Assignment #13 Solutions

Page 412, 48.1 *Let G be the graph in the figure.*

(a) How many different walks are there from a to b?

(b) How many different paths are there from a to b?

(a) There are infinitely many walks from a to b, since a walk could retrace the same set of vertices arbitrarily many times.

(b) There are $5 \times 3 = 15$ paths from a to b.

Page 412, 48.4 *Let $n \geq 2$ be an integer. Form a graph G_n whose vertices are all the two-element subsets of $\{1, 2, \dots, n\}$. In this graph, we have an edge between distinct vertices $\{a, b\}$ and $\{c, d\}$ exactly when $\{a, b\} \cap \{c, d\} = \emptyset$. Please answer:*

(a) How many vertices does G_n have?

(b) How many edges does G_n have?

(c) For which values of $n \geq 2$ is G_n connected? Prove your answer.

(a) There are $\binom{n}{2}$ vertices.

(b) Each vertex is adjacent to $\binom{n-2}{2}$ other vertices, so there are $|E| = \frac{1}{2}\sum_{v \in G_n(v)} d(v) = \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ edges.

(c) G_2 has only one vertex and so is clearly connected. In G_3 and G_4, there is no path from $\{1, 2\}$ to $\{1, 3\}$ (for example), so that G_3 and G_4 are not connected. We will now show that G_n is connected for $n \geq 5$, so that G_n is connected for all $n \geq 2$ except $n = 3, 4$.

Let $n \geq 5$ and $\{a, b\}, \{c, d\} \in V(G_n)$. We will show that G_n is connected by finding a path from $\{a, b\}$ to $\{c, d\}$. We proceed by cases:
Case 1: $\{a, b\} = \{c, d\}$. Then $\{a, b\}$ is a $(\{a, b\}, \{a, b\})$-path.
Case 2: $\{a, b\} \cap \{c, d\} = \emptyset$. Then $\{a, b\} \sim \{c, d\}$ is a $(\{a, b\}, \{c, d\})$-path.
Case 3: $\{a, b\} \cap \{c, d\} \neq \emptyset$, but $\{a, b\} \neq \{c, d\}$. In this case, it must be that $|\{a, b\} \cap \{c, d\}| = 1$, and that $\{a, b, c, d\}$ actually only contains three distinct elements (since one of $a = c$, $a = d$, $b = c$, or $b = d$ holds). Since $n \geq 5$ and $|\{a, b, c, d\}| = 3$, there must be $e, f \in \{1, \dots, n\}$ such that neither e nor f are in $\{a, b, c, d\}$. Then $\{a, b\} \sim \{e, f\} \sim \{c, d\}$ is a $(\{a, b\}, \{c, d\})$-path.

In all cases, we have found a $(\{a, b\}, \{c, d\})$-path. Since $\{a, b\}$ and $\{c, d\}$ were arbitrarily chosen vertices, G_n is connected for $n \geq 5$.

Page 412, 48.8 *Consider the is-connected-to relation on the vertices of a graph. Show that is-connected-to need not be irrefelexive or antisymmetric.*

Consider the graph $G = (V, E)$ where $V = \{a, b\}$ and $E = \{\{a, b\}\}$. Then a is connected to b and b is connected to a. Since is-connected-to is transitive, a is connected to a, so is-connected-to is not irreflexive. Also, $a \neq b$, so is-connected-to is not antisymmetric.

Page 412, 48.9 *Let G be a graph. Prove that G or $\overline{G}$ (or both) must be connected.*

If G is connected, there is nothing to show. Suppose that G is not connected. We will show that $\overline{G}$ is connected.

Let $a, b \in V(\overline{G})$. We will show that there is an (a, b)-path in $\overline{G}$. Since G is not connected, there exist vertices $x, y \in V(G)$ such that there is no (x, y)-path in G. We proceed by cases:
Case 1: Both a and b are in the same component as x in G. Then neither a nor b is connected to y in G. In particular, neither a nor b is adjacent to y in G and so both a and b are adjacent to y in $\overline{G}$. Thus, $a \sim y \sim b$ is an (a, b)-path in $\overline{G}$.
Case 2: Neither a nor b is in the same component as x in G. Then neither a nor b is connected to x in G. In particular, neither a nor b is adjacent to x in G and so both a and b are adjacent to x in $\overline{G}$. Thus, $a \sim x \sim b$ is an (a, b)-path in $\overline{G}$.
Case 3: One of a or b is in the same component as x in G, but the other is not. Without loss of generality, suppose that a is in the same component as x in G. Then a is not in the same component as y in G, so a is not adjacent to y in G and so $a \sim y$ in $\overline{G}$. Also, x is not in the same component of y in G and so is not adjacent to y in G, and so $y \sim x$ in $\overline{G}$. Finally, b is not in the same component as x in G, so is not adjacent to x in G and so $x \sim b$ in $\overline{G}$. Thus, $a \sim y \sim x \sim b$ is a (a, b)-path in $\overline{G}$.

In all cases, we have found a (a, b)-path in $\overline{G}$. Since a and b were arbitrarily chosen vertices, $\overline{G}$ is connected.

Page 420, 49.2 *Let T be a tree. Prove that the average degree of a vertex is less than 2.*

Suppose that T has n vertices. By Theorem 49.12, T has $n - 1$ edges. Then $\sum_{v \in V(T)} d(v) = 2|E(V)| = 2(n-1)$, so that the average degree of a vertex is $\frac{1}{n} \sum_{v \in V(T)} d(v) = \frac{2(n-1)}{n} < 2$.

Page 420, 49.5 *Complete the proof of Theorem 49.4. That is, prove that if G is a graph in which any two vertices are joined by a unique path, then G must be a tree.*

Since any two vertices of G are connected, G is connected. Thus, we need only show that G is acyclic.

Assume for contradiction that G has a cycle. Call it $C = v_0 \sim v_1 \sim \cdots \sim v_\ell$ where $v_0 = v_\ell$. Then $v_1 \sim \cdots \sim v_\ell$ is a (v_1, v_0)-path and $v_1 \sim v_0$ is a (v_1, v_0)-path. Since $\ell \geq 3$ (as all cycles contain at least three vertices), the first of these paths has length at least 2, while the second of these paths has length 1. Thus, they must be distinct (v_1, v_0)-paths, contradicting the fact that G has a unique (v_1, v_0)-path. So, it must be that G has no cycle and so is a tree.

Page 420, 49.7 *Prove the following converse to Proposition 49.8: Let T be a tree with at least two vertices and let $v \in V(T)$. If $T - v$ is a tree, then v is a leaf.*

Let T be a tree with at least two vertices and let $v \in V(T)$. We will proceed by contraposition. Suppose that v is not a leaf–that is, that $d_T(v) \geq 2$ (since clearly $d_T(v) \neq 0$ as T is connected and has at least two vertices); we will show that $T - v$ is not a tree. Since $d_T(v) \geq 2$, v has at least two neighbors: call them x and y. Note that $x \sim v \sim y$ is an (x, y)-path in T, and that this path must be the unique (x, y)-path in T since T is a tree. Thus, $T - v$ does not have an (x, y)-path since any (x, y)-path of $T - v$ would also be an (x, y)-path of T distinct from the (x, y)-path $x \sim v \sim y$ (as v is not a vertex of $T - v$). Since there is no (x, y)-path in $T - v$, it must be that $T - v$ is not connected and so is not a tree.

Page 420, 49.9 *Let G be a forest with n vertices and c components. Find and prove a formula for the number of edges in G.*

Let $n_1, n_2, \ldots, n_c$ be the number of vertices in the c components of G (that is, n_1 is the number of vertices in the first component, n_2 is the number of vertices in the second component, and so on). Then $n_1 + n_2 + \cdots + n_c = n$. Note that each component of G is a tree and so the jth component of G ($1 \leq j \leq c$) has $n_j - 1$ edges by Theorem 49.12. So, in total, G has $(n_1 - 1) + (n_2 - 1) + \cdots + (n_c - 1) = (n_1 + n_2 + \cdots + n_c) - c = n - c$ edges.