1. (30) Let $U = \{e^{i\theta} : \theta \in \mathbb{R}\}$ be the group of complex numbers on the unit circle under complex multiplication, and let $f : U \to U$ be the map given by $f(z) = z^2$. (You may assume without proof that U is a group.)

(i) Prove that f is a homomorphism of groups. (You may use standard properties of complex numbers without proof.)

(ii) Find the kernel of f. (Don't just write down the definition. You need to describe an explicit subset of U.)

(iii) Show that $i = e^{i\pi/2}$ lies in the image (or "range") of f, and describe the image of f as an explicit subset of U.

(iv) Let G and H be groups and let $\phi : G \to H$ be a homomorphism of groups. State the Fundamental Homomorphism Theorem (First Isomorphism Theorem) as it applies to this situation.

(v) Use the earlier parts of the question to find an example of a group G and a normal subgroup $N \trianglelefteq G$ such that $N \neq \{e\}$ and G/N is isomorphic to G.

(vi) Give an example of a finite group G and a normal subgroup N of G such that $N \neq \{e\}$ and G/N is isomorphic to G, or explain why no such example exists.

2. (20) Find all abelian groups of order 400, up to isomorphism. (Points will be deducted for repeated entries!)

3. (30) The symmetric group S_4 has precisely four normal subgroups: S_4 itself, the alternating group A_4, the trivial subgroup $\{e\}$, and a subgroup

$$V_4 = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\}$$

of order 4.

(i) Let $x = (12)$ and $y = (23)$. Prove that the commutator $[x, y] = xyx^{-1}y^{-1}$ has order 3.

(ii) Using your answer to (i), prove that the commutator subgroup of S_4 cannot be V_4 or $\{e\}$.

(iii) Prove that the commutator subgroup of S_4 is A_4.

(iv) Deduce from (iii) that the quotient group S_4/V_4 is nonabelian. To which familiar group is S_4/V_4 isomorphic?

(v) Explain why the center $Z(G)$ of a group G is always abelian.

(vi) Deduce from (v) that $Z(S_4)$ must be either V_4 or $\{e\}$.

(vii) Prove that V_4 is not the center of S_4, and deduce that $Z(S_4) = \{e\}$.

4. (20) True or False. Mark with a "T" or an "F," and provide a brief explanation (a couple of lines), for each part.

(i) ______________The alternating group A_5 has subgroups other than the trivial subgroup $\{e\}$ and A_5 itself.

(ii) ______________The alternating group A_5 has a subgroup of order 30.

(iii) ______________The group of integers $\mathbb{Z}$ has a subgroup of order 4.

(iv) _______________ The group of integers $\mathbb{Z}$ has a subgroup of index 4.

(v) _______________ The permutation $(1\ 2\ 3\ 4)(5\ 6)(7\ 8\ 9)$ in S_9 has order 12.

(vi) _______________ The permutation $(1\ 2\ 3\ 4)(5\ 6)(7\ 8\ 9)$ in S_9 is odd.

Name: ______________________________

University of Colorado

Mathematics 3140: Second In-Class Exam

June 21, 2024

Problem	Points	Score
1	30	
2	20	
3	30	
4	20	
Total	100	