1. (40) Define an operation $*$ on the set of real numbers $\mathbb{R}$ by

$$a * b := \sqrt[3]{a^3 + b^3} = (a^3 + b^3)^{1/3}.$$

(i) Show that $*$ is a binary operation on the set of real numbers, and that $G = (\mathbb{R}, *)$ is an abelian group.

(ii) Show that the set

$$C = \{x^{1/3} : x \in \mathbb{Z}\} = \{\ldots,\ \sqrt[3]{-2},\ \sqrt[3]{-1},\ \sqrt[3]{0},\ \sqrt[3]{1},\ \sqrt[3]{2},\ \ldots\}$$

of all cube roots of integers forms a subgroup of G.

(iii) Show that the set $\mathbb{Z}$ of integers does not form a subgroup of G.

(iv) Prove that G is not cyclic.

2. (20) Let $G = (\mathbb{R}, *)$ be the group of Question 1, and let $(\mathbb{R}, +)$ be the group of real numbers under the usual operation of addition. Let $\phi : (\mathbb{R}, *) \to (\mathbb{R}, +)$ be given by

$$\phi(a) = a^3.$$

(i) Show that $\phi : G \to \mathbb{R}$ is an isomorphism of groups.

(ii) Show that the only finite subgroup of G is the trivial subgroup.

3. (20)

(i) Find the order of the symmetric group S_4, and show that S_4 has a nonabelian subgroup of order 12.

(ii) Give an example of an abelian group of order 8 and a nonabelian group of order 8.

(iii) Show that the symmetric group S_8 has an abelian subgroup of order 8 and a nonabelian subgroup of order 8. (You are not required to find either of the subgroups.)

4. (20) True or False. Mark with a "T" or an "F," and provide a brief explanation (a couple of lines), for each part.

(i) ______________The group $GL(3, \mathbb{R})$ (which consists of 3×3 invertible matrices over $\mathbb{R}$ under matrix multiplication) is nonabelian.

(ii) ______________The group $GL(3, \mathbb{R})$ is cyclic.

(iii) ______________The two-element set $S = \{+1, -1\} \subset \mathbb{Z}$ is a group under the usual addition operation $+$.

(iv) ______________The two-element set $S = \{+1, -1\} \subset \mathbb{Z}$ is a group under the usual division operation $\div$.

(v) ______________The cycle (1 2 3 4 5 6) is an element of the alternating group A_6.

(vi) ______________There exists an infinite group G that has a finite subgroup of every possible order.

Name: ______________________________

University of Colorado

Mathematics 3140: First In-Class Exam

June 11, 2024

Problem	Points	Score
1	40	
2	20	
3	20	
4	20	
Total	100	