Math 2001: Final Exam Review Sheet

The final exam will be cumulative. Roughly one quarter will be from material covered on the first exam (sections 1–9), one quarter will be on material covered on the second exam (sections 10–11, 13–14, 16, 19–21, and 23–24), and one half will be on the material covered since the second exam (sections 25, 29–32, 34–36, 38, 46–49, and 53–55). In addition to what's mentioned below, you should also look at the review sheets that Dr. Monk provided for the first two exams; here we will just list the new things.

Definitions: Be able to reproduce the definitions of the following terms as defined in the book. Your definition doesn't have to be the same word-for-word, but it must be logically equivalent.

- composition of functions
- identity function
- sample space (both the set and the probability function)
- equally likely outcomes
- event, and the probability of an event
- conditional probability of an event given another event
- independence of events
- random variable
- independent random variables
- the quotient q and the remainder r in the expression $a = qb + r$
- congruence modulo n (this was actually from Section 14, but is important here too)
- common divisor and greatest common divisor of two integers
- relatively prime integers
- modular addition, multiplication, subtraction, reciprocals, and division
- graph
- the adjacency relation
- the degree of a vertex
- subgraph, $G - v$, $G - e$, induced subgraph
- clique and clique number, independent set and independence number
- complement of a graph
- walk and path

- the is-connected-to relation
- component, connected graph
- cut vertex and cut edge
- cycle
- forest, tree, and leaf
- partially ordered set (poset) (both the set and the relation on it)
- comparable and incomparable elements
- chains and antichains, height and width of posets
- maximum, minimum, maximal, minimal
- linear order
- isomorphism of posets

Notation: Recognize what each of the following mean:

- $\sum_{a \in A} P(a)$
- $P(A|B)$
- $\gcd(a, b)$
- $\mathbb{Z}_n$, the integers modulo n
- $a \oplus b$, $a \ominus b$, $a \otimes b$, $a \oslash b$
- $d(v)$, the degree of a vertex of a graph
- $\overline{G}$, the complement of a graph
- K_n, the complete graph on n vertices

Formulas: Understand the following formulas. Know the conditions under which they are true and be able to apply them to solve problems.

- If (S, P) is a sample space, then $P(x \geq 0) \, \forall x \in S$ and $\sum_{s \in S} P(s) = 1$.
- If A is an event, then $P(A) = \sum_{a \in A} P(a)$.
- For any two events A and B, we have $P(A) + P(B) = P(A \cup B) + P(A \cap B)$.
- $P(\emptyset) = 0$ and $P(S) = 1$.
- $P(\overline{A}) = 1 - P(A)$.
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) \neq 0$.

- If X and Y are independent random variables, then
$$P(X = a \text{ and } Y = b) = P(X = a)P(Y = b).$$
- For integers a and b, not both zero, we can write $\gcd(a, b) = ax + by$ for some integers x and y.
- Modular arithmetic satisfies commutativity of addition and multiplication and associativity of addition and multiplication. However, it is not true that $a \otimes b = 0$ implies necessarily that $a = 0$ or $b = 0$, or more generally that $a \otimes b = a \otimes c$ implies that $a = 0$ or $b = c$.
- The Fundamental Theorem of Arithmetic.
- If $G = (V, E)$ is a graph, then $\sum_{v \in V} d(v) = 2|E|$.
- The clique number of G is the independence number of $\overline{G}$ and the independence number of G is the clique number of $\overline{G}$.

Concepts and techniques:

- Be able to prove that two functions are equal (proof template 22).
- Be able to describe a probabilistic situation in terms of a sample space and know how to rewrite a sample space with unequal probabilities as one with equal probabilities (for example, the sample space of the sum of two dice has unequal probabilities, but the sample space of the ordered pairs of die rolls has equal probabilities).
- Be able to work with set formulas like $A = (A \cap B) \cup (A \cap \overline{B})$ to compute event probabilities.
- Be able to compute the probability of an event in a sample space where outcomes are equally likely (by using the counting principles that you've covered earlier).
- Be able to express random variables as events.
- Be able to use Euclid's algorithm to compute the greatest common divisor of any two integers a and b and also to find the integers x and y such that $\gcd(a, b) = ax + by$.
- Be able to compute any operations in modular arithmetic and know how to use shortcuts to compute things like $2^{83} \bmod 9$.
- Know how to determine whether an element in $\mathbb{Z}_n$ has a reciprocal and to find it if it exists.
- Know the relation between the prime factorization of two integers and their greatest common divisor.
- Be able to find examples of graphs and subgraphs with various properties (and keep empty graphs, edgeless graphs, and complete graphs in mind as "extreme cases" sometimes good for counterexamples).
- The is-connected-to relation is an equivalence relation.

- Know the various equivalence conditions for a graph being a tree and how to use them to prove that a graph is a tree (see the Recap on p. 419 of your text).
- Be able to prove theorems about trees by leaf deletion (proof template 25).
- Know that every finite nonempty poset has maximal and minimal elements.
- Be able to use isomorphisms to prove things about isomorphic posets.

Practice: Make sure you can do all the homework problems that you have been assigned. Try other problems in those sections and in the Self Tests at the end of the chapters for practice. Also, consider the problems below for more practice:

- Chapter 5 Self Test: 9, 11, 13
- Chapter 6 Self Test: 1–12
- Chapter 7 Self Test: 1, 3–6, 10–11
- Chapter 9 Self Test: 1–3, 5, 7, 9
- Chapter 10 Self Test: 1–4. Also consider 55.3–55.5 on p. 461.