1. (20 pts) Let $V = \text{span}(e^x, e^{-x}, xe^x, xe^{-x})$ be a subspace of C^∞ with basis

$$\mathfrak{B} = \{e^x, e^{-x}, xe^x, xe^{-x}\}.$$

Let $T(f(x)) = 2f(x) - f'(x)$ be a linear transformation from V to V. Find the $\mathfrak{B}$-matrix of T. Is T an isomorphism?

We will compute B column-by-column:

$$[T(e^x)]_{\mathfrak{B}} = [2e^x - e^x]_{\mathfrak{B}} = [e^x]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(e^{-x})]_{\mathfrak{B}} = [2e^{-x} - (-e^{-x})]_{\mathfrak{B}} = [3e^{-x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(xe^x)]_{\mathfrak{B}} = [2xe^x - (xe^x + e^x)]_{\mathfrak{B}} = [xe^x - e^x]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(xe^{-x})]_{\mathfrak{B}} = [2xe^{-x} - (-xe^{-x} + e^{-x})]_{\mathfrak{B}} = [3xe^{-x} - e^{-x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

so that

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

The matrix B is invertible (as it is upper-triangular with nonzero diagonal entries, or by using the zero-trick to see that the columns are linearly independent), so T is an isomorphism.

2. Let $T(A) = A - A^{\mathrm{T}}$ be a linear transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(a) (10 pts) Find a basis of $\operatorname{im}(T)$ and compute $\operatorname{rank}(T)$.

Note $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = (b-c)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Thus, $\operatorname{im}(T)$ is spanned by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and a basis of $\operatorname{im}(T)$ is $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$. So, $\operatorname{rank}(T) = 1$.

(b) (10 pts) Find a basis of $\ker(T)$ and compute $\operatorname{nullity}(T)$.

If we set $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. That is, the matrices in $\ker(T)$ are precisely those with $b = c$, i.e, matrices of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so that $\ker(T) = \operatorname{span}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$ and since these vectors are linearly independent (zero trick), we see that $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ form a basis of $\ker(T)$. So, $\operatorname{nullity}(T) = 3$.

Note that $\operatorname{rank}(T) + \operatorname{nullity}(T) = 4 = \dim(\mathbb{R}^{2\times 2})$ as required by Rank-Nullity.

3. (16 pts) Let $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ be an orthonormal basis of $\mathbb{R}^4$. Let $V = \text{span}(\vec{u_1}, \vec{u_2})$. Note that V is a subspace of $\mathbb{R}^4$. Find the $\mathfrak{B}$-matrix of the linear transformation $T(\vec{x}) = \text{proj}_V \vec{x}$.

Note that $\vec{u_1}, \vec{u_2}$ are linearly independent (as they are orthonormal) and hence they form an orthonormal basis of V. This shows us that $\text{proj}_V(\vec{x}) = (\vec{u_1} \cdot \vec{x})\vec{u_1} + (\vec{u_2} \cdot \vec{x})\vec{u_2}$. We compute the matrix B column-by-column (by noting that $\vec{u_i} \cdot \vec{u_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$):

$$[T(\vec{u_1})]_{\mathfrak{B}} = [\vec{u_1}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(\vec{u_2})]_{\mathfrak{B}} = [\vec{u_2}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(\vec{u_3})]_{\mathfrak{B}} = [\vec{0}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(\vec{u_4})]_{\mathfrak{B}} = [\vec{0}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so that $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

4. (20 pts) Use the Gram-Schmidt process to find an orthonormal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v_1} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 4 \end{bmatrix},$$

and in the process find the QR-factorization of the matrix

$$M = \begin{bmatrix} 0 & 1 & 4 \\ 4 & 3 & 5 \\ 0 & 0 & 8 \\ 0 & 1 & 4 \end{bmatrix}.$$

Please check your work!!!!!!!!!!

$$\vec{u_1} = \frac{1}{4}\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v_2}^{\perp} = \vec{v_2} - 3\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u_2} = \frac{1}{\sqrt{2}}\vec{v_2}^{\perp} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v_3}^{\perp} = \vec{v_3} - 5\vec{u_1} - \frac{8}{\sqrt{2}}\vec{u_2} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

$$\vec{u_3} = \frac{1}{8}\vec{v_3}^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ form an orthonormal basis for V and the QR-factorization is:

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, R = Q^T M = \begin{bmatrix} 4 & 3 & 5 \\ 0 & \sqrt{2} & 4\sqrt{2} \\ 0 & 0 & 8 \end{bmatrix}.$$

Fun Fact: Note that $x_1 = x_4$ in each of the vectors in our basis of V. Since we know the hyperplane $x_1 = x_4$ has dimension 3, it follows that in fact V is this hyperplane.

5. (4 pts each) Decide whether the following statements are true or false. You do not need to show work.

$\mathcal{P}_1$ is isomorphic to $\mathbb{C}$.

$\boxed{\text{True}}$ False

Both spaces have dimension 2, so we can find an isomorphism. For example, $T(a + bx) = a + ıb$ is an isomorphism from $\mathcal{P}_1$ to $\mathbb{C}$.

Let $V = \{f \in C^\infty \mid f'(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$. Then V is a subspace of C^∞.

True $\boxed{\text{False}}$

V is not closed under scalar multiplication (by a negative scalar). For example, $x^3 \in V$, but $-x^3 \notin V$.

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then $\left(\sum_{k=1}^n a_k b_k\right)^2 \leq \sum_{k=1}^n (a_k^2) \sum_{k=1}^n (b_k^2)$.

$\boxed{\text{True}}$ False

Use Cauchy-Schwarz on $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ and square both sides.

Let i and j be integers with $1 \leq i \leq n$ and $1 \leq j \leq n$. Let $V = \{A \in \mathbb{R}^{n \times n} | \vec{e_i}^T A \vec{e_j} = 0\}$.
(Note V is a subspace of $\mathbb{R}^{n \times n}$.) Then $\dim(V) = n^2 - n$.

True $\boxed{\text{False}}$

$\vec{e_i}^T A \vec{e_j} = a_{ij}$, so this condition means that all of the entries of A are free, except that the ijth entry must be zero. So, a basis of V consists of all matrices with precisely one nonzero entry (where the nonzero entry is a 1), except for the matrix with the nonzero entry in the ijth position. Thus, $\dim(V) = n^2 - 1$.

Let A be an $n \times m$ matrix. Then $A^T A$ is a symmetric $m \times m$ matrix.

$\boxed{\text{True}}$ False

This can be seen directly: $(A^T A)^T = A^T (A^T)^T = A^T A$. Alternatively, note that the ijth entry of $A^T A$ is the dot product of the ith and jth columns of A and is thus equal to the jith entry since the dot product is commutative.

Let $\vec{x} \in \mathbb{R}^n$ and let V be a subspace of $\mathbb{R}^n$. Then $\|\operatorname{proj}_V \vec{x}\| \leq \|\vec{x}\|$.

$\boxed{\text{True}}$ False

See Theorem 5.1.10.
