MATH 3110: Extra questions

1. Let M and N be coprime numbers. Prove that every divisor k of MN can uniquely be written in the form $k = k_1 k_2$, where $k_1 | M$ and $k_2 | N$.

2.
   (i) Prove that if the equation $x^2 + y^2 = 3z^2$ has a solution in integers x, y, z, then each of the numbers x, y, and z must be a multiple of 3.
   (ii) Prove that the only solution in integers to the equation $x^2 + y^2 = 3z^2$ is $x = y = z = 0$.
   (iii) Deduce that there are no rational numbers a and b with the property that $a^2 + b^2 = 3$.

3. Prove that the number of ways to tile a $2 \times n$ rectangular board with 2×1 dominoes is F_{n+1}, the $(n+1)$-st Fibonacci number. (Recall that $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$.)

4. Denote the digits in base twelve by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, and B. Design base twelve tests for divisibility by (i) two, (ii) three, (iii) four, (iv) six, (v) eight, (vi) nine, (vii) eleven, and (viii) thirteen.

5. Let N be a Mersenne prime strictly bigger than 7. Prove that, in base twelve, the last two digits of N are either 27 or A7.

6. Let $\chi : \mathbb{N} \to \mathbb{Z}$ be the function given by

$$\chi(n) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \equiv 1 \mod 4; \\ -1 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

(i) Prove that χ is multiplicative.

(ii) You may assume without proof that the function $g(n) := \sum_{d|n} \chi(d)$ is also multiplicative. Prove that we have $g(n) = \tau_1(n) - \tau_3(n)$, where $\tau_i(n)$ is the number of divisors of n that are congruent to i modulo 4.

(iii) Prove that if p and q are primes with $p \equiv 1 \mod 4$ and $q \equiv 3 \mod 4$, then we have $g(2^e) = 0$, $g(p^e) = e + 1$, and

$$g(q^e) = \begin{cases} 1 & \text{if } e \text{ is even, and} \\ 0 & \text{if } e \text{ is odd.} \end{cases}$$

(iv) Suppose that $n \in \mathbb{N}$ has the prime factorization

$$n = 2^e \prod_i p_i^{e_i} \prod_j q_j^{f_j},$$

where $p_i \equiv 1 \mod 4$ for all i and $q_j \equiv 3 \mod 4$ for all j. Prove that

$$g(n) = \begin{cases} \prod_i (e_i + 1) & \text{if } f_j \text{ is even for all } j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$