1. (40) Let $D_6 = \langle \mu, \rho \rangle$ be the dihedral group of order 12 consisting of the symmetries of a regular hexagon, where μ is a reflection and ρ is rotation by $60°$ counterclockwise. You may assume without proof that the center of D_6 is given by $Z(D_6) = \{\iota, \rho^3\}$ (where ι is the identity) and that D_6 has a subgroup $H \cong D_3 \cong S_3$ whose elements are given by

$$H = \{\iota, \rho^2, \rho^4, \mu, \mu\rho^2, \mu\rho^4\}.$$

(i) Explain why we have $Z(D_6) \trianglelefteq D_6$, and determine whether or not we have $H \trianglelefteq D_6$.

(ii) State the Second Isomorphism Theorem for groups, which applies to a group G and its subgroups $H \leq G$ and $N \trianglelefteq G$.

(iii) Now let $G = D_6$, $N = Z(D_6)$, and let H be the subgroup of D_6 given earlier. Apply the Second Isomorphism Theorem to prove that $D_6/Z(D_6)$ is nonabelian.

(iv) Prove that the center of D_6 is not the same as the commutator subgroup of D_6.

(v) (**For 10 points extra credit** up to a maximum of 200 total:) Find the commutator subgroup of D_6, justifying your answer. (Recall that $\rho\mu = \mu\rho^{-1}$.)

2. (40) Let T be the subset of $\mathbb{Q}$ consisting of those fractions whose denominator is odd when written in lowest terms. (For example, $-4/10 = -2/5$ is an element of T, but $7/10$ is not an element of T.)

(i) Prove that T is a subring of $\mathbb{Q}$.

(ii) Is T an ideal of $\mathbb{Q}$? Why or why not?

(iii) Prove that T is an integral domain.

(iv) Prove that $\mathbb{Q}$ is the field of fractions of T.

3. (40) Let T be the ring defined in Question 2, and let $\mathbb{Z}_2$ be the ring $\mathbb{Z}/2\mathbb{Z}$.

(i) Prove that the map $\psi : T \to \mathbb{Z}_2$ given by

$$\psi(a/b) = a \mod 2$$

is a homomorphism of rings, where a/b is written in lowest terms and $b \geq 0$.

(ii) Find the kernel and image of ψ.

(iii) State the First Isomorphism Theorem for rings.

(iv) Let I be the set of all fractions a/b in lowest terms such that a is even and b is odd. Prove that I is an ideal of T.

(v) Determine whether or not I is (a) maximal and/or (b) prime.

(vi) Prove that an element $r \in T$ is a unit if and only if $r \notin I$.

4. (40) Let $f(x) = x^4 - x^2 - 2x - 1$. Find a factorization of $f(x)$ into irreducible polynomials in each of the following rings, justifying your answers briefly:

(i) $\mathbb{Z}_2[x]$ (this can be done without much calculation);

(ii) $\mathbb{Z}_5[x]$;

(iii) $\mathbb{Q}[x]$ (hint: consider factorizations over $\mathbb{Z}$);

(iv) $\mathbb{R}[x]$;

(v) $\mathbb{C}[x]$.

5. (40)

(i) Show that the polynomial $x^3 + x^2 + 1$ is irreducible in $\mathbb{Z}_2[x]$.

(ii) Show that the principal ideal $\langle x^3 + x^2 + 1 \rangle$ of $\mathbb{Z}_2[x]$ (that is, the set of all polynomials $\{(x^3+x^2+1)f(x) : f(x) \in \mathbb{Z}_2[x]\}$) is a maximal ideal of $\mathbb{Z}_2[x]$. (You may assume that this is indeed an ideal.)

(iii) Show that any polynomial in $f(x) \in \mathbb{Z}_2[x]$ can be written in the form

$$f(x) = (x^3 + x^2 + 1)q(x) + (a + bx + cx^2)$$

for some $q(x) \in \mathbb{Z}_2[x]$ and $a, b, c \in \mathbb{Z}_2$.

(iv) Let $I = \langle x^3 + x^2 + 1 \rangle$ be the ideal in part (ii). Show that $E = \mathbb{Z}_2[x]/I$ is a field, and that every element of E may be expressed *uniquely* in the form as $a + bx + cx^2 + I$, where $a, b, c \in \mathbb{Z}_2$.

(v) Show that E has 8 elements, and find the isomorphism type of the group of units, E^*, of E.

(vi) Show that every nonzero element $\gamma \in E$ satisfies $\gamma^7 = 1$, and that every element $\gamma \in E$ satisfies $\gamma^8 = \gamma$. (This does not require any difficult calculations.)

Name: ______________________________

University of Colorado

Mathematics 3140: Final Exam

July 5, 2024

Problem	Points	Score
1	40	
2	40	
3	40	
4	40	
5	40	
Total	200	