Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 1.3

2. rank 3

4. rank 2

14. $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

18. $\begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$

22. To have a unique solution, the rref of coefficient matrix must have a leading one in every column. As there are three rows and three columns, this means that the rref of the coefficient matrix must be $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. See also Thm. 1.3.4.

30. To have rank 1, each row must be a scalar multiple of every other row. It's helpful to find the last row first: solving $5a + 3b - 9c = 1$ (where a, b, c are the entries in the last row) we see that one possible solution is $(a, b, c) = (-1, -1, 1)$ (although infinitely many choices exist since this is one equation in three unknowns). Then, multiply this by 2 for the first row and multiply it by zero for the second row and we get: $\begin{bmatrix} -2 & -2 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. This is not unique, as any other solution to the equation in (a, b, c) will give another possible matrix. For example, $(a, b, c) = (\frac{1}{5}, 0, 0)$ gives the matrix $\begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 \end{bmatrix}$.

34. (a) $A\vec{e_1} = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$, $A\vec{e_2} = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$, $A\vec{e_3} = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$.

    (b) $B\vec{e_i} = \vec{v_i}$ for $i = 1, 2, 3$.

36. This is #34 "in reverse." So, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

54. This is the plane containing the vectors $\vec{v_1}$ and $\vec{v_2}$.

Section 2.1

9. rref $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$, so the matrix is not invertible.

12. $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$.

14. (a) $ad - bc = 2k - 15$. By #13, the matrix is invertible unless $ad - bc = 0$, that is, unless $k = \frac{15}{2}$.

    (b) (Note: this part was optional.)
    By #13, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.
    So, we want all four entries in the matrix on the right to be integers. Instead of checking all four, it's quicker to note that if $\frac{-3}{2k-15}$ and $\frac{2}{2k-15}$ are both integers, then $\frac{3}{2k-15}$ is an integer also, and so $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is also an integer. Conversely, if $\frac{1}{2k-15}$ is an integer, it'll still be an integer if we multiply it by -3, -5, or 2. Thus, all we really need to do is check that $\frac{1}{2k-15}$ and $\frac{k}{2k-15}$ are integers. (The second check is necessary, as k may not be an integer.)

    Now, write $\frac{1}{2k-15} = n$ where n is some integer. Solving for k, we see that $k = \frac{15}{2} + \frac{1}{2n}$. Such k work for all entries except perhaps $\frac{k}{2k-15}$.

    Suppose that $\frac{k}{2k-15}$ is also an integer. This means that $\frac{k}{2k-15} = kn = \frac{15n+1}{2}$ is an integer, which is only the case if n is an odd integer, say $n = 2m + 1$ for any integer m. This will then work for all four entries, so the entries of the inverse matrix will all be integers if and only if $k = \frac{15}{2} + \frac{1}{2(2m+1)} = \frac{15m+8}{2m+1}$ where m is any integer.

32. By inspection, you might guess that $A = 3I_n$ will work. Let's go one step further and prove that this is the *only* matrix that will work. Let $T(\vec{x}) = A\vec{x}$. Then we want $T(\vec{e_i}) = 3\vec{e_i}$ for $i = 1, \ldots, n$. Then, we know that

$$A = \begin{bmatrix} T(\vec{e_1}) & \ldots & T(\vec{e_n}) \end{bmatrix} = 3I_n.$$

37. Let $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$ for some k with $0 < k < 1$. We want to show that $T(\vec{x}) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$ for some k with $0 < k < 1$ (not necessarily the same k as above, although it will turn out to be the same in this case). To do this, take T of both sides: $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v}))$ and use Theorem 2.1.3 repeatedly.

39. Either use Theorem 2.1.3 repeatedly or use Theorems 2.1.2 and 1.3.8.