1. (30) Let A, X, and B be sets, and assume that $A \subseteq B$. Prove that

$$A \cup (X \cap B) = (A \cup X) \cap B.$$

[A Venn diagram is **not** an acceptable proof.]

2. (25) Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers, not including zero. Let

$$X = \{2^{k-1} : k \in \mathbb{N}\} = \{1, 2, 4, 8, \dots\}$$

be the subset of $\mathbb{N}$ consisting of powers of 2, and let

$$Y = \{2n - 1 : n \in \mathbb{N}\} = \{1, 3, 5, 7, \dots\}$$

be the set of positive odd numbers.

Using "strong" induction, or otherwise, prove that every natural number $n = \mathbb{N}$ can be written as the product of an element of X with an element of Y. (For example, if $n = 60$, we can take $x = 4$ and $y = 15$.) In other words, prove that

$$\forall n \in \mathbb{N}, \exists x \in X, \exists y \in Y, n = xy.$$

[In this question, you may assume without proof that every natural number is either even or odd, but never both.]

3. (25) Let $B = \{a, b, c\}$ and $C = \{1, 2, 3, 4, 5, 6, 7, 8\}$. In this question, you may leave your answers as formulas, but you should justify your answers. The notation $\mathcal{P}(B)$ refers to the power set of B.

(i) How many functions are there from B to C?

(ii) How many injective functions are there from B to C? (The book calls injective functions "one-to-one".)

(iii) How many surjective functions are there from B to C? (The book calls surjective functions "onto".)

(iv) How many bijective functions are there from $\mathcal{P}(B)$ to C?

4. (30) Recall that if a and b are integers, the statement "a divides b" (written $a|b$) means that there exists an integer k for which $b = ka$. Let

$$F = \{-42, -21, -14, -7, -6, -3, -2, -1, 1, 2, 3, 6, 7, 14, 21, 42\}$$

be the set of integer divisors of 42, and let

$$D = \{1, 2, 3, 6, 7, 14, 21, 42\}$$

be the set of positive divisors of 42.

(i) Show that the relation $|$ (divides) is not a partial order on F.

(ii) Draw the Hasse diagram of the partially ordered set $(D, |)$. [You may assume without proof that $|$ is a partial order on D.]

(iii) Find two different antichains in $(D, |)$, each having 3 elements.

(iv) Find six different chains in $(D, |)$, each having 4 elements.

(v) Determine whether or not the partial order $|$ on D is a total order, explaining your answer. [“Total order” and “linear order” mean the same thing.]

5. (40) A bag contains 10 balls. These balls are identical, except that they are labeled with the integers 1 up to 10. Three balls are drawn one at a time from the bag, with replacement. (This means that once a ball is drawn, it is tossed back into the bag, where it is hopelessly mixed up with the balls still in the bag. Then the next ball is drawn, tossed back in, and so on.) You should leave your answers to this question as fractions or decimals (not formulae).

(i) Describe the sample space, (S, P), associated with this experiment. (Your answer should make it clear what a typical outcome is, and what the probability of each outcome is.)

(ii) Calculate the probability that all three balls are labeled 1.

(iii) Calculate the probability that all three balls have the same label. (The situation in (ii) is a special case of this.)

(iv) Calculate the probability that two of the balls are labeled 1, and the other is labeled 2 (not necessarily in that order).

(v) Calculate the probability that two of the balls have the same label, and the other has a different label. (The situation in (iv) is a special case of this.)

(vi) Calculate the probability that the balls are labeled 1, 2 and 3 (not necessarily in that order).

(vii) Calculate the probability that the balls all have different labels. (The situation in (vi) is a special case of this.)

(viii) Explain why the probabilities in (iii), (v) and (vii) should add up to 1.

Name: ______________________________

University of Colorado

Mathematics 2001: Final Exam

December 15, 2019

Problem	Points	Score
1	30	
2	25	
3	25	
4	30	
5	40	
Total	150	