1. (15 pts) Use Gauss-Jordan elimination to find all solutions of the system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 5 \\ 2x & + & 4y & - & 3z & = & -4 \\ x & + & 2y & - & 2z & = & -3. \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & 2 & 5 \\ 2 & 4 & -3 & -4 \\ 1 & 2 & -2 & -3 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 2 & 5 \\ 0 & 0 & -7 & -14 \\ 0 & 0 & -4 & -8 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so if we let $y = t$ for an arbitrary scalar t, then the solutions are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-2t \\ t \\ 2 \end{bmatrix}$.

2. For scalars a and b, consider the matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(a) (10 pts) What is the geometrical effect of multiplying $\vec{x}$ by A? (Be specific and give your answer in terms of the scalars a and b.)

Let (r, θ) be the polar coordinates of (a, b). Then A scales by r and rotates counterclockwise by angle θ.

(b) (5 pts) Compute A^{-1} when it exists. For what values of a and b does A^{-1} not exist? (You may use any technique to do this.)

Using the formula for the inverse of a 2×2 matrix,

$$A^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

unless $a^2 + b^2 = 0$ (that is, if $a = b = 0$), in which case A is not invertible.

(c) (5 pts) What is the geometrical effect of multiplying $\vec{x}$ by A^{-1}?

A^{-1} scales by $\frac{1}{r}$ and rotates clockwise by angle θ. This is easiest to see in polar coordinates:

$$A = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, A^{-1} = \frac{1}{r} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

3. Let

$$\vec{w} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \vec{v_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (8 pts) Express $\vec{w}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

We want to find scalars a and b such that $\vec{w} = a\vec{v_1} + b\vec{v_2}$. Setting the first two components equal, we see that $-2b = 6$ and $a = 2$, so $a = 2, b = -3$ is the only possible solution. We then check that this works for the remaining components. (Alternatively, use Gauss-Jordan to find a and b.) So, $\vec{w} = 2\vec{v_1} - 3\vec{v_2}$.

(b) (8 pts) Suppose that T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ such that

$$T(\vec{v_1}) = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \text{ and } T(\vec{v_2}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\vec{w})$.

$$T(\vec{w}) = T(2\vec{v_1} - 3\vec{v_2}) = 2T(\vec{v_1}) - 3T(\vec{v_2}) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

4. Let A be an $n \times m$ matrix and $\vec{b}$ be a vector in $\mathbb{R}^n$. Let $\vec{v_1}$, $\vec{v_2}$, and $\vec{w}$ be vectors in $\mathbb{R}^m$ such that $\vec{v_1}$ and $\vec{v_2}$ are solutions to $A\vec{x} = \vec{b}$ and $\vec{w}$ is a solution to $A\vec{x} = \vec{0}$.

   (a) (5 pts) Show that $(\vec{v_1} + \vec{w})$ is a solution to $A\vec{x} = \vec{b}$.

   We are given that $A\vec{v_1} = \vec{b}$ and $A\vec{w} = \vec{0}$. So,

   $$A(\vec{v_1} + \vec{w}) = A\vec{v_1} + A\vec{w} = \vec{b} + \vec{0} = \vec{b}$$

   and so $(\vec{v_1} + \vec{w})$ is a solution to $A\vec{x} = \vec{b}$.

   (b) (5 pts) Show that $(\vec{v_1} - \vec{v_2})$ is a solution to $A\vec{x} = \vec{0}$.

   We are given that $A\vec{v_1} = \vec{b}$ and $A\vec{v_2} = \vec{b}$. So,

   $$A(\vec{v_1} - \vec{v_2}) = A\vec{v_1} - A\vec{v_2} = \vec{b} - \vec{b} = \vec{0}$$

   and so $(\vec{v_1} - \vec{v_2})$ is a solution to $A\vec{x} = \vec{0}$.

   Aside: This shows us that the solutions of $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{0}$ are often closely related. Namely, if $A\vec{x} = \vec{b}$ is consistent, then there is a certain vector $\vec{v}$ such that every solution of $A\vec{x} = \vec{b}$ can be expressed as $\vec{v} + \vec{w}$ for some solution $\vec{w}$ to $A\vec{x} = \vec{0}$. For this reason, if we want to study the geometry of the solutions to $A\vec{x} = \vec{b}$, it'll be good enough to consider the special case $\vec{b} = \vec{0}$, as you can go from one to the other just by translating everything by the vector $\vec{v}$.

5. The **cross product** of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and define $T(\vec{x}) = \vec{v} \times \vec{x}$ for $\vec{x}$ in $\mathbb{R}^3$.

(a) (8 pts) Show that T is a linear transformation by finding its matrix.

$$A = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

(b) (7 pts) Using part (a), show that $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$ and that $\vec{v} \times (k\vec{w}) = k(\vec{v} \times \vec{w})$ for all vectors $\vec{u}, \vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ and for all scalars k.

$$\vec{v} \times (\vec{u} + \vec{w}) = T(\vec{u} + \vec{w}) = T(\vec{u}) + T(\vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w} \text{ and}$$
$$\vec{v} \times (k\vec{w}) = T(k\vec{w}) = kT(\vec{w}) = k(\vec{v} \times \vec{w}),$$

where the second equality in each of the above chains of equalities is justified by the fact that T is linear.

6. (4 pts each) Decide whether the following statements are true or false. You do not need to show work.

Let A be an $n \times n$ matrix with $\text{rank}(A) = n$. Let $\vec{b}$ in $\mathbb{R}^n$. Then $A\vec{x} = \vec{b}$ has a unique solution.

$\boxed{\text{True}}$ False

Follows from parts b and c of Example 3 on p. 26.

Let A be an $n \times m$ matrix with $\text{rank}(A) < m$. If $A\vec{x} = \vec{b}$ has at least one solution for one choice of $\vec{b}$ in $\mathbb{R}^n$, then it has at least one solution for every choice of $\vec{b}$ in $\mathbb{R}^n$.

True $\boxed{\text{False}}$

Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *has* $\text{rank}(A) = 1 < 2$ *and has at least one solution for* $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, *but has no solution for* $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let A be an $n \times n$ diagonal matrix. Then $\text{rank}(A) = n$.

True $\boxed{\text{False}}$

Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *is diagonal and* $\text{rank}(A) = 1$.

Let A be an $n \times m$ matrix, $\vec{v}$ be a vector in $\mathbb{R}^m$, $\vec{b}$ be a vector in $\mathbb{R}^n$ and k be a scalar. If $A\vec{v} = \vec{b}$, then $A(k\vec{v}) = k^m\vec{b}$.

True $\boxed{\text{False}}$

By Theorem 1.3.10, the correct result is $A(k\vec{v}) = k\vec{b}$. *Note* $k^m\vec{b} \neq k\vec{b}$ *unless* $k = 0$, $m = 1$, *or* $\vec{b} = \vec{0}$, *so this is not generally true.*

If A is a 2×2 matrix representing an orthogonal projection onto a line, then A is invertible.

True $\boxed{\text{False}}$

If A *projects onto the line* L *going through the origin, then every point of the line* $L^\perp$ *that is perpendicular to* L *and goes through the origin will map to* $\vec{0}$, *so we can't find a unique inverse for every point. Alternatively, calculate that the determinant (see 2.2, #38) is equal to zero.*

If A is a 2×2 matrix representing a reflection about a line, then A is invertible.

$\boxed{\text{True}}$ False

We calculated on the practice exam that $A^{-1} = A$ *(that is, you can "undo" a reflection across a line by reflecting across the line again.) Alternatively, calculate that the determinant (see 2.2, #38) is not equal to zero.*
