Assignment #12 Solutions

$\boxed{\text{Page 461, 55.2}}$ *Let n be a positive integer.*

(a) How many different (unequal) linear orders can be formed on the elements $\{1, \ldots, n\}$?

(b) How many different (nonisomorphic) linear orders can be formed on the elements $\{1, \ldots, n\}$?

(a) By Theorem 55.4, all of the linear orders on this set will be isomorphic, so all different orders will be bijections of the order Q described in Theorem 55.4. There are $n!$ bijections on the set $\{1, \ldots, n\}$, so there are $n!$ different linear orders on this set.

(b) By Theorem 55.4, there is only one nonisomorphic linear order on this set.

$\boxed{\text{Page 461, 55.4}}$ *Suppose f is an isomorphism between posets P and Q, and let x and y be elements of P. Prove that x and y are incomparable (in P) if and only if $f(x)$ and $f(y)$ are incomparable (in Q).*

Let $P = (X, \preceq)$ and $Q = (Y, \preceq')$.

Suppose x and y are incomparable in P. We will show that $f(x)$ and $f(y)$ are incomparable in Q. Assume for contradiction that $f(x)$ and $f(y)$ are comparable; without loss of generality, suppose $f(x) \preceq' f(y)$. Then, since f is order-preserving, we have $x \preceq y$, contradicting the fact that x and y are incomparable.

Suppose $f(x)$ and $f(y)$ are incomparable in Q. We will show that x and y are incomparable in P. Assume for contradiction that x and y are comparable; without loss of generality, suppose $x \preceq y$. Then, since f is order-preserving, we have $f(x) \preceq' f(y)$, contradicting the fact that $f(x)$ and $f(y)$ are incomparable. (An alternative way of doing the second half of this proof: use the fact that f^{-1} is an isomorphism from Q to P and apply f^{-1} to everything in the first part of the proof.)

$\boxed{\text{Page 461, 55.5a}}$ *Let P and Q be isomorphic posets and let f be an isomorphism. Let x be an element of the ground set of P. Prove that x is minimum in P if and only if $f(x)$ is minimum in Q.*

Let $P = (X, \preceq)$ and $Q = (Y, \preceq')$.

(i) Suppose x is minimum in P. To show that $f(x)$ is minimum in Q, we need to show that if b is any element of Q, then $f(x) \preceq' b$ in Q. Since f is onto, there is an $a \in X$ with $f(a) = b$. (Note: the proof in the hint in Appendix A of your book says that $a \in P$. This is incorrect; they mean that a is in the ground set of P.) Since x is minimum in P, $x \preceq a$. Thus, $f(x) \preceq' f(a) = b$ since f is order-preserving. Therefore, $f(x)$ is minimum in Q.

(ii) It is possible to do the second half of the proof as an exact parallel of the first half (with f^{-1} instead of f). Instead, let's use an argument by contradiction for variety. Suppose $f(x)$ is minimum in Q and assume for contradiction that x is not minimum in P. Then there is some $a \in X$ such that $a \preceq x$ and $a \neq x$. Since f is order-preserving, $f(a) \preceq' f(x)$ and $f(a) \neq f(x)$

since f is one-to-one and $a \neq x$. This contradicts the minimality of $f(x)$ in Q, so it must be that x is minimum in P.

$\boxed{\text{Page 397, 46.3}}$ *In the map-coloring problem, why do we require that countries be connected (and not in multiple pieces like Russia or Michigan)? Draw a map, in which disconnected countries are permitted, that requires more than four colors.*

A disconnected country is essentially two countries in a connected country color map, except with the additional restriction that they receive the same color. We exclude this case because the primary reason to study this problem is to study "adjacency relations" of countries.

Examples that require more than four colors are not too difficult to find; ask me in class or office hours if you're having trouble finding one.

$\boxed{\text{Page 398, 46.10}}$ *Construct a graph G for which the is-adjacent-to relation, $\sim$, is antisymmetric. Construct a graph G for which the is-adjacent-to relation, $\sim$, is transitive.*

Since we know that is-adjacent-to is symmetric and not reflexive, this relation will be antisymmetric if and only if G is an edgeless graph. So, for example, is-adjacent-to is antisymmetric on the complete graph on one vertex, K_1.
The relation is-adjacent-to will be transitive on any edgeless graph (vacuously, as there are no vertices in the relation) or on any complete graph. (Other more complicated examples are also possible.)

$\boxed{\text{Page 398, 46.12}}$ *Let G be a graph. Prove that there must be an even number of vertices of odd degree.*

By Theorem 46.5, the sum of the degrees of the vertices must be even. If a graph had an odd number of vertices of odd degrees, this sum would be odd.

$\boxed{\text{Page 398, 46.13}}$ *Prove that in any graph with two or more vertices, there must be two vertices of the same degree.*

We will proceed by contradiction: assume that G is a graph with n vertices ($n \geq 2$) for which every vertex has a different degree.
We know that $0 \leq d(v) \leq n-1$ for any vertex $v \in V(G)$, so there are only n different possible vertex degrees. Since G has n vertices, each with a different degree, it must be the case that G has a vertex of each possible degree from 0 to $n-1$. In particular, G has vertices $v_1, v_2 \in V(G)$ such that $d(v_1) = 0$ and $d(v_2) = n-1$. Thus, v_1 is adjacent to no vertex of G, while v_2 is adjacent to every vertex of G (except itself), including v_1. This is a contradiction since the is-adjacent-to relation is symmetric. Thus, G must have two vertices of the same degree.

$\boxed{\text{Page 405, 47.6}}$ *Suppose that G is a subgraph of H. Prove or disprove:*

(a) $\alpha(G) \leq \alpha(H)$.

(b) $\alpha(G) \geq \alpha(H)$.

(c) $\omega(G) \leq \omega(H)$.

(d) $\omega(G) \geq \omega(H)$.

(a) Counterexample: Consider the graphs G and H given by $V(G) = V(H) = \{1, 2\}$ and $E(G) = \emptyset$, $E(H) = \{\{1, 2\}\}$. Then $\alpha(G) = 2$ and $\alpha(H) = 1$.

(b) Counterexample: Consider the graphs G and H given by $V(G) = \{1\}$, $V(H) = \{1, 2\}$ and $E(G) = E(H) = \emptyset$. Then $\alpha(G) = 1$ and $\alpha(H) = 2$.

(c) Proof: Let $k = \omega(G)$. Then k is the clique number of G, so G has a clique of size k; call it S. Note that $S \subseteq V(G) \subseteq V(H)$, so S is a subset of $V(H)$ since $\subseteq$ is transitive. Also, $E(G) \subseteq E(H)$, so all vertices of S are adjacent in H since they are adjacent in G. Thus, S is a clique of H, so we know that H has a clique of size k and so $\omega(H) \geq k$. That is, $\omega(G) \leq \omega(H)$.

(d) Counterexample: Let G and H be given by $V(G) = V(H) = \{1, 2\}$ and $E(G) = \emptyset$, $E(H) = \{\{1, 2\}\}$. Then $\omega(G) = 1$ and $\omega(H) = 2$.

$\boxed{\text{Page 405, 47.8}}$ *Find a graph G on five vertices for which $\omega(G) < 3$ and $\omega(\overline{G}) < 3$.*

Note that we only use the assumption that G has six vertices in the case $d(v) \leq 2$ of the proof of Proposition 47.13, so this can't possibly happen if any vertex v of G has $d(v) \geq 3$. Similarly, if any vertex v of G has $d(v) \leq 1$, then $d_{\overline{G}}(v) \geq 3$, which leads to the same problem. So, if this is possible, G must be a 2-regular graph. In fact, there is essentially only one 2-regular graph on five vertices (up to relabeling the vertices); in the language of Definition 49.1, it is the graph C_5, a cycle on five vertices. It is easy to verify that $\omega(C_5) = 2$ and $\omega(\overline{C_5}) = 2$, so $G = C_5$ will work (and is the only graph on five vertices with this property).