

1. (20)

- (i) Find all abelian groups of order $96 = 3 \times 2^5$, up to isomorphism. (Points will be deducted for repeated entries!)

- (ii) Is the group $S_4 \times V_4$ isomorphic to one of the groups in (i)? If so, identify which one; if not, explain why not.

2. (30) There is a group $T = \{g_1, g_2, \dots, g_{12}\}$ of order 12, and the following table gives the order of each element of T .

element	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
order	1	2	3	3	4	4	4	4	4	4	6	6

(i) Prove that T is not isomorphic to $\mathbb{Z}_{12}$.

(ii) Prove that T is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$.

(iii) Use your answers to (i) and (ii) to prove that T is nonabelian.

(iv) Prove that T is not isomorphic to the dihedral group D_6 of order 12. [Hint: how many elements of order 4 does D_6 have?]

(v) Prove that T is not isomorphic to the alternating group A_4 .

(vi) Prove that T is not a simple group.

3. (30) Let D_6 be the dihedral group of order 12 considered as symmetries of a hexagon, generated by a reflection μ and a rotation ρ by 60° . Let $V_4 = \{e, a, b, c\}$ be the Klein four-group, with identity e .

You may assume without proof that there exists a homomorphism of groups $\psi : D_6 \rightarrow V_4$ satisfying $\psi(\mu) = a$ and $\psi(\rho) = b$.

(i) Prove that ψ is surjective.

(ii) Let G and H be groups and let $\phi : G \rightarrow H$ be a homomorphism of groups. State the Fundamental Homomorphism Theorem (First Isomorphism Theorem) as it applies to this situation.

(iii) Deduce that $\ker(\psi)$ has order 3, and prove that $\ker(\psi) = \{e, \rho^2, \rho^4\}$.

(iv) Prove that $N = \langle \rho^2 \rangle$ is a normal subgroup of D_6 , and that D_6/N is abelian.

(v) Prove that N is the commutator subgroup, $C(D_6)$, of D_6 .

(vi) Prove that the center, $Z(D_6)$, of D_6 is different from the commutator subgroup, $C(D_6)$. [You may use standard results about centers of dihedral groups, if you state them clearly.]

4. (20) True or False. Mark with a “T” or an “F,” and provide a brief explanation (a couple of lines), for each part.

(i) _____ Every subgroup of order 1 is normal.

(ii) _____ Every subgroup of order 2 is normal.

(iii) _____ In the group $\mathbb{Q}/\mathbb{Z}$, the element $\frac{1}{3} + \mathbb{Z}$ has infinite order.

(iv) _____ The symmetric group S_{12} has a subgroup isomorphic to the group T of Question 2.

(v) _____ The subgroup $H = \langle (1\ 2\ 3) \rangle = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ of A_5 is the center, $Z(A_5)$, of A_5 .

(vi) _____ The subgroup $H = \langle (1\ 2\ 3) \rangle = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ of A_5 is the center, $Z(H)$, of H .

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Mathematics 3140: Second In-Class Exam

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Problem	Points	Score
1	20	
2	30	
3	30	
4	20	
Total	100	