m}$ and D be a $p \times m$ matrix with columns $\vec{w_1}, \ldots, \vec{w_m}$. Then

$$\begin{aligned}(j\text{th column of } A(C + D)) &= A(j\text{th column of } C + D)\\ &= A(\vec{v_j} + \vec{w_j})\\ &= A\vec{v_j} + A\vec{w_j}\\ &= A(j\text{th column of } C) + A(j\text{th column of } D)\\ &= (j\text{th column of } (AC + AD))\end{aligned}$$

Or, if you prefer doing this at the level of entries:

$$\begin{aligned}(A(C+D))_{ij} &= \sum_{k=1}^{p} A_{ik}(C+D)_{kj} \\ &= \sum_{k=1}^{p} A_{ik}(C_{kj}+D_{kj}) \\ &= \sum_{k=1}^{p} A_{ik}C_{kj} + \sum_{k=1}^{p} A_{ik}D_{kj} \\ &= (AC)_{ij} + (AD)_{ij} \\ &= (AC+AD)_{ij}\end{aligned}$$

Show $(A+B)C = AC+BC$:

Let A be a $n \times p$ matrix with rows $\vec{v_1}, \ldots, \vec{v_n}$, B be a $n \times p$ matrix with rows $\vec{w_1}, \ldots, \vec{w_n}$, and C be a $p \times m$ matrix. Then

$$\begin{aligned}(i\text{th row of } (A+B)C) &= (i\text{th row of } A+B)C \\ &= (\vec{v_i}+\vec{w_i})C \\ &= \vec{v_i}C + \vec{w_i}C \\ &= (i\text{th row of } A)C + (i\text{th row of } B)C \\ &= (i\text{th row of } AC)+(i\text{th row of } BC) \\ &= (i\text{th row of } AC+BC)\end{aligned}$$

Or, alternatively let A and B be $n \times p$ matrices and C be a $p \times m$ matrix with columns $\vec{v_1}, \ldots, \vec{v_m}$. Then

$$\begin{aligned}(j\text{th column of } (A+B)C) &= (A+B)(j\text{th column of } C) \\ &= (A+B)\vec{v_j} \\ &= A\vec{v_j} + B\vec{v_j} \\ &= A(j\text{th column of } C)) + B(j\text{th column of } C) \\ &= (j\text{th column of } AC) + (j\text{th column of } BC) \\ &= (j\text{th column of } AC+BC)\end{aligned}$$

Note that we haven't actually proven that $(A+B)\vec{v} = A\vec{v} + B\vec{v}$, so if you're using the last proof you should technically justify this fact (although I gave you full credit even if you didn't). The proof is similar to the proof of $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$. Of course, this is a special case of this problem, so if you proved #27 using one of the above techniques, you now know that $(A+B)\vec{v} = A\vec{v}+B\vec{v}$ also. In particular, this gives us an easier way of showing that $S(\vec{x})+T(\vec{x})$ is a linear transformation when S and T are, since if $S(\vec{x}) = A\vec{x}$ and $T(\vec{x}) = B\vec{x}$, then the matrix of $S(\vec{x}) + T(\vec{x})$ is $A+B$.

36. We compute $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$ for any scalar k, so $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$.

Fun fact: If we let $\varphi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, then $\varphi(x+y) = \varphi(x)\varphi(y)$ and any additive calculation on the real numbers can instead by done as a matrix calculation, as long as we switch back at the end of the day. For example,

$$5+7 = \varphi^{-1}(\varphi(5+7)) = \varphi^{-1}(\varphi(5)\varphi(7)) = \varphi^{-1}(\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}) = \varphi^{-1}(\begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}) = 12,$$

where we compute $\varphi^{-1}(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}) = x$. This is an example of an *isomorphism*, which you'll study in a class on Abstract Algebra.