

Assignment #11 Solutions

Page 331, 38.2 Factor the following positive integers into primes.

- a. $25 = 5^2$.
- b. $4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7$.
- c. $10^{10} = 2^{10} \cdot 5^{10}$.
- d. $19 = 19$.
- e. $1 = 1$.

Page 331, 38.4 Suppose a is a positive integer and p is a prime. Prove that $p|a$ if and only if the prime factorization of a contains p .

- (i) Suppose that $p|a$. We will show that the prime factorization of a contains p .
Let $a = q_1 q_2 \dots q_t$ be the prime factorization of a (where it may be the case that $q_i = q_j$ for some $1 \leq i, j \leq t$). Then $p|(q_1 q_2 \dots q_t)$, so $p = q_i$ for some $q \leq i \leq t$ by Lemma 38.3. Thus, the prime factorization of a contains p .
- (ii) Suppose that the prime factorization of a contains p . We will show that $p|a$.
Since the prime factorization of a contains p , we can write $a = pb$ where b contains the rest of the prime factorization of a . Clearly $p|pb$, so $p|a$.

Page 332, 38.6 Prove Lemma 38.3 by induction (or Well-Ordering Principle) using Lemma 38.2.

We will proceed by induction on the number of factors, t . The basis case $t = 1$ is clear since p and q_1 are both primes and a prime has no factors except for 1 and itself. The basis case $t = 2$ follows from Lemma 38.2 since p, q_1 , and q_2 are all primes.

Suppose that the result holds for $t = k$ for some $k \geq 2$. We will prove the result for $t = k + 1$. Let $b = q_1 q_2 \dots q_k$. Then $q_1 q_2 \dots q_{k+1} = bq_{k+1}$. Suppose $p|bq_{k+1}$. Then by Lemma 38.2, either $p|b$ or $p|q_{k+1}$. If $p|b$, then $p = q_i$ for some $1 \leq i \leq k$ by the induction hypothesis. If $p|q_{k+1}$, then $p = q_{k+1}$ since both p and q_{k+1} are primes.

Page 332, 38.8 Let a and b be integers. A common multiple of a and b is an integer n for which $a|n$ and $b|n$. We call an integer m the least common multiple of a and b provided (1) m is positive, (2) m is a common multiple of a and b , and (3) if n is any other positive common multiple of a and b , then $n \geq m$. The notation for the least common multiple of a and b is $\text{lcm}(a, b)$. Please do the following:

- a. Develop a formula for the least common multiple of two positive integers in terms of their prime factorizations; your formula should be similar to the one in Theorem 38.5.
- b. Use your formula to show that if a and b are positive integers, then $ab = \text{gcd}(a, b) \text{lcm}(a, b)$.

a. Let $a = 2^{e_2} 3^{e_3} 5^{e_5} 7^{e_7} \dots$ and $b = 2^{f_2} 3^{f_3} 5^{f_5} 7^{f_7} \dots$. Define

$$m = 2^{\max\{e_2, f_2\}} 3^{\max\{e_3, f_3\}} 5^{\max\{e_5, f_5\}} 7^{\max\{e_7, f_7\}} \dots$$

We claim that $\text{lcm}(a, b) = m$. To verify this, we will check that m satisfies the three conditions in the definition of least common multiple.

- (1) m is a product of positive integers, so m is positive.
- (2) Note $\max\{e_2, f_2\} \geq e_2$ and $\max\{e_2, f_2\} \geq f_2$. Thus, $2^{e_2} | 2^{\max\{e_2, f_2\}}$ and $2^{f_2} | 2^{\max\{e_2, f_2\}}$. Similarly, for each prime p , $p^{e_p} | p^{\max\{e_p, f_p\}}$ and $p^{f_p} | p^{\max\{e_p, f_p\}}$. Thus, the prime factorization of m contains the prime factorization of a and so $a|m$. Similarly, the prime factorization of m contains the prime factorization of b and so $b|m$. Thus, m is a common multiple of a and b .
- (3) Suppose that n is any other positive common multiple of a and b . We want to show that $n \geq m$. Since $a|n$, we know that $2^{e_2} | n$. Since $b|n$, we know that $2^{f_2} | n$. Thus, the prime factorization of n must contain the factor 2 to at least the power $\max\{e_2, f_2\}$. Similarly the factorization of n must contain each prime p to at least the power $\max\{e_p, f_p\}$. That is, $m|n$ and so $n \geq m$.

Thus, we have proven our claim that $\text{lcm}(a, b) = m$.

b. Note that $\min\{e_p, f_p\} + \max\{e_p, f_p\} = e_p + f_p$ for every prime p . Thus,

$$\begin{aligned} \gcd(a, b) \text{lcm}(a, b) &= (2^{\min\{e_2, f_2\}} 3^{\min\{e_3, f_3\}} 5^{\min\{e_5, f_5\}} 7^{\min\{e_7, f_7\}} \dots) (2^{\max\{e_2, f_2\}} 3^{\max\{e_3, f_3\}} 5^{\max\{e_5, f_5\}} 7^{\max\{e_7, f_7\}} \dots) \\ &= 2^{\min\{e_2, f_2\} + \max\{e_2, f_2\}} 3^{\min\{e_3, f_3\} + \max\{e_3, f_3\}} 5^{\min\{e_5, f_5\} + \max\{e_5, f_5\}} 7^{\min\{e_7, f_7\} + \max\{e_7, f_7\}} \dots \\ &= 2^{e_2 + f_2} 3^{e_3 + f_3} 5^{e_5 + f_5} 7^{e_7 + f_7} \dots \\ &= (2^{e_2} 3^{e_3} 5^{e_5} 7^{e_7} \dots) (2^{f_2} 3^{f_3} 5^{f_5} 7^{f_7} \dots) \\ &= ab. \end{aligned}$$

Page 454, 53.1 Let P be the poset in the figure. For each pair of elements x, y listed below, determine whether $x < y$, $y < x$, or x and y are incomparable.

- a. a and b are incomparable.
- b. $a < c$.
- c. c and g are incomparable.
- d. $b < h$.
- e. $c < i$.
- f. $h > d$.

Page 454, 53.2 For the previous problem, please find the following:

- a. The height of the poset and a chain of largest size.

- b. The width of the poset and an antichain of largest size.
 - c. A chain containing three elements that cannot be extended to a larger chain.
 - d. A chain containing two elements that cannot be extended to a larger chain.
 - e. An antichain containing three elements that cannot be extended to a larger antichain.
- a. A chain of largest size is $\{a, c, f, h\}$, so the height of the poset is 4.
 - b. An antichain of largest size is $\{c, d, e, g\}$, so the width of the poset is 4.
 - c. The only chain with this property is $\{a, e, h\}$.
 - d. The only chain with this property is $\{g, j\}$.
 - e. One antichain with this property is $\{h, i, j\}$.

Page 455, 53.9 Which of the various properties of relations does *is comparable to* exhibit? That is, determine (with proof) whether or not it is always reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

Let $P = (X, \preceq)$. Let $x, y \in X$.

1. By Definition 53.1, we always have $x \preceq x$, so x is comparable to x for every $x \in X$. Thus, *is comparable to* is reflexive and so not irreflexive.
2. If x is comparable to y , then we cannot write both $x \not\preceq y$ and $y \not\preceq x$. Thus, we cannot write both $y \not\preceq x$ and $x \not\preceq y$ and so y is comparable to x . Thus, *is comparable to* is symmetric.
3. *is comparable to* need not be antisymmetric, since for example 1 is comparable to 2 and 2 is comparable to 1 in the poset (\mathbb{N}, \leq) , yet $1 \neq 2$.
4. *is comparable to* need not be transitive. For example, consider the poset $(\{a, b, c\}, \{(a, a), (a, b), (a, c), (b, b), (c, c)\})$. Here, b is comparable to a and a is comparable to c , yet b is not comparable to c .

Page 455, 53.10 Which of the various properties of relations does *is incomparable to* exhibit? That is, determine (with proof) whether or not it is always reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

Let $P = (X, \preceq)$. Let $x, y \in X$.

1. By Definition 53.1, we always have $x \preceq x$, so x is comparable to x for every $x \in X$. Thus, *is incomparable to* is irreflexive and so not reflexive.
2. If x is incomparable to y , then we can write both $x \not\preceq y$ and $y \not\preceq x$. Thus, we can write both $y \not\preceq x$ and $x \not\preceq y$ and so y is incomparable to x . Thus, *is incomparable to* is symmetric.
3. *is incomparable to* need not be antisymmetric, since for example a is incomparable to b and b is incomparable to a in the poset $(\{a, b\}, \{(a, a), (b, b)\})$, yet $a \neq b$.

4. *is comparable* to need not be transitive. For example, consider the poset $(\{a, b, c\}, \{(a, a), (a, c), (b, b), (c, c)\})$. Here, a is incomparable to b and b is incomparable to c , yet a is not incomparable to c .

Page 457, 54.2 For each of the following partially ordered sets, determine which elements are maximum, maximal, minimum, and minimal.

- b. The integers $\{1, 2, 3, 4, 5\}$ ordered by divisibility, $|$.
- d. Let $X = \{n \in \mathbb{Z} : n \geq 2\}$. Let $P = (X, |)$; that is, P is the poset of all integers that are greater than 1, ordered by divisibility.
- b. The elements 2 and 3 and 5 are incomparable, 1 lies below 2, and 2 lies below 4, so there is no maximum element. The elements 3, 4, and 5 are maximal. The element 1 is the only minimum element. The element 1 is the only minimal element.
- d. For any element x , the element $2x$ lies above x , so there is no maximum element. For the same reason, there is no maximal element. The primes are incomparable and every other element lies above some prime, so there is no minimum element. No element lies below any prime, so the primes are the minimal elements (and since every composite lies above some prime, no composite is minimal).

Page 457, 54.3 Find a poset that has neither a maximum nor a minimum element.

The most natural example of this is the poset (\mathbb{Z}, \leq) . For any $x \in \mathbb{Z}$, x is neither maximum nor minimum since $x + 1$ lies above it and $x - 1$ lies below it.

A finite example is the poset $(\{a, b\}, \{(a, a), (b, b)\})$. If there were an element that were a maximum or a minimum, it would have to be comparable with every other element of the poset. But the element b is comparable only with itself and so no such element exists.

Page 457, 54.4 Prove or disprove each of the following statements.

- a. If a poset has a maximum element, then it must be unique.
- b. It is possible for a poset to have an element that is both maximum and minimum.
- c. It is possible for a poset to have an element that is both maximal and minimal but is neither maximum nor minimum.
- d. If a poset has exactly one maximal element, then it must be a maximum.
- e. If x is a minimal element in a poset and y is a maximal element in a poset, then $x \leq y$.
- f. If x and y are incomparable, then neither is a minimum.
- g. Distinct (i.e., unequal) maximal elements must be incomparable.
- a. True. Suppose that a and b were both maximum elements of the poset (X, \preceq) . Then $a \preceq b$ since b is a maximum and $b \preceq a$ since a is a maximum. Then, since \preceq is antisymmetric, we have $a = b$. So, there can only be one maximum element.

- b. True. For example, the element x is both a maximum and a minimum in the poset $(\{x\}, \{(x, x)\})$.
- c. True. For example, the element x is both maximal and minimal in the poset $(\{x, y\}, \{(x, x), (y, y)\})$, yet it is neither a maximum nor a minimum since y lies neither below it nor above it.
- d. False. Consider the poset $(\mathbb{Z} \cup \{x\}, \{(a, b) : a, b \in \mathbb{Z} \text{ and } a \leq b, \text{ or } (a, b) = (x, x)\})$; that is, the poset (\mathbb{Z}, \leq) with the added element x that is comparable only to itself. In this poset an integer $n \in \mathbb{Z}$ is not maximal since $n + 1$ lies above n . So, the only maximal element is x . However, x is not a maximum since x does not lie above 0, for example.
- e. False. Note that the element x is maximal and y is minimal in the poset $(\{x, y\}, \{(x, x), (y, y)\})$, yet x and y are not even comparable.
- f. True. If x were a minimum, then y would be above x , which it is not since x and y are not comparable. Similarly, if y were a minimum, then x would be above y , which it is not since x and y are not comparable.
- g. True. Suppose that x and y are distinct maximal elements. Assume for the sake of contradiction that x and y are not incomparable. Then either $x < y$ or $y < x$ (we know that $x \neq y$ since they are distinct). If $x < y$, then x is not maximal. If $y < x$, then y is not maximal. This contradicts the fact that x and y are both maximal. Thus, x and y are incomparable.