Calculus II, Worksheet 3
Name:

Please answer the following questions in the spaces provided, or on your own paper. You may use your textbook, but do not consult any other sources or with each other. This worksheet is due on **August 6th**. As you have plenty of time for this, you will not receive credit for illegible or excessively disorganized work.

1. (5 pts) Maclaurin and Taylor polynomials can sometimes be used to test series for convergence. Compute the 1^{st} Maclaurin polynomial for $\sin x$ and call it $p_1(x)$. Note that $p_1(x)$ is an approximation to $\sin x$ and so should behave similarly to it.

   Determine whether $\displaystyle\sum_{k=1}^{\infty} \sin(\frac{\pi}{k})$ converges by comparing it to $\displaystyle\sum_{k=1}^{\infty} p_1(\frac{\pi}{k})$ via the limit comparison test.

2. (5 pts) Taylor series can sometimes help us find the sum of convergent series. Beginning by differentiating the Maclaurin series for $\frac{1}{1-x}$, show that

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{for } -1 < x < 1.$$

   Then, use this representation to compute $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{2^k}$.

3. (10 pts) Power series can sometimes be used to find solutions to differential equations. Define the **Bessel function** $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$. (*Aside:* This function is useful in physics and engineering. It was first used in analyzing Kepler's laws of planetary motion.)

   (a) Find $J_0'(x)$ and $J_0''(x)$. Express them as sums starting at $k = 0$.

   (b) Show that $y = J_0(x)$ is a solution to the differential equation $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0$.