Answers are provided for all even numbered problems and for some odd numbered problems. If you have a question about a problem that isn't included below, feel free to ask me. If you think you've spotted an error, please let me know.

Section 6.1

22. The determinant of this matrix is 2 for all k (since $\sin^2 k+\cos^2 k = 1$), so it is always invertible.

41. Note that there are only two nonzero patterns in this matrix, so you only need to do the computations for the two of them. You'll end up with 24 as the determinant.

44. For each pattern P in the determinant of A, the corresponding pattern P' of kA satisfies $\text{sgn}(P') = \text{sgn}(P)$ and $\text{prod}(P') = k^n\text{prod}(P)$, so that $\det(kA) = k^n \det(A)$.

46. Use the formula for the inverse of a 2×2 matrix and actually compute the determinant. You'll find $\det(A^{-1}) = \frac{1}{\det(A)}$. Or, see Theorem 6.2.8, which proves this result more generally for $n \times n$ matrices.

48. Examples are not hard to find. Pick matrices A, B, C, D where the determinants are easy to compute and see what happens. For example, if you let A and D be the zero matrix and B and C be I_2, then $\det(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = 1$, while $\det(A)\det(D) - \det(B)\det(C) = -1$.

56. There is only one nonzero pattern P in this matrix (choose the entries which are equal to 1). For that pattern, $\text{prod}(P) = 1$ and $\text{sgn}(P) = (-1)^{(n-1)+(n-2)+\cdots+1} = (-1)^{\frac{n(n-1)}{2}}$. Thus, $\det(M_n) = (-1)^{\frac{n(n-1)}{2}}$.

57. A permutation matrix has one nonzero pattern with product 1. The signature can be either 1 or -1 depending on the permutation matrix, so the determinant of a permutation matrix will always be either 1 or -1.

Section 6.2

30. (a) Computing the determinant (or, at least the patterns that contain the t^2 terms) shows that the coefficient is $b - a$.

    (b) A matrix with two equal columns is not invertible and so has determinant 0. Thus, $f(a) = f(b) = 0$. Since we know that f is a polynomial of degree 2, it must be the case that $f(t) = k(t - a)(t - b)$ for some constant k. But, this constant is just the coefficient of t^2 when we FOIL this out, so by part (a) we know that $k = b - a$.

    (c) The matrix is invertible whenever $f(t) \neq 0$, in other words, for all values of t except for a and b.

    Aside: Consider the case $t = c$. Then the determinant is $(b - a)(c - a)(c - b)$, which is zero if and only if two or more of a, b, c are equal. (The original problem assumed $a \neq b$, but we can show that this is the determinant even if $a = b$.) This is the $n = 3$ case of the Vandermonde determinant, which is explored more generally in the (unassigned) problem #31 following this.

Aside 2: Suppose we want to show that $1, t, t^2$ are linearly independent. We could do this by writing $c_1 + c_2 t + c_3 t^2 = 0$ and letting $t = a$, $t = b$, and $t = c$ for three distinct scalars a, b, c to find three equations relating c_1, c_2, c_3. The matrix of this system of three equations in three unknowns is the transpose of the matrix consider in #30, so the fact that $\det(A^T) = \det(A)$ tells us that this matrix is invertible, so that $1, t, t^2$ are linearly independent. The general case in #31 can be used similarly to show that $1, t, \ldots, t^{n-1}$ are linearly independent.

Aside 3: Suppose we want to show that a nonzero polynomial of degree ≤ 2 has at most two roots. Write $f(t) = c_0 + c_1 t + c_2 t^2$ and suppose $f(a) = 0, f(b) = 0$, and $f(c) = 0$. Interpreting this as three equations relating c_0, c_1, c_2, written $A\vec{x} = \vec{0}$, we again get that the coefficient matrix A is the transpose of the matrix above. Since this matrix is invertible for a, b, c distinct, this tells us that there is a unique solution, which will be $\vec{x} = \vec{0}$ since the system is homogeneous. Thus, the only polynomial of degree ≤ 2 with three roots is the zero polynomial. The general case in #31 can be used similarly to show that any nonzero polynomial of degree $\leq n$ has at most n roots.

37. Note that if A and A^{-1} have integer entries, then $\det(A)$ and $\det(A^{-1})$ are integers. Since $\det(A^{-1}) = \frac{1}{\det(A)}$, this can only happen if $\det(A) = \pm 1$.

38. $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 = 9$.

39. Proceed as in #38 to see that $\det(A^T A) \geq 0$. If A is invertible, then $\det(A) \neq 0$, so we can say in fact that $\det(A^T A)$ is positive.

44. (a) This will be $\vec{0}$ if and only if $\vec{v_2}, \ldots, \vec{v_n}$ are not linearly independent (since this is the only way that the matrix $\begin{bmatrix} \vec{x} & \vec{v_2} & \ldots & \vec{v_n} \end{bmatrix}$ will fail to be invertible for every choice of $\vec{x}$).
   (b) Compute this using the strategy suggested in the problem. You'll find $\vec{e_2} \times \cdots \times \vec{e_n} = \vec{e_1}$.
   (c) We want to show that $\vec{v_i} \cdot (\vec{v_2} \times \ldots \vec{v_n}) = 0$ for $i = 2, \ldots, n$. But $\vec{v_i} \cdot (\vec{v_2} \times \ldots \vec{v_n}) = \det \begin{bmatrix} \vec{v_i} & \vec{v_2} & \ldots & \vec{v_n} \end{bmatrix} = 0$ since the matrix has two equal columns.
   (d) $\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n} = -\vec{v_3} \times \vec{v_2} \times \cdots \times \vec{v_n}$ since swapping two columns in a determinant multiplies the determinant by -1 (we saw this for rows, but it also holds for columns since $\det(A^T) = \det(A)$).
   (e) $\det \begin{bmatrix} \vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n} & \vec{v_2} & \ldots & \vec{v_n} \end{bmatrix} = (\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n}) \cdot (\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n}) = \|\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n}\|^2$.
   (f) There are two ways to do this. One is to use Definition A.9. The above parts tell us various things about what the cross product in $\mathbb{R}^n$ does with respect to the direction and length of the vectors in the cross product, and you can use these to show that the three properties in the definition hold (and, since these three properties uniquely determine the vector, this will show that the two definitions are equivalent). Another strategy is to use the explicit computation in Theorem A.11. This tells us how the cross product in $\mathbb{R}^3$ is defined explicitly, so we just need to compute the cross product defined in this problem for the vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. You can do this using the same technique as part (b), and will find that the general cross product is the usual cross product in the case $n = 3$.

Aside: Note that this problem lets us do the same nice things with cross products that we've previously seen in $\mathbb{R}^3$ in $\mathbb{R}^n$ as well.

57. Before we prove this, let's look at why we should care. You used the Implicit Function Theorem in Calc I and Calc III (if you've taken it) when doing implicit differentiation. In order for the slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ to make sense, the implicit function $F(x, y) = 0$ has to be an ordinary differentiable function in the neighborhood of the point where you're calculating the implicit derivative ("neighborhood" is a precise mathematical term, but essentially mean some open interval containing the point in the Calc I case and some open region containing the point in the Calc III case). In other words, there has to be a piece of $F(x, y) = 0$ going through the point where you want to calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ which can be described as $y = f(x)$ for some function f. (For example, if you start with $F(x, y) = x^2 + y^2 - 1 = 0$, then any point on the top half lies on the curve $y = f(x) = \sqrt{1 - x^2}$.) The same thing can be done in more than two variables using vector valued functions: if $F(\vec{x}, \vec{y}) = \vec{0}$, then we might be able to find a function f such that $\vec{y} = f(\vec{x})$ is a portion of the surface $F(\vec{x}, \vec{y}) = \vec{0}$. In Calc III, you considered the case $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} z \end{bmatrix}$. Of course, it isn't always possible to find such an f. You could come up with a really horrible function F where no portion of $F(\vec{x}, \vec{y}) = 0$ can be described as $\vec{y} = f(\vec{x})$ for any function f. It turns out that this does work if F is a continuously differentiable function (that is, F is differentiable and F' is continuous) and if the portion of the Jacobian of F involving the $\vec{y}$ variables is an invertible matrix. (Your Calc III instructor probably just told you that the Implicit Function Theorem applies if the function F is "fairly nice" since most Calc III students don't know what invertibility is. Now that you've taken Linear Algebra, you get the precise version.) The full theorem is a bit tricky to prove, but the special case where F is a linear transformation is easier to prove, and that's what this problem does. (Note also that the conditions on F are simpler to check when F is a linear transformation.)

Let $T(\vec{x}) = A\vec{x}$ be a linear transformation from $\mathbb{R}^{m+n}$ to $\mathbb{R}^m$. (Here, T is playing the role of F above.) Write $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ where A_1 is $m \times m$ and A_2 is $m \times n$. Suppose that $\det(A_1) \neq 0$ (this is the equivalent of having the portion of the Jacobian be invertible above). Consider the equation $T(\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix}) = \vec{0}$ (where we think of $\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix}$ as a $(m+n) \times 1$ block matrix). We want to show that for every vector $\vec{x}$, there is a unique vector $\vec{y}$ for which this holds. Using Theorem 1.3.8 or Theorem 2.3.9, we compute that $T(\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix}) = A_1\vec{y} + A_2\vec{x}$. So, let's look at $T(\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix}) = \vec{0}$ and solve for $\vec{y}$. Since $\det(A_1) \neq 0$, we know that A_1 is invertible, so that we can rearrange the equation $A_1\vec{y} + A_2\vec{x} = \vec{0}$ as $\vec{y} = -A_1^{-1}A_2\vec{x}$. This shows that there is indeed a unique choice of $\vec{y}$ for every $\vec{x}$ such that $T(\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix}) = \vec{0}$.

Now, consider the transformation $L(\vec{x}) = \vec{y}$ (which plays the role of f above). To finish the problem, we want to show that L is linear by finding its matrix, M. But, we saw above that $\vec{y} = -A_1^{-1}A_2\vec{x}$ so that $L(\vec{x}) = -A_1^{-1}A_2\vec{x}$ and $M = -A_1^{-1}A_2$.