Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 7.1

2. Write $A\vec{v} = \lambda\vec{v}$. Note that $\lambda \neq 0$ since A is invertible. Then, multiply both sides by $\frac{1}{\lambda}A^{-1}$ to see that $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$, so that $\vec{v}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

4. Similarly to #2, $\vec{v}$ is an eigenvector with eigenvalue 7λ.

6. $\vec{v}$ is an eigenvector of AB. If $A\vec{v} = \lambda_1\vec{v}$ and $B\vec{v} = \lambda_2\vec{v}$, then $AB\vec{v} = \lambda_1\lambda_2\vec{v}$.

23. See Theorem 7.4.1.

46. (a) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $a + 2b = \lambda$ and $c + 2d = 2\lambda$, so that $c = 2a + 4b - 2d$. Thus, V is the space of all matrices of the form $\begin{bmatrix} a & b \\ 2a+4b-2d & d \end{bmatrix}$, so that $V = \text{span}(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix})$. If you solved for different variables, other forms are possible, such as $V = \text{span}(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix})$.

    (b) For A in V, we have $T(A) = A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_A\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where λ_A is a scalar dependent on the matrix A. As such $\text{im}(T) = \text{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$ and $\text{rank}(T) = 1$. If we set $T(A) = \vec{0}$, we have $\begin{bmatrix} a & b \\ 2a+4b-2d & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solving as in (a) yields the new condition $a = -2b$, so that $\ker(T) = \text{span}(\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$.

    (c) Note that $\ker(L)$ is the set of all matrices for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector and for which $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 0. You can either calculate it directly, or use the same technique as in (b). Either way, you'll end up with $\ker(L) = \text{span}(\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix})$. You can calculate $\text{im}(L)$ directly if you wish (start by writing $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but it's simpler to note that $\text{rank}(L) = 2$ by Rank-Nullity, so that $\text{im}(L) = \mathbb{R}^2$ so that $\vec{e_1}, \vec{e_2}$ is a basis of $\text{im}(L)$.

50. Let $\vec{x}(t) = \begin{bmatrix} h(t) \\ f(t) \end{bmatrix}$.

(a) $\vec{x}(t) = 2^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

(b) $\vec{x}(t) = 3^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

(c) Note that $\begin{bmatrix} 600 \\ 500 \end{bmatrix} = 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, so that Theorem 7.1.3 tells us that $\vec{x}(t) = 4(2^t) \begin{bmatrix} 100 \\ 100 \end{bmatrix} + (3^t) \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

Section 7.2

22. $f_{A^T}(\lambda) = \det(A^T - \lambda I_n) = \det(A^T - (\lambda I_n)^T) = \det((A - \lambda I_n)^T) = \det(A - \lambda I_n) = f_A(\lambda)$, so the characteristic polynomials are the same. Thus, A and A^T have the same eigenvalues with the same algebraic multiplicities.

38. The simplest way to do this is to note that $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 - 5\lambda - 14$, so that factoring gives the eigenvalues 7 and -2. Alternatively, use the equations $\mathrm{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1 \lambda_2$ to solve for the eigenvalues.

40. AB and BA will have the same addends on the diagonal (but not in the same order or grouped in the same way), so that both will have the same trace.

44. #43 suggests a strategy to use here. Let's use traces to show that the two sides can't be equal. If we do it directly, we find $\mathrm{tr}(AB - BA) = 0$, so that $\mathrm{tr}(A) = 0$. Unfortunately, this isn't good enough, since it's possible for an invertible matrix to have trace 0. But, maybe we can modify the given equation to end up with one that's clearly impossible. There are at least a couple ways to do this:

    One strategy is to multiply both sides by A^{-1} on the left. Then, $B - A^{-1}BA = I_n$. Note that similar matrices have the same trace (see #41), so that $\mathrm{tr}(B - A^{-1}BA) = 0$, while $\mathrm{tr}(I_n) = n$. Since $0 \neq n$, no such matrices A and B can possibly exist.

    Another strategy is to note that if such matrices A and B existed, then we could define matrices $C = BA$ and $D = A^{-1}$. Then $CD - DC = B - A^{-1}BA = I_n$. But, we saw in #43 that the equation $CD - DC = I_n$ has no solutions. Thus, no such matrices C and D can exist, and so no such matrices A and B can exist either.

Section 7.3

22. Note that in particular, $\vec{e_1}$ and $\vec{e_2}$ must be eigenvectors of A with eigenvalue 7. Then, the first column of A is $A\vec{e_1} = 7\vec{e_1}$, and the second column of A is $A\vec{e_2} = 7\vec{e_2}$. Thus, the only matrix with this property is $7I_2$.

28. The characteristic polynomial of $J_n(k)$ is $(k - \lambda)^n$, so the only eigenvalue is k with algebraic multiplicity n. Note $E_k = \ker(\begin{bmatrix} \vec{0} & \vec{e_1} & \dots & \vec{e_{n-1}} \end{bmatrix}) = \mathrm{span}(\vec{e_1})$ has dimension 1, so the geometric multiplicity of k is 1.

38. (Oops. This one wasn't actually assigned, but I'd already written the solution before I realized this, so here it is anyway.) First, we want to show that 1 is an eigenvalue of T. Since A is an orthogonal matrix, we know that the only possible eigenvalues of T are 1 and -1. Thus, the only way that 1 could fail to be an eigenvalue of T is if -1 is an eigenvalue with algebraic multiplicity 3, that is, if $f_A(\lambda) = (-1-\lambda)^3$. In this case, $f_A(0) = -1$. But we know that $f_A(0) = \det(A) = 1$, so this can't equal -1. As such, it must be that either -1 is not an eigenvalue or the algebraic multiplicity of -1 is less than 3. In either case, 1 is an eigenvalue of T. So, let $\vec{v}$ be an eigenvector associated to the eigenvalue 1. Then $T(\vec{v}) = \vec{v}$ as required.

48. (a) $\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}(t)$, so $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) We want to find an eigenbasis for A. Note that $f_A(\lambda) = \lambda^2 - \lambda - 1$, so that the eigenvalues of A are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Note that λ_1 and λ_2 are conjugates, so that $\lambda_1\lambda_2 = -1$, $\lambda_1 + \lambda_2 = 1$, and both are roots of $f_A(\lambda)$. Using these relations, we can do the calculations symbolically (which is nice, because they're incredibly messy otherwise).

We compute $E_{\lambda_1} = \text{span}(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix})$ and $E_{\lambda_2} = \text{span}(\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix})$. Thus, we have an eigenbasis, and so can apply Theorem 7.1.3.

We have initial condition $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$, so that Theorem 7.1.3 tells us that

$$\vec{x}(t) = \frac{1}{\sqrt{5}}(\lambda_1^t) \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}(\lambda_2^t) \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Then, $a(t)$ is the first component of $\vec{x}(t)$, and $j(t)$ is the second component. Written out in full, this tells us that:

$$a(t) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{t+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{t+1},$$

$$j(t) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{t} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{t}.$$

The formula for $j(t)$ is known at Binet's Formula.

Note that $a(t)$ and $j(t)$ are always integers. So, even though we have $\sqrt{5}$ all over the place, we know that they have to all cancel out nicely (or multiply to become a 5), so that we just end up with integers at the end of the day.

(c) Note that $|\lambda_2| < 1$, so that $\lambda_2^t \to 0$ as $t \to \infty$. Thus, the second term in the numerator and denominator of $\frac{a(t)}{j(t)}$ will go to zero as $t \to \infty$, so that $\lim_{t\to\infty} \frac{a(t)}{j(t)} = \frac{1+\sqrt{5}}{2}$.

52. We want to calculate the determinant of

$$A - \lambda I_n = \begin{bmatrix} -\lambda & 0 & 0 & \dots & 0 & a_0 \\ 1 & -\lambda & 0 & \dots & 0 & a_1 \\ 0 & 1 & -\lambda & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & a_{n-1} - \lambda \end{bmatrix}.$$

Check that if you pick any entry from the last column, there is precisely one nonzero pattern that contains that entry (for example, the only nonzero pattern containing a_0 is the one containing all of the 1's directly below the diagonal in the first $n-1$ columns and containing a_0 in the nth column). Since every pattern contains some entry from the last column, this means there are only n nonzero patterns in the matrix. Let P_k be the pattern containing a_k (or $a_{n-1} - \lambda$ in the case of P_{n-1}). We compute that $\mathrm{sgn}(P_k) = (-1)^{n-1-k}$ and $\mathrm{prod}(P_k) = (-\lambda)^k (1)^{n-2-k} a_k = (-1)^k \lambda^k a_k$ for $k \neq n-1$ and $\mathrm{prod}(P_{n-1}) = (-\lambda)^{n-1}(a_{n-1} - \lambda = (-1)^n \lambda^n + (-1)^{n-1} a_{n-1} \lambda^{n-1}$. Thus, $\mathrm{sgn}(P_k)\mathrm{prod}(P_k) = (-1)^{n-1} a_k$ for $k \neq n-1$ and $\mathrm{sgn}(P_n)\mathrm{prod}(P_n) = (-1)^n \lambda^n + (-1)^{n-1} a_{n-1} \lambda^{n-1}$. Summing (and factoring $(-1)^n$ from each term to simplify the notation), we see that

$$f_A(\lambda) = (-1)^n \left(\lambda^n - a_{n-1}\lambda^{n-1} - \dots - a_1 \lambda - a_0 \right).$$

54. First, a word on notation. If we have a polynomial such as $f(x) = x^2 - 2x + 3$, we can almost imagine putting a matrix in for x instead of a number, except for the last term (since we can't add a matrix to a scalar). However, if we replace the constant term with the matrix $3I_n$, so that $f(A) = A^2 - 2A + 3I_n$, then it makes perfect sense to talk about f as a function from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$, which is what this problem is about. To make sure you understand the definition, verify that $f(I_n) = 2I_n$ for the polynomial f above. In this problem, our goal is to show that every polynomial satisfies its characteristic polynomial. That is, that $f_A(A)$ is the zero matrix.

   (a) B_{11} is the matrix in #52 (except $m \times m$ instead of $n \times n$). B_{21} is the zero matrix.

   (b) Note that A and B are both the matrix of T, but with respect to different bases. As such, A is similar to B. Thus, $f_A(\lambda) = f_B(\lambda)$. By Theorem 6.1.5, $f_B(\lambda) = f_{B_{22}}(\lambda) f_{B_{11}}(\lambda)$. As we've computed $f_{B_{11}}(\lambda)$ in #52, we can combine all of this to say that

   $$f_A(\lambda) = (-1)^m f_{B_{22}}(\lambda)(\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0).$$

   (c) $f_A(A)\vec{v} = (-1)^m f_{B_{22}}(A)(A^m \vec{v} - a_{m-1}A^{m-1}\vec{v} - \dots - a_1 A\vec{v} - a_0\vec{v})$ and since $A^m\vec{v} = a_0\vec{v} + \dots + a_{m-1}A^{m-1}$, we have $f_A(A)\vec{v} = (-1)^m f_{B_{22}}(A)(\vec{0}) = \vec{0}$.

   (d) Note that the equation in the previous part holds for every vector $\vec{v}$. Recall that $f_A(A)$ is some matrix. What the previous part tells us is that every nonzero vector in $\mathbb{R}^n$ is an eigenvector of $f_A(A)$ with eigenvalue 0. In particular, taking $\vec{v} = \vec{e_j}$ in part (c) tells us that the jth column of $f_A(A)$ is $\vec{0}$. Letting $j = 1, \dots, n$, we see that $f_A(A)$ is the zero matrix as claimed.

Section 7.4

37. Yes, $\lambda^2 - 7\lambda + 7$ is the characteristic polynomial for both, so they have the same real eigenvalues $\lambda_{1,2} = \dfrac{7 \pm \sqrt{21}}{2}$ and so are both similar to the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (by Theorem 7.4.4) and so A is similar to B by Theorem 3.4.6 (parts b and c).

38. No! For example, consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

    But wait! Isn't this the same problem as #37 with the numbers changed?
    Not quite. Note that $\lambda^2 - 7\lambda + 7$ had two distinct roots (and so two distinct eigenvalues) while $\lambda^2 - 4\lambda + 4$ has only one distinct root (and so the eigenvalue 2 has algebraic multiplicity 2, but in the counterexample given above it has a geometric multiplicity of only 1 in one case).

42. Check your class notes.

58. (a) Let $\vec{x} \in \text{im}(A)$. We want to show that $\vec{x} \in \ker(A)$. As $\vec{x} \in \text{im}(A)$, there is some vector $\vec{v}$ such that $\vec{x} = A\vec{v}$. Then $A\vec{x} = A^2\vec{v} = \vec{0}$, so that $\vec{x} \in \ker(A)$. As $\vec{x}$ was an arbitrary element of $\text{im}(A)$, this shows us that $\text{im}(A) \subseteq \ker(A)$.

    (b) We must have $\text{rank}(A) \leq \text{nullity}(A)$ by part(a) and $\text{rank}(A) + \text{nullity}(A) = 3$ by Rank-Nullity. Since A is not the zero matrix, $\text{rank}(A) \neq 0$. The only way that all of these conditions can be satisfied is if $\text{rank}(A) = 1$ and $\text{nullity}(A) = 2$.

    (c) As we know that $\mathbb{R}^3$ has dimension 3, it is sufficient to show that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent. Consider the relation $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$. Note that $A\vec{v_1} = A\vec{v_3} = \vec{0}$ by part (a), while $A\vec{v_2} = \vec{v_1} \neq \vec{0}$ by definition, so multiplying both sides of the relation by A shows us that $c_2 = 0$. Then, since we chose $\vec{v_3}$ so that it is not a scalar multiple of $\vec{v_1}$, it must be that $c_1 = c_3 = 0$ as well. Thus, $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent and so a basis of $\mathbb{R}^3$.

    (d) We compute the $\mathfrak{B}$-matrix column by column in the usual way and find

    $$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

    Aside: One reason that this is interesting is that it shows that any 3×3 matrix A such that $A^2 = 0$ is similar to B. So, if we find two 3×3 matrices A and B such that $A^2 = B^2 = 0$, we know right away that A is similar to B by this problem.

Section 7.5

12. First verify that $\overline{z + w} = \bar{z} + \overline{w}$, $\overline{zw} = \bar{z}\bar{w}$, and $\overline{(z^n)} = \bar{z}^n$. (These are easy to check, from the definitions.)

    Suppose that λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ where the coefficients $a_n, \ldots, a_0$ are real. Then $a_n\lambda_0^n + \cdots + a_1\lambda_0 + a_0 = 0$ and the result follows by taking the conjugate of both sides and using the above properties. (Note that since a_i is real, $\overline{a_i} = a_i$.)

27. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A (repeated according to algebraic multiplicity, so $\lambda_1 = \lambda_2 \neq \lambda_3$). Then $\text{tr}(A) = 1 = 2\lambda_2 + \lambda_3$ and $\det(A) = 3 = \lambda_2^2\lambda_3$. Solving for λ_2, λ_3, we see $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$.

42. (Note that this problem shows that complex eigenvalues and eigenvectors of real matrices always come in conjugate pairs. This is often useful. Compare this to #12 applied to the characteristic polynomial.)

    (a) Recall $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \bar{z}\bar{w}$ from #12. Then the ijth entry of $\overline{AB}$ is

    $$\overline{\sum_{k=1}^{p} a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}}\,\overline{b_{kj}},$$

    which is the ijth entry of $\bar{A}\bar{B}$.

    (b) We will use part (a), where B is the $n \times 1$ matrix $\vec{v} + \imath\vec{w}$. Let $\lambda = p + \imath q$. Then $AB = \lambda B$ and so $A\overline{B} = \overline{AB} = \overline{\lambda B} = \bar{\lambda}\overline{B}$, so $A(\vec{v} - \imath\vec{w}) = (p - \imath q)(\vec{v} - \imath\vec{w})$.