Local Analysis of Deformation Quantization

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Zusammenfassung

In dieser Arbeit wird mit deformationstheoretischen Methoden der Begriff der Quantisierung untersucht. Zuerst werden dabei Deformationen nichtkommutativer geringter Räume über einem kommutativen lokal geringten Parameterraum ähnlich wie in der Algebraischen Geometrie und Komplexen Analysis definiert. Dann wird geklärt, was unter einer DIRAC Quantisierung eines kommutativen geringten Raums mit einer POISSON Struktur, also einem Raum klassischer Observablen, zu verstehen ist. Anschließend konstruieren wir auf dem POISSON Raum der polynomialen Observablen über einem Kotangentialbündel eine Quantisierung im obigen Sinne. Mit Hilfe eines vollständigen Symbolkalküls, der ebenfalls in der Arbeit erklärt wird, gelingt es, die Quantisierung polynomialer Observablen auf Kotangentialbündeln auszudehnen zu einer Quantisierung von Symbolen auf Kotangentialbündeln. Außerdem werden funktorielle Eigenschaften dieser sogenannten Normalordnungsquantisierung untersucht. Insgesamt wird somit gezeigt, daß ein deformationstheoretischer Zugang zur Quantisierung nicht nur im formalen sondern auch im konkreten Sinne möglich ist.

abstract

In this work we give a deformation theoretical approach to the problem of quantization. First the notion of a deformation of a noncommutative ringed space over a commutative locally ringed space is introduced similarly to the one in Algebraic Geometry or Complex Analysis. Then we clarify the notion of a DIRAC quantization of a commutative ringed space with a POISSON structure, i.e. of a space of classical observables. Afterwards a DIRAC quantization for the POISSON space of classical polynomial observables on a cotangent bundle is constructed. By using a complete symbol calculus which is also introduced in this work we succeed in extending the above quantization of polynomial observables to a quantization of a POISSON space of symbols over a cotangent bundle. Furthermore we consider functorial properties of this normal order quantization. Altogether it is shown that a deformation theoretical approach to quantization is possible not only in a formal sense but also such that the deformation parameter $\hbar$ can attain any real value.

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Introduction

One of the major problems in mathematical physics is to find an appropriate mathematical language for various quantization schemes. "In physics, the term 'quantization' refers to the process of passing from a classical-mechanical description of a physical system to a quantum-mechanical description of the same system" (see ALAN WEINSTEIN [57], p. 31). Classical systems like the harmonic oscillator or a free particle in EUCLIDEAN space can be quantized in a mathematical rigorous way, but a general and satisfactory scheme for quantizing all classical systems "living" on an arbitrary symplectic manifold is still lacking. So the "quantization problem" is not completely solved, though important partial results have been achieved.

Dirac's quantization scheme

After quantum theory was developped in the twentieth of this century by HEISENBERG, SCHRÖDINGER, BOHR and others, DIRAC [12] probably was the first who tried to establish a general quantization scheme, which usually is refered to as canonical quantization. Though DIRAC'S ideas are very convincing and fruitful from the point of view of physics, they are somewhat lacking of mathematical rigor. So a lot of attempts have been undertaken to make his ideas mathematically precise. This is also the goal of this work, where we will give a proposal how to dress canonical quantization in a rigorous mathematical language by using deformation theory. Before explaining our approach in more detail, let us describe DIRAC'S idea of canonical quantization.

As already mentioned above, a classical mechanical system is modelled on a symplectic manifold, the phase space. The smooth functions on the symplectic manifold serve as observables of the system. Now the space of these functions carries two important structures, first that one of a commutative algebra and second that one of a LIE algebra, where the bracket is given by the POISSON bracket on the symplectic manifold. Passing to the quantum setting, the algebra of observables becomes a noncommutative algebra acting on a certain HILBERT space, which is interpreted as the space of pure states. It is assumed that to every classical observable there corresponds a quantized one such that selfadjoint, i. e. realvalued, classical observables are mapped to selfadjoint operators. According to DIRAC the algebra structure of the space of quantized observables should be determined by the following two conditions:

(*i*) The commutator of two quantized observables is given by the product of $\hbar/i$ and the POISSON bracket of its classical counterparts:

$$[\mathfrak{q}(a), \mathfrak{q}(b)] = \frac{\hbar}{i} \mathfrak{q}(\{a, b\}). \tag{0.1}$$

Here $\hbar$ is PLANCK'S constant, a, b are classical observables and $\mathfrak{q}$ is the quantization map.

(*ii*) The quantization map $\mathfrak{q}$ preserves the unit element:

$$\mathfrak{q}(1) = 1. \tag{0.2}$$

In other words this means there should be a LIE algebra morphism of the space of classical observables to the LIE algebra of quantum observables, where the bracket operation for quantum observables is given by the commutator times $\hbar/i$. The last assumption in DIRAC'S canonical quantization scheme is that the quantized observables have a common admissable domain in the HILBERT space of quantum states and that the action of the algebra of quantized observables is irreducible. Now GROENEWOLD [25] and VAN HOVE [55] have shown the remarkable result that there is no quantization scheme in the above sense, if one wants to quantize all observables. See also ABRAHAM, MARSDEN [1] as well as GUILLEMIN, STERNBERG [26] for a proof and discussion of this fact.

Geometric and formal deformation quantization

By relaxing DIRAC'S original requirements, it is possible to circumvent the theorem of GROENEWOLD and VAN HOVE and set up a mathematically rigorous apparatus for the quantization of a classical physical system. Without going into details let us give two examples.

In geometric quantization by KOSTANT and SOURIAU (see for example WOODHOUSE [66]) one first constructs a certain HERMITIAN line bundle L with connection ∇ over M

for every quantizable symplectic manifold M, i.e. for every symplectic M with symplectic form ω such that $\hbar\omega$ is integral. The line bundle L gives rise to a HILBERT space $L^2(L)$ of square integrable sections of L on which a classical observable a acts as differential operator of the form $\mathfrak{q}(a) = \frac{\hbar}{i}\nabla_{X_a} + a$. Now the HILBERT space $L^2(L)$ is too large. By introduction of a so-called polarization on M one can cut down $L^2(L)$ to the "correct" HILBERT space $\mathcal{H}$ of square integrable sections of L which are covariantly constant along the leaves of the polarization. Unfortunately this implies that one also has to cut down the space of quantizable functions resp. quantizable classical observables to those whose associated differential operator $\frac{\hbar}{i}\nabla_{X_a} + a$ leaves $\mathcal{H}$ invariant. It may now happen that the space of quantizable functions on M is much too small. For example it might contain only the space of smooth functions linear in momentum and thus not contain essential functions like the HAMILTONIAN which often is quadratic in momentum. But we are definitely interested in the quantization of the HAMILTONIAN. Therefore we cannot consider geometric quantization as the perfect solution to the quantization problem, though geometric quantization gives a very beautiful approach and works very well in a lot of cases. See for example RAWNSLEY [41] or BLATTNER [7] for more information on the topic of geometric quantization and its relation to other quantization schemes.

In the second approach by BAYEN ET AL. [3], every classical observable can be quantized, but here PLANCK'S constant $\hbar$ which governs quantization is considered only as a formal parameter. More precisely BAYEN ET AL. [3] regard a quantization as a formal deformation with formal parameter $\hbar$ in the sense of GERSTENHABER [19, 20]. The space to be deformed is the algebra $\mathcal{C}^\infty(M)$ (resp. the sheaf $\mathcal{C}^\infty_M$) of smooth functions on a symplectic manifold M. The deformed algebra is given by the linear space $\mathcal{C}^\infty(M)[[\hbar]]$ of formal power series in the variable $\hbar$ with coefficients in $\mathcal{C}^\infty(M)$ together with a noncommutative product $\star$ on $\mathcal{C}^\infty(M)[[\hbar]]$:

$$\left(\sum_{k\in\mathbb{N}} a_k\hbar^k\right) \star \left(\sum_{l\in\mathbb{N}} b_l\hbar^l\right) = \sum_{n\in\mathbb{N}} \sum_{k+l+m=n} \alpha_m(a_k, b_l)\, \hbar^n. \tag{0.3}$$

Here the a_k, b_l are elements of $\mathcal{C}^\infty(M)$ and the bilinear mappings $\alpha_m : \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ have to be appropriately chosen, such that $\alpha_0(a, b) = ab$ and such that associativity is fulfilled. Furthermore the quantization map $\mathfrak{q}$ is given by the canonical embedding $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[[\hbar]]$, $a \mapsto a$, and it is assumed that DIRAC'S quantization condition $[\mathfrak{q}(a), \mathfrak{q}(b)] = \frac{\hbar}{i}\mathfrak{q}(\{a, b\})$ holds for every $a, b \in \mathcal{C}^\infty(M)$ up to first order in $\hbar$. In particular one wants to write

$$\mathfrak{q}(a) \star \mathfrak{q}(b) \;=\; ab + \frac{\hbar}{2i}\,\{a,b\} + \hbar^2\,\alpha_2(a,b) + \ldots + \hbar^n\,\alpha_n\,(a,b) + \ldots \tag{0.4}$$

for every $a, b \in \mathcal{C}^\infty(M)$. It is highly nontrivial to see that such formal deformations exist, but by a result of De Wilde and Lecomte [65] one knows that every symplectic manifold can be quantized resp. deformed in the formal sense. A geometric proof of this fact is given by Fedosov [17]. The major set-back of formal deformation quantization now is that Planck's constant is regarded only as a formal parameter, whereas experimental phycisists tell us that $\hbar$ has a certain measurable numerical value. Therefore the formal quantization scheme is inappropriate for certain aspects of quantization.

Though both of the two outlined approaches do not give a completely satisfactory answer to the quantization problem, they are nevertheless very useful for a better understanding of quantum theory and its mathematical description. In particular they have to be understood as important and essential steps on the tough way to the final and right mathematical language (which we all hope to exist) for quantization.

Concrete deformation quantization

In this work we will give a new proposal for a mathematically rigorous quantization scheme and try to make Dirac's ideas precise. The quantization scheme we introduce is based on deformation theory but differs from the formal approach in certain important aspects. As already mentioned Bayen et al. [3] use Planck's constant $\hbar$ only in a formal sense to give a quantization of the algebra $\mathcal{C}^\infty(M)$ of classical observables on a symplectic manifold M. Now I was interested in the question whether it is possible in their approach to give $\hbar$ a real or complex value such that the formal power series

$$\mathfrak{q}(a) \star \mathfrak{q}(b) \;=\; ab + \frac{\hbar}{2i}\,\{a,b\} + \hbar^2\,\alpha_2(a,b) + \ldots + \hbar^n\,\alpha_n\,(a,b) + \ldots \tag{0.5}$$

for the product of two quantized observables converges for every $a, b \in \mathcal{C}^\infty(M)$ and this particular value of $\hbar$. Unfortunately even in the flat case of $M = \mathbb{R}^{2n}$ with its canonical symplectic structure one can show that the only $\hbar$ fulfilling the above requirement is $\hbar = 0$. It even seems in general to be impossible to construct a nontrivial subalgebra of observables $a, b \in \mathcal{C}^\infty(M)$ for which the series in Eq. (0.5) does not have a vanishing convergence radius. The reason of this is that the bilinear mappings $\alpha_m : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ governing the product expansion in Eq. (0.5) are constructed by a complicated cohomological argument, which does not allow any considerations about the growth of the α_m. This suggests that one should consider another path for deformation quantization and leads us to our main question:

> Is it possible to find a quantization scheme using concrete deformations over real or complex parameter spaces (and not formal ones) such that the algebras of classical observables which can be quantized are large enough for purposes of theoretical physics and such that calculations can be carried out?

It is the main result of this work that we can give a positive answer to the above question. In particular we prove that a concrete deformation quantization of the algebra of classical polynomial observables on a cotangent bundle exists and also one of the algebra of classical observables coming from a symbol space. Hereby we use a formalism which is closer to the deformation theory in complex analysis resp. algebraic geometry than to the deformation theory of GERSTENHABER [19, 20].

Before explaining our approach in more detail let us mention that the problem to find a concrete resp. converging deformation quantization is also mentioned several times in the mathematical physics literature, for example in BLATTNER [7], RIEFFEL [45], VAISMAN [54] or WEINSTEIN [58]. Whereas BLATTNER, VAISMAN and WEINSTEIN do not further pursue this problem, RIEFFEL [45] tackles it by using a different approach to ours, namely the language of C^*-algebras.

Several difficulties arise when one tries to establish a quantization scheme built upon concrete deformations. As I intended to include local aspects in our quantization scheme, it was necessary to use the language of sheaves resp. ringed spaces for spaces of classical and quantized observables as well as for parameter spaces. With the help of this language it is no problem anymore to formulate what a family of algebras varying over a parameter space i.e. a deformation of an algebra should be; it is just a certain sheaf over the parameter space. But, as the regarded family of algebras in general does not comprise a bundle of finite dimensional vector spaces, it is not yet immediately clear how to express the requirement that the family of algebras is in a certain sense a "continuous" one. The notion of a flat morphism of algebras now gives the right tool for this. More precisely we require that the algebra morphisms between the stalks of the parameter space and those of the family of algebras are flat. In concrete models of our quantization scheme we therefore always have to prove that the flatness or some slightly weaker condition is fulfilled.

A second problem in a concrete deformation theory is that unlike in the case of formal deformations one does not have a natural quantization map which associates to any classical observable a quantized one. This suggests that a quantization is more than just a deformation and that one needs an appropriate definition of what a deformation quantization should be. In particular we had to clarify in the definition where the quantization map comes in and which properties it should have. In the context of C^*-algebras such a definition was given by MARC RIEFFEL in [42, 43, 44, 45] with his concept of a "strict deformation quantization". But the C^*-algebra setting turned out not to be applicable for our purposes, as our approach first uses sheaves and second at the best algebras with a FRÉCHET-$*$-structure but not a with C^*-structure. So a new notion of a deformation quantization appropriate for our needs had to be found. The outcome of these attempts is a definition of a deformation quantization which on a first glance looks quite complicated, but is capable to comprise smooth, holomorphic, real analytic or even formal deformations and quantizations in one language. Another point is that though the definition of a quantization is complicated, calculations in concrete models of our approach are a lot easier than for example in geometric quantization. Furthermore they are closer to the methods used in theoretical physics. Compare for example the quantization of the HAMILTONIAN of a free particle moving on a RIEMANNIAN manifold in section 2.4 with the one in WOODHOUSE, *Geometric Quantization* [66]. Finally we succeed in giving a definition of what a category of quantizations should be, which opens up a new way to study functorial properties of quantizations.

Contents

In chapter 1 we recall the definition of a (flat) deformation of a not necessarily commutative ringed space. The parameter spaces of those deformations are commutative locally ringed spaces. A quantization is then introduced as a deformation together with an appropriate family of isomorphisms which linearly map the ringed space to be deformed to the fibers of the deformation. In more physical terms do these isomorphisms associate to any classical observable one and only one quantized observable for any value of the deformation or quantization parameter. The quantization is called DIRAC, if the space to be quantized is a POISSON space and the commutator of two quantized variables is equal to the quantization of the POISSON bracket of the two observables up to higher orders of PLANCK'S constant which serves as deformation parameter.

As forming the product of two classical mechanical resp. quantum systems induces a natural monoidal structure on the corresponding category of state spaces, we will consider in section 1.3 the question, whether the categories of deformations and quantizations also have a product resp. a monoidal structure. We show the remarkable and new result that this is the case indeed for appropriate categories of deformation quantizations.

Our goal in the following chapter is to define a DIRAC quantization of the POISSON space of polynomial observables associated to the cotangent bundle T^*X of a RIEMANNIAN manifold X. In other words we want to construct a model for the abstract definition of a quantization given in the first chapter. The space of polynomial observables which thereby shall be quantized is large enough to contain all physically interesting observables. It is not so much a problem to find a deformation of the space of polynomial observables, though the construction of the deformation is somewhat technical, but more to find an appropriate quantization map. I now used the fact that X carries a natural connection which gives rise to normal coordinates on X. Then, roughly speaking, a classical polynomial observable is quantized by first expanding the observable over a point of X in normal coordinates of that point. Afterwards its momentum and spacial components are quantized separately in such a way that momentum observables - expressed in normal coordinates - are written to the right of the spacial observables. Therefore this quantization corresponds to what physicists would call a normal order quantization. After having constructed a DIRAC quantization for polynomial observables I could show that it even gives rise to a functor going from the category of RIEMANNIAN manifolds to differentiable DIRAC quantizations such that the natural monoidal structures on these categories are preserved. Though normal order quantization is well known in physics for $M = T^*\mathbb{R}^n = \mathbb{R}^{2n}$, it thus appears first in a curved, deformation theoretical and functorial context.

In a following section of the second chapter we explain how the deformed polynomial observables act in a natural way as differential operators on the space of smooth functions on X. With the help of this action we succeed in giving an easier formula for the quantization map. This fact is then used to calculate the quantizations of some classical physical systems, namely that one of the harmonic oscillator, a free particle moving on a RIEMANNIAN manifold and the Hydrogen atom.

In case X is a complex manifold we also define a deformation of the sheaf of polynomial observables over T^*X and then extend it in section 2.5 to a deformation of the ringed space $(X, \mathcal{D}_0^\infty)$ of holomorphic functions on T^*X which are of bounded growth. Now the sheaf $\mathcal{D}_0^\infty$ has the same underlying linear structure as the sheaf of infinite order differential operators on X, which is a well-known object in microlocal analysis (see BJÖRK [6]). Despite this fact I could not find a proof in the microlocal analysis literature for the nontrivial fact that $\mathcal{D}_0^\infty$ resp. the sheaf of infinite order differential operators on X is well-defined indeed. Therefore I included a proof of this and also showed that $\mathcal{D}_0^\infty$ can be flatly deformed, in particular that it can be flatly deformed to the sheaf of infinite order differential operators on X. To my knowledge this last result is new.

In chapter 3 we leave our main path and introduce the so-called complete symbol calculus on manifolds. To put it shortly, a symbol calculus associates to any differential or pseudo-differential operator A on a RIEMANNIAN manifold X a certain smooth function σ_A on the cotangent bundle T^*X, such that A is (almost) completely characterized by its symbol σ_A. In a certain way one can then regard the symbol σ_A as the classical observable and the operator A as its quantization. It is not too difficult to define a symbol calculus on EUCLIDEAN space $\mathbb{R}^n$, so one could try to set up a symbol calculus on a curved manifold by first defining it locally in charts. Unfortunately those locally defined symbols need not always fit together, so we have to find a global and chart independent definition of a complete symbol. Using ideas of WIDOM [63, 64] we succeed in giving such a definition which also provides the essential tools for our further quantization purposes.

In section 3.1 we will give an introduction to symbols over conic bundles. In the following section we then define a new notion of a FOURIER transform on RIEMANNIAN manifolds. This and the symbols of section 3.1 will give us the means to construct a canonical integral representation for pseudo-differential operators on manifolds.

From section 3.3 on ideas of WIDOM [63, 64] come into play. We thus construct the complete symbol of a pseudo-differential operator by applying it to the exponential of a so-called phase function. But we use a different phase function than WIDOM [63, 64] does. The reason is that because of our special choice of a phase function we can explicitely construct an inverse (up to smoothing symbols resp. operators) of the symbol map by using the integral formula of section 3.2.

In section 3.4 we derive an asymptotic expansion for the symbol of the product of two pseudo-differential operators. The results are analogous to the ones in WIDOM [63, 64], but we give different and complete proofs.

By means of the complete symbol calculus we first will be able in chapter 4 to extend the deformation theoretic quantization scheme for polynomial observables to larger classes of classical observables, namely to certain symbol spaces. Secondly the complete symbol calculus will allow us to write down a concrete formula for the product of two quantized observables. This is very helpful for calculational purposes and an advantage over the definition of normal order quantization in chapter 2, where only a very abstract characterization of the product of two quantized observables could be given. The complete symbol calculus sheds some new light on the quantization process, as it opens a rigorous path to apply asymptotic expansions, which are already very usefull in the theory of partial differential equations and geometry, for the purposes of quantum physics. Finally in this last chapter we consider some questions of functoriality and naturality and show that the "normal order quantization" of symbol spaces is a functor.

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Notation

In the following we will explain some notation used in this thesis.

k	field
$(X, \mathcal{A})$	ringed space with structure sheaf $\mathcal{A}$ on a topological space X, p. 1
$\mathcal{C}_X^\infty$, $\mathcal{C}^\infty$	sheaf of smooth functions on a differentiable manifold X, p. 2
$\mathcal{O}_X$, $\mathcal{O}$	sheaf of holomorphic functions on a complex manifold X, p. 2
$\pi : E \to X$, $E \to X$	smooth or holomorphic vector bundle over a smooth resp. complex manifold X with total space E and projection π
TX, T^*X	tangent bundle resp. cotangent bundle of a smooth or complex manifold X
$\mathcal{C}_X^\infty(\cdot, E)$, $\mathcal{C}^\infty(\cdot, E)$	sheaf of smooth sections of a differentiable vector bundle $E \to X$
$\mathcal{O}_X(\cdot, E)$, $\mathcal{O}(\cdot, E)$	sheaf of holomorphic sections of a holomorphic vector bundle $E \to X$
$\mathcal{C}_{X,E}^\infty(\nu)$, $\mathcal{C}_E^\infty(\nu)$, $\mathcal{C}^\infty(\nu)$	sheaf defined on X, consisting of fiberwise ν-homogeneous smooth functions on a differentiable vector bundle $E \to X$, p. 21 and 25
$\mathcal{O}_{X,E}(\nu)$, $\mathcal{O}_E(\nu)$, $\mathcal{O}(\nu)$	sheaf defined on X, consisting of fiberwise ν-homogeneous holomorphic functions on a holomorphic vector bundle $E \to X$, p. 21 and 25

$\mathcal{D}_0$	sheaf of classical polynomial observables or polynomial symbols on a smooth resp. complex manifold X, p. 22
$\mathcal{D}_\hbar$	sheaf of $\hbar$-deformed polynomial observables on a smooth resp. complex manifold X, p. 24
$\mathcal{D}_{\text{đ}}$	sheaf of universally deformed polynomial observables on a smooth resp. complex manifold X, p. 26
$\mathcal{D}^\infty$	sheaf of classical observables of infinite order in momentum on a complex manifold X, p. 40 and 44
$\mathcal{D}^\infty_\hbar$	sheaf of $\hbar$-deformed observables of infinite order in momentum on a complex manifold X, p. 44
$\mathcal{D}^\infty_\hbar$	sheaf of universally deformed observables of infinite order in momentum on a complex manifold X, p. 45
$L^2(X)$	HILBERT space of square-integrable functions on a RIEMANNIAN manifold X, p. 34
$\mathrm{S}^\mu_{\rho,\delta}(\cdot, E)$, $\mathrm{S}^\mu_{\rho,\delta}(\cdot, X)$	sheaf of symbols on a smooth conic or vector bundle $E \to X$ resp. on a cotangent bundle T^*X of order μ and type (ρ, δ), p. 48 and 49
$\mathrm{S}^\infty_{\rho,\delta}(\cdot, E)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, X)$	sheaf of symbols on a smooth conic or vector bundle $E \to X$ resp. on a cotangent bundle T^*X of arbitrary order and type (ρ, δ), p. 48 and 49
$\mathrm{S}^{-\infty}(\cdot, E)$, $\mathrm{S}^{-\infty}(\cdot, X)$	sheaf of smoothing symbols on a smooth conic or vector bundle $E \to X$ resp. on a cotangent bundle T^*X, p. 48 and 49
$\mathrm{S}^\infty_{\rho,\delta;\hbar}(\cdot, X)$	sheaf of $\hbar$-deformed symbols of type (ρ, δ) on a smooth manifold X, p. 70
$\mathrm{S}^\infty_{\rho,\delta;\text{đ}}(\cdot, X)$	sheaf of universally deformed symbols of type (ρ, δ) on a smooth manifold X, p. 72

$\underline{\mathbb{C}\text{-}\mathsf{Vec}}_{Hilb}$	category of complex Hilbert spaces and isometric linear mappings
$\underline{\mathsf{Def}}^{flat}_{*,(P,\mathcal{S})}$, $\underline{\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$	category of flat resp. flatly filtered deformations with distinguished point $*$ over a commutative locally k-ringed space $(P,\mathcal{S})$, p. 5
$\underline{k\text{-}\mathsf{Esp}}_{Poisson}$	category of Poisson k-spaces, p. 12
$\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$, $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$	category of smooth resp. holomorphic $\mathbb{C}$-ringed spaces, p. 18 and 19
$\underline{\mathsf{Quant}}^{flat}_{*,(P,\mathcal{S})}$, $\underline{\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$	category of quantizations over a flat resp. a flatly filtered deformation over a commutative locally k-ringed space $(P,\mathcal{S})$ with distinguished point $*$, p. 10
$\underline{\mathsf{Quant}}^{diff.}_{Dirac}$, $\underline{\mathsf{Quant}}^{hol.}_{Dirac}$, $\underline{\mathsf{Quant}}^{\mathbb{R}\text{-}anal.}_{Dirac}$, $\underline{\mathsf{Quant}}^{form.}_{Dirac}$	category of differentiable (resp. holomorphic, real analytic, formal) Dirac quantizations, p. 14
$\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$, $\underline{\mathcal{O}\text{-}\mathsf{Def}}^{flat}_{*,(P,\mathcal{S})}$	category of flatly filtered resp. flat deformations over $(P,\mathcal{S})$ in the category of smooth resp. holomorphic ringed spaces, p. 18 and 19
$\underline{\mathcal{C}^\infty\text{-}\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$, $\underline{\mathcal{O}\text{-}\mathsf{Quant}}^{flat}_{*,(P,\mathcal{S})}$	category of flatly filtered resp. flat quantizations over $(P,\mathcal{S})$ in the category of smooth resp. holomorphic ringed spaces, p. 19
$\underline{\mathsf{Rep}}_{Hilb}$	category of Hilbert space representations of $\mathbb{C}$-algebras with intertwining operators as morphisms, p. 34

$\underline{\mathsf{Diff}}_{symp}$	category of symplectic manifolds and symplectomorphisms, p. 12 and 76
$\underline{\mathsf{Diff}}_{Riem}$, $\underline{\mathsf{Diff}}^{emb.}_{Riem}$	category of RIEMANNIAN manifolds and isometric smooth mappings resp. isometric embeddings as morphisms, p. 31
$\underline{\mathsf{Diff}}_{or.Riem}$, $\underline{\mathsf{Diff}}^{emb.}_{or.Riem}$	category of orientable RIEMANNIAN manifolds and isometric smooth mappings resp. isometric embeddings as morphisms, p. 34

Chapter 1

Foundations

1.1 Deformation theory

In this paragraph we introduce our concept of a deformation. We keep the language close to algebraic geometry (see HARTSHORNE [27] plus references therein) and complex analysis (see GINDIKIN, KHENKIN [22]). That means we use sheaves of algebras respectively ringed spaces as the spaces to be deformed and as parameter spaces. The new aspect is that sheaves of noncommutative algebras are considered for deformation though the parameter spaces stay commutative.

First we recall the notion of a ringed space. A **ringed space** is a pair $(X, \mathcal{A})$ where X is a topological space and $\mathcal{A}$ a sheaf of rings on X. $\mathcal{A}$ is called the **structure sheaf** of the ringed space. It gives the topological space X an additional structure like that of a smooth manifold, complex space, scheme or supermanifold and can be regarded as a space of germs of "admissible generalized functions" on X. If the stalks $\mathcal{A}_x$ with x in X are all commutative (resp. noncommutative, local) we call $(X, \mathcal{A})$ a commutative (resp. noncommutative, locally) ringed space. If all the sections $\mathcal{A}(U)$ with U open in X are k-algebras, where k is a field, one says $(X, \mathcal{A})$ to be a k-ringed space or a ringed space over k. More generally if $\underline{\mathsf{A}}$ is a subcategory of the category of rings and all the stalks $\mathcal{A}_x$ take values in $\underline{\mathsf{A}}$, $(X, \mathcal{A})$ is said to be a ringed space with values in $\underline{\mathsf{A}}$.

A **morphism of ringed spaces** from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is a pair $F = (f, \phi)$, where $f : X \to Y$ is a continuous map and $\phi : \mathcal{B} \to f_*(\mathcal{A})$ a morphism of sheaves on Y with values in the category of rings. Here $f_*(\mathcal{A})$ is the direct image of $\mathcal{A}$ via f, that means $f_*(\mathcal{A}(U)) = \mathcal{A}\left(f^{-1}(U)\right)$ for all open $U \subset Y$. If the ringed spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are locally ringed (resp. take values in $\underline{\mathsf{A}}$), we require $\phi_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ to be local (resp. to be a morphism in $\underline{\mathsf{A}}$) for all x in X. In case $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are ringed spaces over k, it is assumed that $\phi(U) : \mathcal{B}(U) \to \mathcal{A}(f^{-1}(U))$ is a homomorphism of k-algebras for all $U \subset X$ open.

It is obvious that ringed spaces (resp. ringed spaces with values in $\underline{\mathsf{A}}$) and their morphisms form a subcategory of the category of sheaves on topological spaces.

Let us now write down some special commutative ringed spaces:

$(\mathbb{R}^n, \mathcal{C})$,	where	$\mathcal{C}$	-	sheaf of continuous functions on $\mathbb{R}^n$, $n \in \mathbb{N}$,
$(\mathbb{R}^n, \mathcal{C}^r)$,	where	$\mathcal{C}^r$	-	sheaf of r-times continuously differentiable functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, $r \in \mathbb{N}^* \cup \{\infty\}$,
$(\mathbb{R}^n, \mathcal{C}^\omega)$,	where	$\mathcal{C}^\omega$	-	sheaf of real analytic functions on $\mathbb{R}^n$, $n \in \mathbb{N}$,
$(\mathbb{C}^n, \mathcal{O})$,	where	$\mathcal{O}$	-	sheaf of holomorphic functions on $\mathbb{C}^n$, $n \in \mathbb{N}$.

These examples of commutative ringed spaces can be generalized.

Example 1.1.1 Let $n \in \mathbb{N}$ and $r \in \mathbb{N}^* \cup \{\infty\}$. A ringed space $(X, \mathcal{A})$ is called a

(*i*) topological manifold of dimension n, if $(X, \mathcal{A})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{C})$,

(*ii*) differentiable r-manifold of dimension n, if $(X, \mathcal{A})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^r)$,

(*iii*) real analytic manifold of dimension n, if $(X, \mathcal{A})$ is locally isomorphic to $(U, \mathcal{C}^\omega)$, where $U \subset \mathbb{R}^n$ is open,

(*iv*) complex manifold of dimension n, if $(X, \mathcal{A})$ is locally isomorphic to $(U, \mathcal{O}_n|_U)$, where $U \subset \mathbb{C}^n$ is open,

(*v*) scheme, if for every $x \in X$ there exists an open neighbourhood U of x, such that $(U, \mathcal{A}|_U)$ is isomorphic to an affine scheme.

The structure sheaf of a topological (resp. r-times differentiable, real analytic or complex) manifold X is denoted by $\mathcal{C}_X$ (resp. $\mathcal{C}^r_X$, $\mathcal{C}^\omega_X$ or $\mathcal{O}_X$).

Supersymmetric structures (see Wess, Bagger [60]) are our first examples of non-commutative ringed spaces. Most easily this can be seen with the definition of superspaces according to Manin [34].

Definition 1.1.2 *A* **superspace** *consists of a pair* $(M, \mathcal{O}_M)$*, where* M *is a topological space and* $\mathcal{O}_M$ *a sheaf of supercommutative rings, such that all stalks* $\mathcal{O}_{M,x}$ *,* $x \in M$ *are local.*

Supermanifolds are superspaces which locally split into an even and odd part such that the splitting is differentiable and the odd part is a locally free module sheaf over the even part.

Example 1.1.3 Superspaces and supermanifolds are noncommutative ringed spaces.

The notion of a fibered morphism of ringed spaces is crucial for the definition of a deformation.

Definition 1.1.4 *A morphism* $F = (f, \phi) : (X, \mathcal{A}) \to (P, \mathcal{S})$ *of ringed spaces is called* **fibered***, if the following conditions are fulfilled:*

(i) $(P, \mathcal{S})$ *is a commutative locally ringed space.*

(ii) $f : X \to P$ *is surjective.*

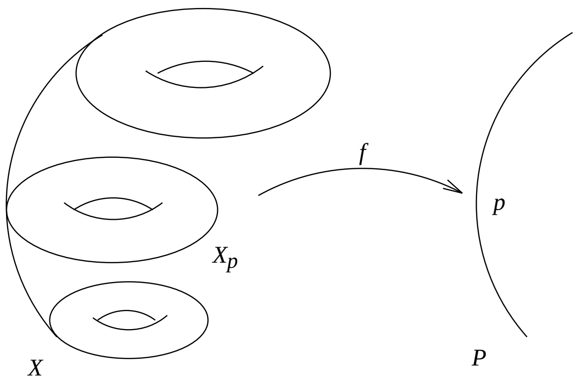
(iii) $\phi_x : \mathcal{S}_{f(x)} \to \mathcal{A}_x$ *maps* $\mathcal{S}_{f(x)}$ *into the center of* $\mathcal{A}_x$ *for all* $x \in X$.

The **fiber** *of* F *over a point* p *of* P *then is the ringed space* $(X_p, \mathcal{A}_p)$ *defined by*

$$X_p = f^{-1}(p), \quad \mathcal{A}_p = \mathcal{A}|_{f^{-1}(p)}/\mathfrak{m}_p\mathcal{A}|_{f^{-1}(p)},$$

where $\mathfrak{m}_p$ *is the maximal ideal in* $\mathcal{S}_p$ *which acts on* $\mathcal{A}|_{f^{-1}(p)}$ *via* ϕ.

A fibered morphism of ringed spaces can be pictured in the following way:



Additionally to this intuitive picture on the "space level", the sheaf morphism ϕ and condition (iii) in 1.1.4 guarantee that the structure sheaves on X, P and the X_p fit to the above picture. More precisely does condition (iii) imply that the stalks $\mathcal{A}_x$ are central extensions of $\mathcal{A}_x/\mathfrak{m}_{f(x)}\mathcal{A}_x$ by $\mathcal{S}_{f(x)}$.

Remark 1.1.5 By using the language of sheaves resp. ringed spaces we allow the considered spaces to have singularities.

Example 1.1.6 (i) Let P and Y be smooth manifolds, $X = Y \times P$ the product of the two and $f = pr_2 : X \to P$ the projection on the second cooordinate. Then $(f, f_*) : (X, \mathcal{C}^\infty_X) \to (P, \mathcal{C}^\infty_P)$ gives rise to a fibered morphism, which we call **trivial**.

(ii) More generally let $f : X \to P$ be a smooth fiber bundle. Then it induces a fibered morphism $(f, f_*) : (X, \mathcal{C}^\infty_X) \to (P, \mathcal{C}^\infty_P)$.

A deformation of a ringed space $(X, \mathcal{A})$ now consists of a fibered morphism $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ such that one of its fibers is isomorphic to $(X, \mathcal{A})$ and such that the fibers vary "continuously" in a certain sense over the parameter space. As the fibers of D in general neither comprise a (smooth) fiber bundle nor are finite dimensional, we have to express the above continuity condition by another notion, namely the one of (algebraic) flatness. It now is a well-known fact in algebraic geometry and complex analysis (see HARTSHORNE [27]) that under certain conditions flatness is a good algebraic substitute for the geometric notions of local triviality resp. of a continuous family. Therefore we will require that the morphism $\Delta : \mathcal{S} \to \mathcal{B}$ is flat resp. flatly filtered. Here we use flatly filtered morphisms to include the case, where the parameter spaces are not algebraic or complex but come from smooth manifolds. Flatness then is not anymore an appropriate substitute for "continuity" or "local triviality", but flat filteredness is. In abstract terms we use flat filteredness for the case when the stalks of the parameter space are not HAUSDORFF with respect to the KRULL topology.

Definition 1.1.7 *Let $(P, \mathcal{S})$ be a commutative locally ringed space over a field k, $*$ a distinguished point in P, and $(X, \mathcal{A})$ a k-ringed space. A* **flat deformation** *of $(X, \mathcal{A})$ over the* **parameter space** *$(P, \mathcal{S})$ (with distinguished point $*$) is a fibered morphism $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ over k together with an isomorphism $I = (i, \iota) : (X, \mathcal{A}) \to (Y_*, \mathcal{B}_*)$ such that for all $p \in P$ and $x \in d^{-1}(p)$ the homomorphism $\Delta_x : \mathcal{S}_p \to \mathcal{B}_x$ is flat.*

A **flatly filtered deformation** *of $(X, \mathcal{A})$ is given by a fibered morphism $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ over k together with an isomorphism $I = (i, \iota) : (X, \mathcal{A}) \to (Y_*, \mathcal{B}_*)$ such that for all $p \in P$ and $y \in d^{-1}(p)$ the following condition holds:*

$\Delta_y : \mathcal{S}_p \to \mathcal{B}_y$ induces for every $m \in \mathbb{N}$ a flat morphism $\Delta_y^m : \mathcal{S}_p/\mathfrak{m}_p^m \to \mathcal{B}_y/\mathfrak{m}_p^m\mathcal{B}_y$, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{S}_p$, and $\mathcal{B}_y$ is filtered by $\mathcal{B}_y \subset \mathfrak{m}_p\mathcal{B}_y \subset \ldots \subset \mathfrak{m}_p^m\mathcal{B}_y \subset \ldots$.

Now let $\tilde{D} = (\tilde{d}, \tilde{\Delta}) : (\tilde{Y}, \tilde{\mathcal{B}}) \to (P, \mathcal{S})$, $\tilde{I} = (\tilde{i}, \tilde{\imath}) : (X, \mathcal{A}) \to (\tilde{Y}_*, \tilde{\mathcal{B}}_*)$ be a deformation over $(P, \mathcal{S})$ of a second k-ringed space $(\tilde{X}, \tilde{\mathcal{A}})$. Furthermore assume that we have a morphism $F = (f, \mathcal{F}) : (X, \mathcal{A}) \to (\tilde{X}, \tilde{\mathcal{A}})$ of k-ringed spaces. Then a **morphism of deformations** *from D to $\tilde{D}$ over F is given by a morphism of k-ringed spaces $B = (b, \beta) : (Y, \mathcal{B}) \to (\tilde{Y}, \tilde{\mathcal{B}})$ such that the diagrams*

$$\begin{tikzcd}
(Y,\mathcal{B}) \arrow[rr, "B"] \arrow[dr, "D"'] & & (\tilde{Y},\tilde{\mathcal{B}}) \arrow[dl, "\tilde{D}"] \\
& (P,\mathcal{S}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,\mathcal{A}) \arrow[r, "F"] \arrow[d, "I"'] & (\tilde{X},\tilde{\mathcal{A}}) \arrow[d, "\tilde{I}"] \\
(Y_*,\mathcal{B}_*) \arrow[r, "B_*"'] & (\tilde{Y}_*,\tilde{\mathcal{B}}_*)
\end{tikzcd}$$

commute.

It is obvious by definition that the flat resp. flatly filtered deformations over a commutative locally k-ringed space $(P, \mathcal{S})$ with distinguished point $*$ form a category $\underline{\mathsf{Def}}^{flat}_{*,(P,\mathcal{S})}$ resp. $\underline{\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$.

In case all the ringed spaces $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and $(P, \mathcal{S})$ of a deformation $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ have the form $(\{*\}, A)$, $(\{*\}, B)$ resp. $(\{*\}, S)$, where $*$ is a point and A, B, and S are k-algebras, we sometimes say by abuse of language, that the algebra B or the algebra morphism $\Delta : S \to B$ is a deformation of A over S.

Example 1.1.8 (*i*) (Products of k-ringed spaces) Let $(X, \mathcal{A})$ be any k-ringed space and $(P, \mathcal{S})$ a k-scheme. Then the product $(X \times P, \mathcal{B}) = (X, \mathcal{A}) \times_k (P, \mathcal{S})$ is a flat deformation of $(X, \mathcal{A})$ with distinguished point $*$ for any closed point $* \in P$. This can be seen easily from the fact that $\mathcal{B}_{(x,p)} = \mathcal{A}_x \otimes_k \mathcal{S}_p$ for every $x \in X$ and $p \in P$.

(*ii*) (First order deformation of algebras) Consider a k-algebra A and the algebra $S = k[\varepsilon]/\varepsilon^2 k[\varepsilon]$. Furthermore let $\alpha : A \times A \to A$ be a HOCHSCHILD 2-cocycle of A, in other words let the relation

$$a\,\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)\,c \;=\; 0 \tag{1.1}$$

hold for all $a, b, c \in A$. Then we can define a new k-algebra B, whose underlying linear structure is isomorphic to $A \otimes_k S$ and whose product is given by the following construction: Any element $\text{ظ} \in B$ can be written uniquely in the form $\text{ظ} = a + b\,\varepsilon = a \otimes 1 + b \otimes \varepsilon$ with $a, b \in A$. Then the product of $\text{ظ} = a + b\,\varepsilon \in B$ and $\text{ع} = c + d\,\varepsilon \in B$ is given by

$$\text{ظ}\,\text{ع} \;=\; a\,c + [\alpha(a, c) + ad + bc]\,\varepsilon. \tag{1.2}$$

By condition (1.1) this product is associative. We thus receive a flat deformation $\Delta : S \to B$ of the algebra A and call it the **first order deformation** of A through the HOCHSCHILD cocycle α. See GERSTENHABER, SCHACK [21] for further information on this and the connection between deformation theory and HOCHSCHILD cohomology.

(iii) (Formal deformation of an algebra) Let us generalize the preceeding example. Assume again A to be an arbitrary k-algebra and choose bilinear maps $\alpha_n : A \times A \to A$ for $n \in \mathbb{N}$ such that α_0 is the product on A and α_1 a HOCHSCHILD cocycle. Furthermore let S be the algebra $k[[t]]$ of formal power series in one variable over k. Then define on the linear space $B = A[[t]]$ of formal power series in one variable with coefficients in A the following bilinear map:

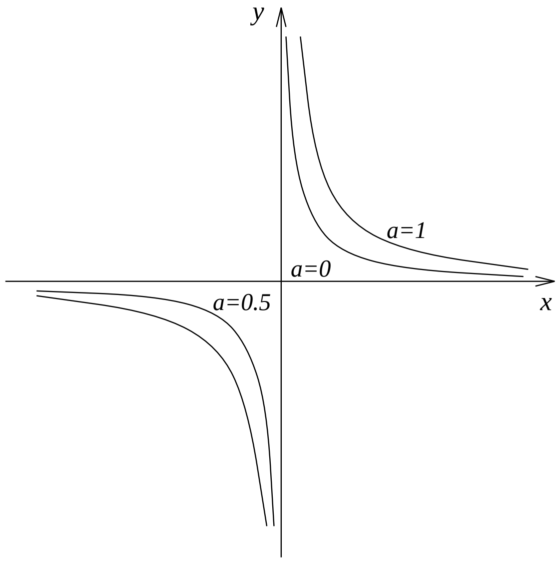
$$\star : B \times B \to B, \;\left(\sum_{n \in \mathbb{N}} a_n\, t^n, \sum_{n \in \mathbb{N}} b_n\, t^n \right) \mapsto \sum_{n \in \mathbb{N}} \;\sum_{\substack{k,l,m \in \mathbb{N} \\ k+l+m=n}} \alpha_m(a_k, b_l)\, t^n. \tag{1.3}$$

If B together with $\star$ becomes a k-algebra or in other words if $\star$ is associative, one can easily see that it gives a flat deformation of A over $S = k[[t]]$. In that case we say that B is a **formal deformation** of A through the family $(\alpha_n)_{n \in \mathbb{N}}$. Contrarily to the preceeding example there might not exist for every HOCHSCHILD cocycle α on A a formal deformation B of A defined by a family $(\alpha_n)_{n \in \mathbb{N}}$ such that $\alpha_1 = \alpha$. In case it exists, we call the deformation B of A **induced** by α. If the third HOCHSCHILD cohomology group $\mathrm{HH}^3(A, A)$ vanishes, there exists for every HOCHSCHILD cocycle α on A a deformation B of A induced by α. See again GERSTENHABER, SCHACK [21] for further information.

(iv) (Formal deformation on symplectic manifolds) Let us consider the last two examples for the case $A = \mathcal{C}^\infty(M)$ where M is a symplectic manifold. Then the POISSON bracket $\{\,,\,\}$ gives a HOCHSCHILD cocycle on $\mathcal{C}^\infty(M)$. There now exists a first order deformation of $\mathcal{C}^\infty(M)$ through $\frac{1}{2i}\{\;,\;\}$ and, though $\mathrm{HH}^3(A, A)$ might not always

vanish, even a formal deformation of $\mathcal{C}^\infty(M)$ induced by $\frac{1}{2i}\{\ ,\ \}$. This fact was first proven by DEWILDE and LECOMTE [65]. A second and shorter proof was given by FEDOSOV [17].

(*v*) (Deformation of a scheme) Consider the k-scheme $Y = \operatorname{Spec} k[x, y, t]/(xy - t]$. It gives rise to a fibration $Y \to \operatorname{Spec} k[t]$, whose fibers Y_a with $a \in k$ are hyperbola $xy = a$, when $a \neq 0$, and consist of the two axes $x = 0$ and $y = 0$, when $a = 0$. Let us illustrate this deformation in a picture.



For further information on this and similar examples see HARTSHORNE [27], in particular Example 3.3.2.

(*vi*) (Quantized universal enveloping algebras according to DRINFELD [14]) A **quantized universal enveloping algebra** for a complex LIE algebra $\mathfrak{g}$ is a HOPF algebra A over $\mathbb{C}[[\hbar]]$ such that A is a topologically free $\mathbb{C}[[\hbar]]$-module (i.e. $A = (A/\hbar A)[[\hbar]]$ as left $\mathbb{C}[[\hbar]]$-module) and $A/\hbar A$ is the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$. Because A is a topologically free $\mathbb{C}[[\hbar]]$-module, A is a flat $\mathbb{C}[[\hbar]]$-module and thus a flat deformation of $U\mathfrak{g}$ over $\mathbb{C}[[\hbar]]$. See DRINFELD [14] or KASSEL [31] for details and examples of quantized universal enveloping algebras.

(*vii*) (Quantum plane) Consider the tensor algebra $T = \bigoplus_{n\in\mathbb{N}} (\mathbb{R}^2)^{\otimes n}$ of the 2-dimensional real vector space $\mathbb{R}^2$, and let (x, y) be the canonical basis of $\mathbb{R}^2$. Now form the tensor product sheaf $T_{\mathbb{C}^*} = T \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{C}^*}$ of T with the sheaf of holomorphic functions on $\mathbb{C}^*$

and let $I_{\mathbb{C}^*}$ be the ideal sheaf in $T_{\mathbb{C}^*}$ generated by the relations

$$x \otimes y - z\, y \otimes x \;=\; 0, \tag{1.4}$$

where $z : \mathbb{C}^* \to \mathbb{C}^*$ is the identity function. The quotient sheaf $B = B_{\mathbb{C}^*} = T_{\mathbb{C}^*}/I_{\mathbb{C}^*}$ then is a sheaf of $\mathbb{C}$-algebras and an $\mathcal{O}_{\mathbb{C}^*}$-module. Now move all occurances of x in an element of $B_{\mathbb{C}^*}$ by Eq. (1.4) to the right of all y's. By using the fact that $\frac{1}{z}$ is well-defined over $\mathbb{C}^*$ one can thus show that $B_{\mathbb{C}^*}$ is a free $\mathcal{O}_{\mathbb{C}^*}$-module. Hence $B_{\mathbb{C}^*}$ is flat over $\mathcal{O}_{\mathbb{C}^*}$. Further it is easy to see that for every $q \in \mathbb{C}^*$ the $\mathbb{C}$-algebra $A_q = B_q/\mathfrak{m}_q A_q$ is freely generated by elements x, y with relations

$$x \otimes y - q\, y \otimes x \;=\; 0. \tag{1.5}$$

We call A_q the **q-deformed quantum plane** and B the **over $\mathbb{C}^*$ universally deformed quantum plane**. Altogether we can now interpret B as a flat deformation of A_q over $\mathbb{C}^*$, in particular as a flat deformation of $A_1 = T \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x, y]$, the algebra of complex polynomials in two generators.

In the same way one can deform function algebras on higher dimensional vector spaces as well as function algebras on certain LIE groups. We thus receive for example the quantum group $SU_q(2)$ as a deformation of a HOPF algebra of functions on $SU(2)$. See for example FADDEEV, RESHETIKHIN, TAKHTAJAN [16], MANIN [35] and WESS, ZUMINO [61] for more information on q-deformations of vector spaces, LIE groups, differential calculi and all that.

(viii) (Noncommutative Fiber Bundles) Let $SU_q(2)$ be the q-deformed $SU(2, \mathbb{C})$. Then it is possible to construct a sheaf $\mathcal{M}$ over S^4 and for every $k \in \mathbb{Z}$ a sheaf $\mathcal{P}^k$ over S^4 together with a morphism $\rho_k : \mathcal{M} \to \mathcal{P}^k$ such that $\rho_k : \mathcal{M} \to \mathcal{P}^k$ can be regarded as a q-deformation of the classical $SU(2, \mathbb{C})$-bundle over S^4 with index k. $\rho_k : \mathcal{M} \to \mathcal{P}^k$ is therefore called the **noncommutative $SU_q(2)$-fiber bundle over $\mathcal{M}$** with index $k \in \mathbb{Z}$. Note that both sheaves $\mathcal{M}$ and $\mathcal{P}^k$ are not commutative for $q \neq 1$. Here we cannot go into the details of the construction of noncommutative fiber bundles but refer the interested reader to PFLAUM [38, 37] for exact definitions and statements.

Every deformation gives rise to an exact sequence

$$0 \longrightarrow \mathfrak{m}_* \mathcal{B}|_{Y_*} \longrightarrow \mathcal{B}|_{Y_*} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0 \tag{1.6}$$

of sheaves on Y_*. A **trivialization** of D then is a multiplicative right inverse $\rho : \mathcal{A} \to \mathcal{B}|_{Y_*}$ of π. In case such a trivialization exists, we call the deformation D **trivial**.

Example 1.1.9 *(i)* Consider Example 1.1.8 (i). Obviously it has a trivialization induced by all maps $\mathcal{A}_x \to \mathcal{B}_{(x,p)}$, $a \mapsto a \otimes_k 1$ with $x \in X$ and $p \in P$.

(ii) If the HOCHSCHILD cocycle α in Example 1.1.8 (ii) is zero, the deformation B of A through α is obviously trivial. More generally any first order deformation of A has a trivialization, if $\mathrm{HH}^2(A, A) = 0$. In the same way any formal deformation of A is trivial, if $\mathrm{HH}^2(A, A)$ vanishes. See GERSTENHABER, SCHACK [21] and references therein for details.

1.2 Quantization

Let us now give the abstract looking definition of a deformation quantization and explain it afterwards in some more down to earth terms.

Definition 1.2.1 *A* **deformation quantization** *of a k-ringed space $(X, \mathcal{A})$ consists of a flat (resp. flatly filtered) deformation $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ of $(X, \mathcal{A})$ with distinguished point $*$ and a linear morphism $Q = (q, \mathfrak{q}) : (Y, \mathcal{B}) \to (X, \mathcal{A})$ of k-ringed spaces, such that:*

(i) $q : Y \to X$ is onto and transversal to d in the sense that every fiber $q^{-1}(x)$ with $x \in X$ intersects Y_p for any $p \in P$ in exactly one point.

(ii) (Correspondence principle) Q is a linear left inverse to the canonical inclusion $(X, \mathcal{A}) \to (Y_*, \mathcal{B}_*) \to (Y, \mathcal{B})$.

We sometimes say that Q is a (flat resp. flatly filtered) quantization of $(X, \mathcal{A})$ over the deformation D, or that (Q, D) is a (flat resp. flatly filtered) quantization of $(X, \mathcal{A})$.

Now let $(\tilde{Q}, \tilde{D})$ with $\tilde{D} = (\tilde{d}, \tilde{\Delta}) : (\tilde{Y}, \tilde{\mathcal{B}}) \to (P, \mathcal{S})$, $\tilde{I} = (\tilde{i}, \tilde{\imath}) : (\tilde{X}, \tilde{\mathcal{A}}) \to (\tilde{Y}_*, \tilde{\mathcal{B}}_*)$ and $\tilde{Q} = (\tilde{q}, \tilde{\mathfrak{q}}) : (\tilde{Y}, \tilde{\mathcal{B}}) \to (\tilde{X}, \tilde{\mathcal{A}})$ be a quantization of a second k-ringed space $(\tilde{X}, \tilde{\mathcal{A}})$. Furthermore assume that we have a morphism $F = (f, \mathcal{F}) : (X, \mathcal{A}) \to (\tilde{X}, \tilde{\mathcal{A}})$ of k-ringed spaces. Then a **morphism of deformation quantizations** *from (Q, D) to $(\tilde{Q}, \tilde{D})$ over F is given by a morphism $B = (b, \beta) : (Y, \mathcal{B}) \to (\tilde{Y}, \tilde{\mathcal{B}})$ such that B gives rise to a morphism of deformations from D to $\tilde{D}$ over F, and such that the diagram*

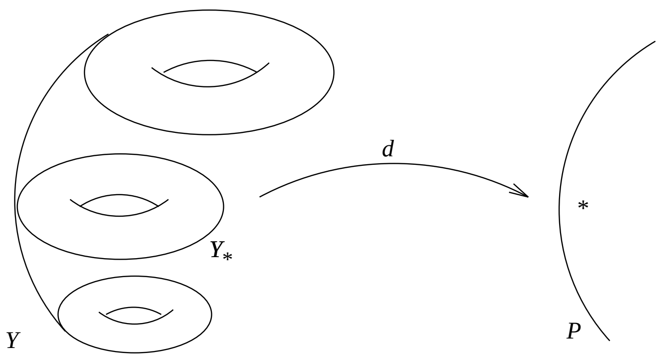
$$
\begin{tikzcd}
(Y, \mathcal{B}) \arrow[r, "B"] \arrow[d, "Q"'] & (\tilde{Y}, \tilde{\mathcal{B}}) \arrow[d, "\tilde{Q}"] \\
(X, \mathcal{A}) \arrow[r, "F"'] & (\tilde{X}, \tilde{\mathcal{A}})
\end{tikzcd}
$$

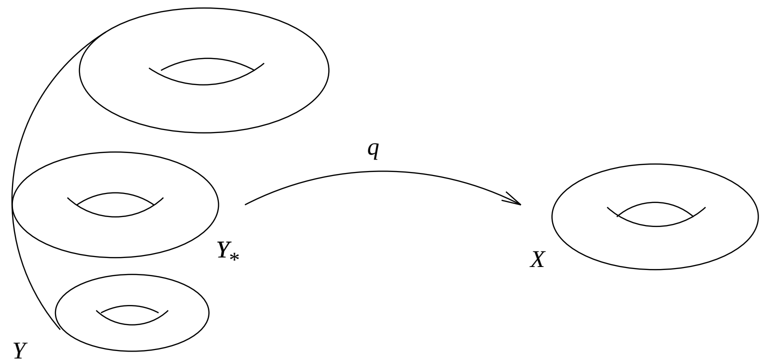
commutes.

It is obvious by definition that deformation quantizations over flat resp. flatly filtered deformations over a commutative locally k-ringed space $(P, \mathcal{S})$ with distinguished point $*$ form a category $\underline{\mathsf{Quant}}^{flat}_{*,(P,\mathcal{S})}$ resp. $\underline{\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$.

Note that, as all quantizations regarded in this work are deformation quantizations anyway, we usually just say quantization instead of deformation quantization.

Let us now illustrate the "spacial part" of the definition in a little picture.





By condition (i) above Y almost looks like the product $X \times P$, and q and d can be regarded as the projections on the first and second coordinates respectively. But more important than this technical observation is condition (ii). It namely expresses the fact that the morphism $\mathfrak{q} : \mathcal{A} \to \mathcal{B}$ is a quantization map, that is associates to any observable

$f \in \mathcal{A}(U)$ with $U \subset X$ open a universally quantized observable $\mathfrak{q}(f) \in \mathcal{B}(q^{-1}(U))$ such that

$$\mathfrak{q}(f)|_U = f \tag{1.7}$$

holds. In other words this means that the classical limit of the universal quantization of a classical observable is again the original classical observable, i.e. that the correspondence principle is fulfilled. If one now wants to quantize a classical observable $f \in \mathcal{A}(U)$ for a specific quantization parameter $p \in P$, one can form the universal quantization $\mathfrak{q}(f)$ of f and then project this object down to the fiber lying over p. The resulting element $\mathfrak{q}_p(f) \in \mathcal{B}_p\left(q^{-1}(U) \cap Y_p\right)$ is the quantization of f for the specific parameter p. The quantization $\mathfrak{q}(f)$ thus gives all possible quantizations of f, and $\mathfrak{q}_p(f)$ only the one for a specific point p in P.

Example 1.2.2 (*i*) Consider the trivial deformation $D = (d, \Delta) : (X \times P, \mathcal{B}) \to (P, \mathcal{S})$ of a k-ringed space $(X, \mathcal{A})$ in Example 1.1.8 (*i*). Now let $Q = (q, \mathfrak{q}) : (X \times P, \mathcal{B}) \to (X, \mathcal{A})$ be the canonical morphism given by

$$q : X \times P \to X, \;\; (x, p) \mapsto x \tag{1.8}$$

$$\mathfrak{q}_{(x,p)} : \mathcal{A}_x \to \mathcal{B}_{(x,p)} = \mathcal{A}_x \otimes_k \mathcal{S}_p, \;\; f \mapsto f \otimes 1. \tag{1.9}$$

Then (Q, D) obviously is a quantization of $(X, \mathcal{A})$ which we call a **trivial** or an **unphysical** one.

(*ii*) Consider the formal deformation B of a k-algebra A in Example 1.1.8 (*iii*). The natural injection $\mathfrak{q} : A \to B$ together with the trivial map $q : \{*\} \to \{*\}$ then give rise to a quantization of the ringed space $(\{*\}, A)$ over the (formal) parameter space $(\{*\}, k[[\hbar]])$. In that case we usally call B a **formal deformation quantization** of A over $k[[\hbar]]$.

In mathematical physics we are mainly interested in the quantization of a certain class of ringed spaces, namely those which can be interpreted as a classical phase space plus a sheaf of observables on that phase space. These ringed spaces are called POISSON spaces and will be defined in the following.

Definition 1.2.3 *A* **Poisson** k**-space** *is a commutative k-ringed space $(X, \mathcal{A})$ together with a bilinear map $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called* **Poisson bracket**, *such that the following conditions hold:*

(i) *(anti-symmetry)*
$\{f,g\} = -\{g,f\}$ *for* $f,g \in \mathcal{A}(U)$, $U \subset X$ *open.*

(ii) *(*JACOBI *identity)*
$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$ *for* $f,g,h \in \mathcal{A}(U)$.

(iii) *(*LEIBNIZ *rule)*
$\{f,gh\} = g\{f,h\} + \{f,g\}h$ *for* $f,g,h \in \mathcal{A}(U)$.

A **morphism of Poisson k-spaces** *is a morphism* $(f,\phi) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ *of the underlying k-ringed spaces such that the equation*

$$\phi\Big(\{a,b\}_Y\Big) \;=\; \{\phi(a),\phi(b)\}_X \tag{1.10}$$

holds for all $a,b \in \mathcal{B}(U)$, $U \subset Y$ *open.*

Note that by definition the POISSON k-spaces form a category $\underline{k\text{-}\mathsf{Esp}}_{Poisson}$. As in the following we will mainly consider the case $k = \mathbb{C}$, we just say POISSON space instead of POISSON $\mathbb{C}$-space.

Example 1.2.4 (i) Consider the even dimensional EUCLIDEAN space $\mathbb{R}^{2n}$ and the sheaf $\mathcal{C}^\infty_{\mathbb{R}^{2n}}$ of smooth functions on it. Denote any element of $\mathbb{R}^{2n}$ by a $2n$-tupel $(x_1,...,x_n,\xi_1,...,\xi_n)$. Then the equation

$$\{f,g\} \;=\; \sum_{1\le k\le n} \frac{\partial f}{\partial x_k}\frac{\partial g}{\partial \xi_k} - \frac{\partial f}{\partial \xi_k}\frac{\partial g}{\partial x_k} \tag{1.11}$$

with $f,g \in \mathcal{C}^\infty_{\mathbb{R}^{2n}}(U)$ and $U \subset \mathbb{R}^{2n}$ open defines on the ringed space $\Big(\mathbb{R}^{2n},\mathcal{C}^\infty_{\mathbb{R}^{2n}}\Big)$ the structure of a POISSON space.

(ii) Let Y be symplectic manifold with symplectic form ω. If $\mathcal{C}^\infty$ denotes the sheaf of differentiable functions on Y, define $\{\ ,\ \} : \mathcal{C}^\infty \times \mathcal{C}^\infty \to \mathcal{C}^\infty$ by $\{f,g\} = \omega(X_f,X_g)$ with $f,g \in \mathcal{C}^\infty(U)$, $U \in X$ open and X_f, X_g the Hamiltonian vector fields over U associated to f resp. g. Then $(Y,\mathcal{C}^\infty)$ together with $\{\ ,\ \}$ forms a POISSON space. By setting $Y = \mathbb{R}^{2n}$ (i) is a special case of this example. Now let Y' be another symplectic manifold and $\Omega : Y \to Y'$ a symplectomorphism. Then the pull-back $\Omega^* : \mathcal{C}^\infty_{Y'} \to \mathcal{C}^\infty_{Y}$ gives rise to a POISSON morphism from $(Y,\mathcal{C}^\infty_Y)$ to $(Y,\mathcal{C}^\infty_{Y'})$. We thus receive a functor from the category $\underline{\mathsf{Diff}}_{symp}$ to the category $\underline{\mathbb{C}\text{-}\mathsf{Esp}}_{Poisson}$ of POISSON spaces.

(*iii*) Let X be any manifold, ω the canonical symplectic form on T^*X and $\mathcal{D}$ be the direct image sheaf $\pi_*\mathcal{C}^\infty_{T^*X}$, where $\pi : T^*X \to X$ is the canonical projection and $\mathcal{C}^\infty_{T^*X}$ the sheaf of differentiable functions on T^*X. In more down to earth terms we then have $\mathcal{D}(U) = \mathcal{C}^\infty(\pi^{-1}(U))$ for every $U \subset X$ open. Now define a Poisson bracket on $\mathcal{D}$ similar to (*i*) by $\{f, g\} = \omega(X_f, X_g)$ for $f, g \in \mathcal{P}(U) = \mathcal{C}^\infty_{T^*X}(\pi^{-1}(U))$, $U \in X$ open. This bracket gives rise to a Poisson structure on the ringed space $(X, \mathcal{D})$.

(*iv*) A **Poisson manifold** is a manifold Y together with a Poisson bracket $\{\ ,\ \}$ on the sheaf of smooth functions on Y. Obviously a Poisson manifold is a Poisson spacel. See Vaisman [54] as a reference on Poisson manifolds.

(*v*) More generally take a symplectic manifold Y with (real) polarization P. Define X to be the space of leaves of P equipped with the quotient topology defined by the canonical projection $\pi : Y \to X$. Let $\mathcal{P}$ be the direct image $\pi_*\mathcal{C}^\infty_Y$ and pull down the Poisson bracket on $\mathcal{C}^\infty_Y$ to a bracket $\{\ ,\ \}$ on $\mathcal{P}$ via π_*. We thus receive a Poisson space $(X, \mathcal{P})$ with bracket $\{\ ,\ \}$ and call it associated to the symplectic manifold (Y, ω) and polarization P.

(*vi*) A **Poisson Lie group** is a Lie group G together with a Poisson bracket $\{,\}$ on the sheaf $\mathcal{C}^\infty_G$ of smooth functions on G such that the multiplication on G gives rise to a morphism of Poisson spaces. Thus by definition every Poisson Lie group forms a Poisson space. See Chari, Pressley [10] for further information on Poisson Lie groups.

(*vii*) Let G be a symmetry Lie group of a Hamiltonian system (M, ω, H), where M is a smooth manifold with a symplectic form ω, and H a smooth function on M. Further assume that G has an equivariant moment map $m : M \to \mathfrak{g}^*$. For any regular value $\xi \in \operatorname{im} m \subset \mathfrak{g}^*$ the set $M_\xi = m^{-1}(\xi)$ is a submanifold of M. Now there exists a foliation on M generated by the "kernel distribution" $\omega^\perp$ of ω. This gives rise to the quotient space Q_ξ of all leaves of the foliation and to the sheaf $\mathcal{M}_\xi$ of all smooth functions on M_ξ which are constant on the leaves. Note that Q_ξ in general neither needs to be a smooth manifold nor even a Hausdorff topological space. Then the push-down $\mathcal{Q} = \pi_*(\mathcal{M}_\xi)$, where $\pi : M_\xi \to Q$ is the canonical projection, is a sheaf on Q, which canonically inherites a Poisson bracket from the one on $\mathcal{M}_\xi$. We thus receive a new Poisson space $(Q, \mathcal{Q})$, the so-called **Marsden-Weinstein**

quotient.

Definition 1.2.5 *Let* $(X, \mathcal{A})$ *be a* Poisson *space and* $(P, \mathcal{S})$ *be one of the following parameter spaces:*

$$\begin{aligned}
(P, \mathcal{S}) &= (\mathbb{C}, \mathcal{O}_{\mathbb{C}}) && (\textit{holomorphic case}) \\
&= (\mathbb{R}, \mathcal{C}^{\omega}_{\mathbb{R}}) && (\textit{real analytic case}) \\
&= (\mathbb{R}, \mathcal{C}^{\infty}_{\mathbb{R}}), (\mathbb{C}, \mathcal{C}^{\infty}_{\mathbb{C}}) && (\textit{differentiable case}) \\
&= (*, \mathcal{F}_1) && (\textit{formal case}).
\end{aligned}$$

Suppose further we have a deformation $D : (Y, \mathcal{B}) \to (P, \mathcal{S})$ *of* $(X, \mathcal{A})$. *A holomorphic (resp. real analytic, differentiable or formal)* **Dirac quantization** *of* $(X, \mathcal{A})$ *is then given by a flat quantization (resp. flatly filtered quantization in the differentiable case)* $Q = (q, \mathfrak{q}) : (Y, \mathcal{B}) \to (X, \mathcal{A})$ *of* $(X, \mathcal{A})$ *over* $D = (d, \Delta)$ *with distinguished point* 0 *(resp.* $*$ *in the formal case) such that for every* $f, g \in \mathcal{A}(U)$, $U \subset X$ *open the commutator* $[\mathfrak{q}(f), \mathfrak{q}(g)]$ *can be expanded in the form*

$$[\mathfrak{q}(f), \mathfrak{q}(g)] = -i\, \bar{d}\, \mathfrak{q}(\{f, g\}) + \bar{d}^2\, r_{f,g} \tag{1.12}$$

with $r_{f,g} \in \mathcal{A}(U \times \mathbb{C})$. *Here* $\bar{d}$ *denotes in the holomorphic, real analytic and differentiable case the function* $\bar{d} = \Delta(id_{\mathbb{C}}) : U \times \mathbb{C} \to \mathbb{C}, (x, \hbar) \to \hbar$ *and in the formal case the element* $\Delta(\hbar)$ *with* $\hbar$ *being the generator of* $\mathcal{F}_1 = \mathbb{C}[[\hbar]]$.

A morphism of holomorphic (resp. real analytic, differentiable or formal) Dirac *quantizations is given by a morphism of quantizations with parameter space* $(P, \mathcal{S})$ *over a morphism of* Poisson *spaces.*

It is obvious by definition that holomorphic (resp. real analytic, differentiable or formal) quantizations form a category $\underline{\mathsf{Quant}}^{hol.}_{Dirac}$ (resp. $\underline{\mathsf{Quant}}^{\mathbb{R}\text{-}anal.}_{Dirac}$, $\underline{\mathsf{Quant}}^{diff.}_{Dirac}$ or $\underline{\mathsf{Quant}}^{form.}_{Dirac}$).

Remark 1.2.6 In case the quantization is holomorphic, real analytic or differentiable, formula (1.12) gives rise to the equation

$$[\mathfrak{q}(f), \mathfrak{q}(g)]_{\hbar} = -i\, \hbar\, \mathfrak{q}(\{f, g\})_{\hbar} + \hbar^2\, (r_{f,g})_{\hbar} \tag{1.13}$$

over the fiber $Y_{\hbar} = d^{-1}(\hbar)$ of $\hbar \in \mathbb{C}$. This is the more familiar form of the Dirac quantization (modulo higher orders of $\hbar$).

Let us now give some examples for quantizations of certain POISSON spaces in order to clarify and justify the definition of a DIRAC quantization.

Example 1.2.7 (WEYL calculus and MOYAL product) Consider the symplectic manifold $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with its canonical symplectic structure $\omega = \sum_{k=1}^{n} dx_k \wedge d\xi_k$, where $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ are canonical coordinates. We want to show that the MOYAL product, which will be defined in the following with the help of the WEYL calculus, gives rise to a quantization of a certain POISSON space of so-called symbols on $\mathbb{R}^n$. For the required definitions and theorems about symbol spaces, asymptotic expansions and all that we refer the reader to section 3, in particular Definition 3.1.5, or to GRIGIS, SJÖSTRAND [24] and SHUBIN [47].

The POISSON space to be quantized in this example is $(\mathbb{R}^n, \mathrm{S}^\infty_{1,0})$, where the symbol sheaf $\mathrm{S}^\infty_{1,0}$ is defined by $\mathrm{S}^\infty_{1,0}(U) = \bigcup_{\mu\in\mathbb{R}} \mathrm{S}^\mu_{1,0}(U)$ for every $U \subset \mathbb{R}^n$ open and

$$\begin{aligned}\mathrm{S}^\mu_{1,0}(U) \;=\; &\Big\{b \in \mathcal{C}^\infty(U \times \mathbb{R}^n) : \forall K \subset U \text{ compact}, \alpha, \beta \in \mathbb{N}^n \;\exists C_{K,\alpha,\beta} > 0 \text{ s.t.}\\ &\forall x \in K\, \forall \xi \in \mathbb{R}^n \sup_{x\in K} \left|\partial_x^\alpha \partial_\xi^\beta\, a(x,\xi)\right| \leq C_{K,\alpha,\beta}\,(1 + |\xi|)^{\mu-\alpha}\Big\}\end{aligned} \tag{1.14}$$

for every $\mu \in \mathbb{R}$. The POISSON bracket on $\mathrm{S}^\infty_{1,0}(U)$ is the one induced by ω (cf. Examples 1.2.4 (i) and (iii)). Now we can define for any symbol $a \in \mathrm{S}^\infty_{1,0}(U)$ and any $\hbar \in \mathbb{R}^*$ the pseudo-differential operator $\mathrm{Op}_\hbar(a) \in \Psi^\mu_{1,0}(U)$ by the following oscillatory integral:

$$[\mathrm{Op}_\hbar(a)\, u]\,(x) \;=\; \frac{1}{(2\pi)^n} \int\!\!\int e^{i\hbar<\xi,(x-y)>}\, a\left(\frac{x+y}{2}, \xi\right)\, u(y)\, dy\, d\xi, \tag{1.15}$$

where $u \in \mathcal{C}^\infty_0(U)$. $\mathrm{Op}_\hbar(a)$ is called the **Weyl quantization** of a for $\hbar$. Note that in physics $\mathrm{Op}_\hbar(a)$ is usually formally written as

$$\mathrm{Op}_\hbar(a) \;=\; \frac{1}{(2\pi)^n} \int\!\!\int \hat{a}(y,\zeta)\, e^{i\sum_{k=1}^n y_k\cdot\mathfrak{q}_\hbar(\xi_k) + \zeta_k\cdot\mathfrak{q}_\hbar(x_k)} dy\, d\zeta, \tag{1.16}$$

where $\hat{a}$ is the FOURIER transform $\hat{a}(y,\zeta) = \frac{1}{(2\pi)^n}\int\int e^{-i(<\xi,y>+<\zeta,x>)}\, a\,(x,\xi)\; dx\, d\xi$, $\mathfrak{q}_\hbar(\xi_k) = -i\hbar\frac{\partial}{\partial x_k}$ and $\mathfrak{q}_\hbar(x_k)$ is the multplication operator with x_k. Let us consider the product of $\mathrm{Op}_\hbar(a)$ and $\mathrm{Op}_\hbar(b)$ with $a, b \in \mathrm{S}^\infty_{1,0}(U)$. By methods of the theory of pseudo-differential operators given for example in SHUBIN [47] it is a well-known fact that $\mathrm{Op}_\hbar$ has an inverse $\sigma_{W,\hbar} : \Psi^\mu_{1,0}(U) \to \mathrm{S}^\mu_{1,0}(U)$, which asssociates to every pseudo-differential operator its so-called **Weyl symbol**. Furthermore one knows that for symbols a, b and $\hbar \neq 0$ the relation

$$\mathrm{Op}_\hbar(a) \cdot \mathrm{Op}_\hbar(b) \;=\; \mathrm{Op}_\hbar(c) \tag{1.17}$$

holds, where the symbol c has the following asymptotic expansion:

$$c(x,\xi) \;\sim\; \sum_{\alpha,\beta\in\mathbb{N}^n} (-1)^{|\beta|} \left(\frac{-i\hbar}{2}\right)^{|\alpha+\beta|} \left(\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\frac{\partial^{|\beta|}}{\partial x^\beta} a(x,\xi)\right) \left(\frac{\partial^{|\beta|}}{\partial\xi^\beta}\frac{\partial^{|\alpha|}}{\partial x^\alpha} b(x,\xi)\right). \quad (1.18)$$

In particular we have

$$[\mathrm{Op}_\hbar(a), \mathrm{Op}_\hbar(b)] \;=\; -i\hbar\, \mathrm{Op}_\hbar(\{a,b\}) + \hbar^2 R, \quad R \in \Psi^\infty_{1,0}(U), \quad (1.19)$$

which justifies to call $\mathrm{Op}_\hbar$ a quantization map. MOYAL now used Eq. (1.19) and the above symbol map to define an $\hbar$-dependent family of noncommutative products on a space of smooth functions on $\mathbb{R}^{2n}$. In particular MOYAL thus constructed a deformation respectively a quantization of a space of classsical observables on the symplectic manifold $\mathbb{R}^{2n}$.

Let us translate in the following MOYAL'S ideas to our deformation theoretical language. The arguments will be rather sketchy, as similar ones will be used later on in section 4 in a more general context. Define on the sheaf $\mathrm{S}^\infty_{1,0}$ for $\hbar \neq 0$ a noncommutative associative product by

$$\mathrm{S}^\infty_{1,0}(U) \times \mathrm{S}^\infty_{1,0}(U) \ni (a,b) \mapsto \sigma_{W,\hbar}\left(\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)\right) \in \mathrm{S}^\infty_{1,0}(U). \quad (1.20)$$

Denote the resulting sheaf of algebras by $\mathrm{S}^\infty_{1,0;\hbar}$, and by $\mathrm{S}^\infty_{1,0;0}$ or just $\mathrm{S}^\infty_{1,0}$ the sheaf $\mathrm{S}^\infty_{1,0}$ with its canonical (commutative) algebra structure of pointwise multiplication of functions. Next construct the sheaf $\mathrm{S}^\infty_{1,0;\hbar\!\!\!/}$ on $\mathbb{R}^{n+1}$, such that $\mathrm{S}^\infty_{1,0;\hbar\!\!\!/}(U \times O)$ for $U \subset \mathbb{R}^n$ and $O \subset \mathbb{R}$ open consists of all smooth functions $U \times \mathbb{R}^n \times O \to \mathbb{C}$ whose restriction to $U \times \mathbb{R}^n$ (uniformly) belongs to $\mathrm{S}^\infty_{1,0}(U)$. Embed $\mathcal{C}^\infty_\mathbb{R}$ into $\mathrm{S}^\infty_{1,0;\hbar\!\!\!/}$ by $f \overset{\Delta}{\mapsto} f \circ pr_{2n+1}$ and check that one can give $\mathrm{S}^\infty_{1,0;\hbar\!\!\!/}$ a unique structure of a sheaf of algebras such that the fibers of the morphism $\left(pr_{n+1}, \Delta\right) : \left(\mathbb{R}^{n+1}, \mathrm{S}^\infty_{1,0;\hbar\!\!\!/}\right) \to (\mathbb{R}, \mathcal{C}^\infty)$ are isomorphic to $\mathrm{S}^\infty_{1,0;\hbar}$. Thus we receive a flatly filtered deformation of $\mathrm{S}^\infty_{1,0}$ and by Eq. (1.19) even a differentiable DIRAC quantization of $\left(\mathbb{R}^n, \mathrm{S}^\infty_{1,0}\right)$.

Theorem 1.2.8 (DEWILDE, LECOMTE; FEDOSOV) *There exists a formal* DIRAC *quantization of the sheaf of smooth functions on a symplectic manifold M*

PROOF: By DEWILDE, LECOMTE [65] or FEDOSOV [17] there exists a noncommutative associative product $m_\hbar$ (cf. Example 1.1.8 (iv)) on the sheaf $\mathcal{A} = \mathcal{C}^\infty_M[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $\mathcal{C}^\infty_M$ such that the following conditions hold:

(i) $m_\hbar(f,g) \;=\; fg + \frac{\hbar}{2i}\{f,g\} + ...$ for all $f,g \in \mathcal{C}^\infty_M(U)$ and $U \subset M$ open.

(ii) $m_\hbar(\hbar^k f, \hbar^l g) \;=\; \hbar^{k+l} m_\hbar(f,g)$ for all $k,l \in \mathbb{N}$, $f,g \in \mathcal{C}^\infty_M(U)$ and $U \subset M$ open.

(iii) $m_\hbar(z,f) \;=\; m_\hbar(f,z) \;=\; 0$ for all constant functions $z \in \mathbb{C}$ and all $f \in \mathcal{C}^\infty_M(U)$ with $U \subset M$ open.

Now the canonical embeddings $\phi_U : \mathbb{C}[[\hbar]] \to \mathcal{A}(U)$ with $U \subset M$ open induces a fibered morphism $(f,\phi) : (M,\mathcal{A}) \to \big(\{*\}, \mathbb{C}[[\hbar]]\big)$, where $f : M \to \{*\}$ is the constant function. The stalk $\mathcal{A}_m$ with $m \in M$ is a $\mathbb{C}[[\hbar]]$-module. The maximal ideal $\mathfrak{m} = \hbar\,\mathbb{C}[[\hbar]]$ induces on $\mathcal{A}_m$ a topological structure, namely the Krull-topology with respect to the filtration

$$\mathcal{A}_m \supset \mathfrak{m}\,\mathcal{A}_m \supset \mathfrak{m}^2\,\mathcal{A}_m \supset ... \supset \mathfrak{m}^n\,\mathcal{A}_m \supset ... \,. \tag{1.21}$$

Now the Krull-topology on $\mathcal{A}_m$ is Hausdorff, and $\mathcal{A}_m/\mathfrak{m}^n\mathcal{A}_m \cong \mathcal{C}^\infty_m \otimes_\mathbb{C} \mathbb{C}[[\hbar]]/\mathfrak{m}^n$ for all $n \in \mathbb{N}$. So $\mathcal{A}_m/\mathfrak{m}^n\mathcal{A}_m$ is a flat $\mathbb{C}[[\hbar]]/\mathfrak{m}^n$-module. As $\mathbb{C}[[\hbar]]$ is Noetherian, Theorem 1 in Chapter III, §5.2 of Bourbaki [9], *Commutative Algebra*, implies that $\mathcal{A}_m$ is a flat $\mathbb{C}[[\hbar]]$-module. Furthermore we have $\mathcal{A}_m \big/ \mathfrak{m}\mathcal{A}_m \cong \mathcal{C}^\infty_m$, so the fiber $(M, \mathcal{A}_*)$ is isomorphic to $(M, \mathcal{C}^\infty_M)$. The quantization morphism $\mathfrak{q} : \mathcal{C}^\infty_M \to \mathcal{A}$ is canonically given by $\mathfrak{q}(f) \;=\; f$ for any $f \in C^\infty_M(U)$. Hence by the above condition (i) Dirac's quantization condition holds, and the claim now follows. □

Remark 1.2.9 Further results on the formal quantization of Poisson spaces are given in Donin [13].

Note 1.2.10 Formal quantizations lack of physical applicability as $\hbar$ cannot be given a nonvanishing numerical value. We therefore try to circumvent this obstacle by establishing concrete resp. convergent deformation quantizations. This will be the main object of discussion from chapter 2 on.

1.3 Product and monoidal structures

It is a well-known principle in quantum physics that the state space of two coupled systems is the tensor product of the state spaces of both systems. In classical mechanics a similar situation holds: the phase space resp. configuration space of two systems is the Cartesian product of both phase spaces resp. configuartion spaces. A quantization theory now should obey these principles or, to put it differently, the category of deformation quantizations resp. appropriate subcategories of it should have a natural monoidal structure. We will now show in this section that this is the case indeed.

Let us first define the category $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ of smooth ringed spaces. The class of objects consists of all ringed spaces $(X, \mathcal{A})$, such that X is a smooth manifold and such that the structure sheaf $\mathcal{A}$ carries besides its algebra structure that one of a locally free left $\mathcal{C}^\infty_X$-module. The morphisms (f, φ) in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ all have the form $\varphi = f_*$, where f is a smooth map between manifolds. In $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ there exists a natural monoidal structure given by the following. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be objects in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$. Then regard the projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ on the first resp. second coordinate and define the external tensor product $\mathcal{A} \boxtimes \mathcal{B}$ by

$$\mathcal{A} \boxtimes \mathcal{B} \;=\; pr_1^{-1}\mathcal{A} \otimes pr_2^{-1}\mathcal{B}. \tag{1.22}$$

Defining

$$(X, \mathcal{A}) \times (Y, \mathcal{B}) \;=\; (X \times Y, \mathcal{A} \boxtimes \mathcal{B}) \tag{1.23}$$

we now get a product $\times$ on $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ with projections $P_1 = (pr_1, \mathfrak{p}_1) : (X \times Y, \mathcal{A} \boxtimes \mathcal{B}) \to (X, \mathcal{A})$ and $P_2 = (pr_2, \mathfrak{p}_2) : (X \times Y, \mathcal{A} \boxtimes \mathcal{B}) \to (Y, \mathcal{B})$, where $\mathfrak{p}_1(f) = f \otimes 1$ and $\mathfrak{p}_2(g) = 1 \otimes g$ for $f \in \mathcal{A}(U)$, $g \in \mathcal{B}(V)$ and $U \subset X$, $V \subset Y$ open. The product $\times$ also gives rise to a tensor product functor on $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$, where the unit is obviously given by the one point space $(\{*\}, \mathbb{C})$ with its canonical structure of a zero dimensional manifold.

Next let $(P, \mathcal{S}) = (P, \mathcal{C}^\infty_P)$ be a smooth manifold and $(f, \phi) : (X, \mathcal{A}) \to (P, \mathcal{S})$ a fibered morphism. We then call (f, ϕ) a **smooth fibered morphism**, if it is a morphism in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ and if f is a submersion. Note that $(P, \mathcal{S})$ is a commutative locally ringed space and an object in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$. Now consider the subcategory $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$ of $\underline{\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$ consisting of all flatly filtered deformations such that the underlying fibered morphism is a smooth one. If then $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ and $\tilde{D} = (\tilde{d}, \tilde{\Delta}) : (\tilde{Y}, \tilde{\mathcal{B}}) \to (P, \mathcal{S})$ denote deformations in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$ of $(X, \mathcal{A})$ resp. $(\tilde{X}, \tilde{\mathcal{A}})$, the fiber product $(Y \times_P \tilde{Y}, \mathcal{B} \boxtimes_{\mathcal{S}} \tilde{\mathcal{B}}) = (Y, \mathcal{B}) \times_{(P,\mathcal{S})} (\tilde{Y}, \tilde{\mathcal{B}})$ in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ of D and $\tilde{D}$ is well-defined, as $\mathcal{S}$ lies in the center of $\mathcal{B}$ and $\tilde{\mathcal{B}}$. Note that hereby

$$\mathcal{B} \boxtimes_{\mathcal{S}} \tilde{\mathcal{B}} \;=\; pr_1^{-1}\mathcal{B} \otimes_{\mathcal{S}_p} pr_2^{-1}\tilde{\mathcal{B}}. \tag{1.24}$$

Furthermore $(Y, \mathcal{B}) \times_{(P,\mathcal{S})} (\tilde{Y}, \tilde{\mathcal{B}}) \to (P, \mathcal{S})$ is a smooth fibered morphism such that the fiber over $*$ is isomorphic to the product $(X, \mathcal{A}) \times (\tilde{X}, \tilde{\mathcal{A}})$. If we can now show that the condition of flat filteredness is fulfilled for $(Y, \mathcal{B}) \times_{(P,\mathcal{S})} (\tilde{Y}, \tilde{\mathcal{B}}) \to (P, \mathcal{S})$, this morphism even comprises a flatly filtered deformation of $(X, \mathcal{A}) \times (\tilde{X}, \tilde{\mathcal{A}})$ and can be interpreted as the product of D and $\tilde{D}$ in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$. So let us show the flat filteredness and first prove a little lemma.

Lemma 1.3.1 *Let $(f, \phi) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ be a smooth fibered morphism. Then the sheaf $\mathcal{A}$ is flatly filtered over $\mathcal{S}$.*

Proof: As $\mathcal{B}$ is a locally free $\mathcal{C}^\infty_Y$-module sheaf, it suffices to show that $\mathcal{C}^\infty_Y$ is flatly filtered over $\mathcal{S} = \mathcal{C}^\infty_P$. So assume $p \in P$, $y \in f^{-1}(p)$, $m \in \mathbb{N}$ and let $\mathfrak{m}_p$ be the maximal ideal in $\mathcal{C}^\infty_p$. Then as f is a submersion, there exists an open neighborhood U of y in Y such that $X = f^{-1}(p) \cap U$ is a submanifold of Y. So the stalk $\mathcal{C}^\infty_{X,y}$ of smooth functions on X at y is well-defined, and we have

$$\mathcal{C}^\infty_y \Big/ \mathfrak{m}_p^m \mathcal{C}^\infty_y \;\cong\; \mathcal{C}^\infty_{X,y} \otimes_{\mathbb{C}} \left(\mathcal{S}_p \Big/ \mathfrak{m}_p^m \right). \tag{1.25}$$

But the latter $\mathcal{S}_p \big/ \mathfrak{m}_p^m$-module is flat, hence the claim follows. □

Now $\mathcal{B} \boxtimes_{(P,\mathcal{S})} \tilde{\mathcal{B}}$ is a locally free $\mathcal{C}^\infty_{Y \times_P \tilde{Y}}$-module sheaf, as $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are locally free over $\mathcal{C}^\infty_Y$ resp. $\mathcal{C}^\infty_{\tilde{Y}}$. But then the above lemma entails that $\mathcal{B} \boxtimes_{(P,\mathcal{S})} \tilde{\mathcal{B}}$ is flatly filtered over $\mathcal{S}$. Thus we can construct for two deformations D and $\tilde{D}$ their product $D \times \tilde{D}$ in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$. Together with the **null deformation** $D_{\mathrm{n}} = 1_{(P,\mathcal{S})} : (P, \mathcal{S}) \to (P, \mathcal{S})$ as unit we thus receive a tensor product $\times$ on $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$.

The same idea now leads to a product and a tensor product in the subcategory $\underline{\mathcal{C}^\infty\text{-}\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$ of $\underline{\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$ consisting of quantizations (D, Q) such that D lies in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Def}}^{filt.}_{*,(P,\mathcal{S})}$ and Q is a morphism in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$. From quantizations $Q = (q, \mathfrak{q}) : (Y, \mathcal{B}) \to (X, \mathcal{A})$ and $\tilde{Q} = (\tilde{q}, \tilde{\mathfrak{q}}) : (\tilde{Y}, \tilde{\mathcal{B}}) \to (\tilde{X}, \tilde{\mathcal{A}})$ in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$ over D resp. $\tilde{D}$ construct $Q \times \tilde{Q} : (Y, \mathcal{B}) \times_{(P,\mathcal{S})} (\tilde{Y}, \tilde{\mathcal{B}}) \to (X, \mathcal{A}) \times (\tilde{X}, \tilde{\mathcal{A}})$ and check that together with $D \times \tilde{D}$ this forms the claimed product. The product functor $\times$ in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Quant}}^{filt.}_{*,(P,\mathcal{S})}$ is also a tensor product with unit given by the **null quantization** $Q_{\mathrm{n}} : (P, \mathcal{S}) \to (*, \mathbb{C})$ over the null deformation.

In the same way like above we can construct the category $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$ of holomorphic ringed spaces and define the categories $\underline{\mathcal{O}\text{-}\mathsf{Def}}^{flat}_{*,(P,\mathcal{S})}$ and $\underline{\mathcal{O}\text{-}\mathsf{Quant}}^{flat}_{*,(P,\mathcal{S})}$ of flat deformations and quantizations in $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$. Like for the smooth case there exist canonical products respectively tensor products on $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$, $\underline{\mathcal{O}\text{-}\mathsf{Def}}^{flat}_{*,(P,\mathcal{S})}$ and $\underline{\mathcal{O}\text{-}\mathsf{Quant}}^{flat}_{*,(P,\mathcal{S})}$.

Remark 1.3.2 We have introduced the categories $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ and $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$ to have a notion of noncommutatively ringed spaces such that the structure sheaves have certain common features, namely are locally free modules of smooth resp. holomorphic functions. There are more and easier to carry out constructions - like for example the construction of a tensor product - in $\underline{\mathcal{C}^\infty\text{-}\mathsf{Esp}}$ and $\underline{\mathcal{O}\text{-}\mathsf{Esp}}$ than in the huge category of arbitrary ringed spaces.

Chapter 2

Quantization of a Cotangent Bundle

We regard a smooth or complex manifold X as the **configuration space** of a classical physical system and the symplectic manifold T^*X with its canonical symplectic structure as the **phase space** or **space of states** of the considered system. In the next step we then have to explain in mathematical terms what the observables of the system are. In our approach a certain class of **classical observables** is given by a sheaf $\mathcal{A}$ of commutative algebras on X consisting of smooth functions locally defined on T^*X such that the ringed space $(X, \mathcal{A})$ forms a Poisson space. The natural question which arises now is, whether the Poisson space $(X, \mathcal{A})$ can be deformed and quantized. In this chapter we will tackle that question and work out the above program for a special sheaf $\mathcal{A}$ on X, namely the one of polynomial observables on X.

2.1 Deformed polynomial observables

In this first section of chapter 2 we will show, how the sheaf of polynomial observables on the cotangent bundle T^*X can be deformed. The parameter space $(P, \mathcal{S})$ will be given by $P = \mathbb{R}$ or $\mathbb{C}$ and by the sheaf $\mathcal{S}$ of smooth resp. holomorphic functions on P.

So let X be differentiable (resp. complex) manifold, $\mathcal{E}$ the sheaf of smooth complex valued (resp. holomorphic) functions on X and $\mathcal{E}_{T^*X}(\nu)$ the sheaf $\mathcal{C}^\infty_{T^*X}(\nu)$ (resp. $\mathcal{O}_{T^*X}(\nu)$) on X consisting of complex valued smooth (resp. holomorphic) functions on T^*X which are ν-homogenous in the fibers of T^*X. In more detail, if $\pi : T^*X \to X$ denotes the canonical projection, then $\mathcal{E}(\nu)(U)$ is given by

$$\mathcal{E}(\nu)(U) = \{a \in \mathcal{E}(\pi^{-1}(U)) : a \text{ is } \nu\text{-homogeneous on every fiber}\}. \tag{2.1}$$

Definition 2.1.1 *Let $U \subset X$ be open and define $\mathcal{D}_0(U)$ to be the $\mathbb{C}$-algebra $\bigoplus_{\nu\in\mathbb{N}} \mathcal{E}(\nu)(U)$ of polynomial functions on $\pi^{-1}(U)$. It is called the* **algebra of classical polynomial observables** *over U. Variation of U over all open sets in X defines a sheaf $\mathcal{D}_0$ of $\mathbb{C}$-algebras on X. The ringed space $(X, \mathcal{D}_0)$ is the space of* **classical polynomial observables** *or of* **polynomial symbols** *on X.*

It is our goal to quantize $\mathcal{D}_0$ by a deformation through a complex parameter $\hbar$. To achieve that we first choose a coordinate patch U in X with coordinates $z = (z_1, ..., z_n) : U \to \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. They induce a chart $(z, \xi) = (z_1, ..., z_n, \xi_1, ..., \xi_n) : T^*U \to \mathbb{K}^{2n}$. In these local coordinates $\mathcal{D}_0(U)$ is isomorphic to the polynomial ring $\mathcal{E}(U)[\xi_1, ..., \xi_n]$ over $\mathcal{E}(U)$ in n variables.

Now one can define a chart dependent algebra $\mathcal{D}_\hbar(z)$ as the algebra of all sums $a = \sum_{\nu\in\mathbb{N}} a_\nu$ with the following properties:

(i) $a_\nu \in \mathcal{E}(\nu)(U)$ for all $\nu \in \mathbb{N}$.

(ii) There exists an integer ν_0 such that $a_\nu = 0$ for all $\nu \geq \nu_0$.

(iii) The multiplication is given by

$$\left(\sum_\nu a_\nu\right) \cdot_\hbar \left(\sum_\mu b_\mu\right) = \left(\sum_\rho c_\rho\right) \text{ with } c_\rho = \sum_{\nu+\mu-|\alpha|=\rho} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} a_\nu \frac{\partial^{|\alpha|}}{\partial z^\alpha} b_\mu.$$

Then $\mathcal{D}_\hbar(z)$ carries the structure of a $\mathbb{C}$-algebra such that the diagram

$$\begin{tikzcd} \mathcal{E}(U) \arrow[rr] & & \mathcal{D}_\hbar(z) \\ & \mathbb{C} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. Only the associativity of $\cdot_\hbar$ is not immediately clear, but follows from

$$\begin{aligned}\left(\sum_{\nu\in\mathbb{N}} a_\nu\right) \cdot_\hbar \left(\left(\sum_{\nu\in\mathbb{N}} b_\nu\right) \cdot_\hbar \left(\sum_{\nu\in\mathbb{N}} c_\nu\right)\right) &= \\ = \sum_\nu \sum_{\substack{\rho,\mu,\alpha \\ \rho+\mu-|\alpha|=\nu}} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} a_\rho \frac{\partial^{|\alpha|}}{\partial z^\alpha} &\left(\sum_{\sigma,\eta,\beta} \frac{(-i\hbar)^{|\beta|}}{\beta!} \frac{\partial^{|\beta|}}{\partial \xi^\beta} b_\sigma \frac{\partial^{|\beta|}}{\partial z^\beta} c_\eta\right)\end{aligned}$$

$$= \sum_{\nu} \sum_{\substack{\rho,\sigma,\eta,\beta,\gamma,\delta \\ \rho+\sigma+\eta-|\beta|-|\gamma|-|\delta|=\nu}} \frac{(-i\hbar)^{|\beta|+|\gamma|+|\delta|}}{\beta!\,\gamma!\,\delta!} \frac{\partial^{|\gamma+\delta|}}{\partial\xi^{\gamma+\delta}} a_\rho \frac{\partial^{|\gamma|}}{\partial z^\gamma} \frac{\partial^{|\beta|}}{\partial\xi^\beta} b_\sigma \frac{\partial^{|\beta+\delta|}}{\partial z^{\beta+\delta}} c_\eta \tag{2.2}$$

$$= \sum_{\nu} \sum_{\substack{\mu,\eta,\alpha \\ \mu+\eta-|\alpha|=\nu}} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial\xi^\alpha} \left(\sum_{\substack{\sigma,\rho,\gamma, \\ \sigma+\rho-|\gamma|=\mu}} \frac{(-i\hbar)^{|\gamma|}}{\gamma!} \frac{\partial^{|\gamma|}}{\partial\xi^\gamma} a_\rho \frac{\partial^{|\gamma|}}{\partial z^\gamma} b_\sigma \right) \frac{\partial^{|\alpha|}}{\partial z^\alpha} c_\eta$$

$$= \left(\left(\sum_{\nu\in\mathbb{N}} a_\nu \right) \cdot_\hbar \left(\sum_{\nu\in\mathbb{N}} b_\nu \right) \right) \cdot_\hbar \left(\sum_{\nu\in\mathbb{N}} c_\nu \right)$$

Additionally the following lemma holds.

Lemma 2.1.2 *Fix $\hbar \in \mathbb{C}$. Then $\mathcal{D}_\hbar(z)$ is isomorphic as $\mathbb{C}$-algebra to the $\mathbb{C}$-algebra generated by $\mathcal{E}(U)$ and the elements $\xi_1, ..., \xi_n$ modulo the relations*

$$\xi_k \cdot \xi_l = \xi_l \cdot \xi_k \quad \textit{and} \quad \xi_k \cdot f = \left(-i\hbar \frac{\partial}{\partial z_k} f \right) + f \cdot \xi_k, \tag{2.3}$$

where $f \in \mathcal{E}(U)$ and $1 \le k, l \le n$.

Proof: It is easy to see that $\mathcal{E}(U)$ and $\xi_1, ..., \xi_n$ generate $\mathcal{D}_\hbar(z)$ and that the relations (2.3), which will be denoted in the following by (R), hold. Therefore we have a surjective morphism $\mathbb{C} < \mathcal{E}(U), \xi_1, ..., \xi_n/R > \to \mathcal{D}_\hbar(z)$. On the other hand define $\mathcal{D}_\hbar(z) \to \mathbb{C} < \mathcal{E}(U), \xi_1, ..., \xi_n/R >$ by $\sum_{\nu\in N} a_\nu \mapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha \xi^\alpha$ where $a_\nu = \sum_{\alpha\in\mathbb{N}^n,\, |\alpha|=\nu} (a_\alpha \circ \pi)\, \xi^\alpha$ with $a_\alpha \in \mathcal{E}(U)$. By using the Leibniz rule one realizes that this mapping is in fact a morphism of $\mathbb{C}$-algebras and inverse to $\mathbb{C} < \mathcal{E}(U), \xi_1, ..., \xi_n/R > \longrightarrow \mathcal{D}_\hbar(U, z)$. □

Now assume $\tilde{z} : U \xrightarrow{\sim} \tilde{V} \subset \mathbb{K}^n$ to be another coordinate system on U. Then we have in $\mathcal{D}_\hbar(\tilde{z})$

$$\begin{aligned} \left(\sum_l \frac{\partial \tilde{z}_l}{\partial z_k} \tilde{\xi}_l \right) \cdot_\hbar f &= \sum_l \frac{\partial \tilde{z}_l}{\partial z_k} \left(-i\hbar \frac{\partial}{\partial \tilde{z}_l} f \right) + f \sum_l \frac{\partial \tilde{z}_l}{\partial z_k} \tilde{\xi}_l \\ &= -i\hbar \frac{\partial f}{\partial z_k} + f \sum_l \frac{\partial \tilde{z}_l}{\partial z_k} \tilde{\xi}_l \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \left(\sum_r \frac{\partial \tilde{z}_r}{\partial z_k} \tilde{\xi}_r \right) \cdot_\hbar \left(\sum_s \frac{\partial \tilde{z}_s}{\partial z_l} \tilde{\xi}_s \right) &= \sum_{r,s} \left(\frac{\partial \tilde{z}_r}{\partial z_k} \frac{\partial \tilde{z}_s}{\partial z_l} \right) \tilde{\xi}_r \, \tilde{\xi}_s + \sum_{r,s} \frac{\partial \tilde{z}_r}{\partial z_k} \left(-i\hbar \frac{\partial}{\partial \tilde{z}_r} \frac{\partial \tilde{z}_s}{\partial z_l} \right) \tilde{\xi}_s \\ &= \sum_{r,s} \left(\frac{\partial \tilde{z}_r}{\partial z_k} \frac{\partial \tilde{z}_s}{\partial z_l} \right) \tilde{\xi}_r \, \tilde{\xi}_s + \sum_s \left(-i\hbar \frac{\partial^2 \tilde{z}_s}{\partial z_k \partial z_l} \right) \tilde{\xi}_s \\ &= \left(\sum_s \frac{\partial \tilde{z}_s}{\partial z_l} \tilde{\xi}_s \right) \cdot_\hbar \left(\sum_r \frac{\partial \tilde{z}_r}{\partial z_k} \tilde{\xi}_r \right). \end{aligned} \tag{2.5}$$

Hence the next lemma follows easily.

Lemma 2.1.3 *There exists a unique homomorphism*

$$\mathcal{D}_\hbar(\tilde{z}, z) : \mathcal{D}_\hbar(z) \to \mathcal{D}_\hbar(\tilde{z}) \tag{2.6}$$

satisfying $\mathcal{D}_\hbar(\tilde{z}, z)(f) = f$ *for* $f \in \mathcal{E}(U)$ *and* $\mathcal{D}_\hbar(\tilde{z}, z)(\xi_k) = \sum_l \frac{\partial \tilde{z}_l}{\partial z_k} \tilde{\xi}_l$ *for* $k = 1, ..., n$. *The* $\mathcal{D}_\hbar(\tilde{z}, z)$ *are isomorphisms such that*

$$\mathcal{D}_\hbar(\hat{z}, z) = \mathcal{D}_\hbar(\hat{z}, \tilde{z}) \circ \mathcal{D}_\hbar(\tilde{z}, z) \quad \textit{and} \quad \mathcal{D}_\hbar(z, z) = id \tag{2.7}$$

holds for any additional coordinate map $\hat{z} : U \xrightarrow{\sim} \hat{V} \subset \mathbb{K}^n$.

PROOF: Equations (2.4) and (2.1) and Lemma 2.1.2 entail that $\mathcal{D}_\hbar(\tilde{z}, z)$ respects the relations defining $\mathcal{D}_\hbar(z)$, so $\mathcal{D}_\hbar(\tilde{z}, z)$ is well-defined. The rest is obvious. □

Note 2.1.4 Evaluation $\mathcal{D}_\hbar(z) \ni a \overset{e_\zeta}{\mapsto} a(\zeta) \in \mathbb{C}$ at $\zeta \in T^*U$ gives a linear functional on $\mathcal{D}_\hbar(z)$ which is a character iff $\hbar = 0$.

The proposition shows that one can introduce an equivalence relation between elements $\sum_\nu a_\nu \in \mathcal{D}_\hbar(z)$ and $\sum_\nu b_\nu \in \mathcal{D}_\hbar(\tilde{z})$ by $\sum_\nu a_\nu \sim \sum_\nu b_\nu$ iff $\mathcal{D}_\hbar(\tilde{z}, z)\left(\sum_\nu a_\nu\right) = \sum_\nu b_\nu$. As $\mathcal{D}_\hbar(\tilde{z}, z)$ is linear and multiplicative, the set $\mathcal{D}_\hbar(U)$ of equivalence classes $[\sum_\nu a_\nu]$ carries in a natural way the structure of a $\mathbb{C}$-algebra. It becomes an $\mathcal{E}(U)$-module by the algebra map $\mathcal{E}(U) \ni f \to [f] \in \mathcal{D}_\hbar(U)$. The $\mathcal{D}_\hbar(U)$ can be interpreted as spaces of sections of a sheaf $\mathcal{D}_\hbar$ of $\mathbb{C}$-algebras on X. The restriction morphisms are given by $\mathcal{D}_\hbar(U) \ni [\sum_\nu a_\nu] \to [\sum_\nu a_\nu|_V] \in \mathcal{D}_\hbar(V)$ for $V \subset U$ open. $\mathcal{D}_\hbar$ is unique up to isomorphism and carries the structure of an $\mathcal{E}_X$-module because the coordinate patches U form a basis of the topology of X. Furthermore $\mathcal{D}_0$ is isomorphic to the sheaf of classical polynomial observables defined in Definition 2.1.1.

Definition 2.1.5 *The ringed space* $(X, \mathcal{D}_\hbar)$ *is called the* **space of $\hbar$-deformed polynomial observables** *on* X.

If $\mathcal{V}$ is a smooth resp. holomorphic vector field over U, we want to assign to $\mathcal{V}$ an element $\mathcal{V}^\hbar \in \mathcal{D}_\hbar(U)$ such that for any additional smooth resp. holomorphic vector field $\mathcal{W}$ over U the equation

$$\mathcal{V}^\hbar \cdot \mathcal{W}^\hbar - \mathcal{W}^\hbar \cdot \mathcal{V}^\hbar = -i\hbar\, [\mathcal{V}\, ,\, \mathcal{W}]^\hbar \tag{2.8}$$

is valid in $\mathcal{D}_\hbar(U)$. Writing $\mathcal{V} = \sum\, v_k\, \frac{\partial}{\partial z_k}$ define $\mathcal{V}^\hbar$ to be the equivalence class of $\sum\, i\, v_k\, \xi_k \in \mathcal{D}_\hbar(U)$. It is obvious by the definition of $\mathcal{D}_\hbar(U)$ that $\mathcal{V}^\hbar$ is independent of the specially chosen coordinate system z. If $\mathcal{W} = \sum\, w_k\, \frac{\partial}{\partial z_k}$, we have in the Lie algebra of vector fields

$$[\mathcal{V}\,,\,\mathcal{W}] \;=\; \sum_{k,l} \left(v_k\, \frac{\partial w_l}{\partial z_k} - w_k\, \frac{\partial v_l}{\partial z_k} \right) \frac{\partial}{\partial z_l} \tag{2.9}$$

and in $\mathcal{D}_\hbar(z)$

$$\left(\sum_k\, i\, v_k\, \xi_k \right) \cdot \left(\sum_l\, i\, w_l\, \xi_l \right) - \left(\sum_k\, i\, w_k\, \xi_k \right) \cdot \left(\sum_l\, i\, v_l\, \xi_l \right) = -i\hbar \sum_{k,l} i\, \left(v_k \frac{\partial w_l}{\partial z_k} - w_k\, \frac{\partial v_l}{\partial z_k} \right) \xi_l. \tag{2.10}$$

Hence equation (2.8) follows.

The generators $\xi_1, ..., \xi_n$ of $\mathcal{D}_\hbar(z)$ correspond to the elements $-i\frac{\partial^\hbar}{\partial z_1}, ..., -i\frac{\partial^\hbar}{\partial z_n} \in \mathcal{D}_\hbar(U)$. Now if $a = \sum\limits_\nu a_\nu$ is a sum of fiberwise ν-homogeneous holomorphic functions a_ν over $\pi^{-1}(U)$, we therefore sometimes write $a(-i\frac{\partial^\hbar}{\partial z}) = \sum\limits_\nu a_\nu(-i\frac{\partial^\hbar}{\partial z})$ for the equivalence class in $\mathcal{D}_\hbar(U)$ defined by a . Furthermore we denote by $\frac{\partial^{\hbar\,|\alpha|}}{\partial z^\alpha}$, $\alpha \in \mathbb{N}^n$ the element $\left(\frac{\partial^\hbar}{\partial z_1}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{\partial^\hbar}{\partial z_n}\right)^{\alpha_n} \in \mathcal{D}_\hbar(U)$.

Let $(e_1, ..., e_n)$ be a frame of TX over $U \subset X$. Then the vector fields $\frac{\partial}{\partial z_k}$, $k = 1, ..., n$ can be expanded as linear combinations of $(e_1, ..., e_n)$. Hence the elements $(e_1^\hbar, ..., e_n^\hbar)$ together with $\mathcal{E}(U)$ generate the algebra $\mathcal{D}_\hbar(U)$. Note that because of equation (2.8) the commutator $e_k^\hbar \cdot e_l^\hbar - e_l^\hbar \cdot e_k^\hbar$ is in general not zero for $k \neq l$.

Varying the parameter $\hbar$ over $\mathbb{C}$, the $\mathcal{D}_\hbar$ shall be interpreted as fibers of a certain deformation $(X \times \mathbb{C}, \mathcal{D}_{\bar{d}}) \to (\mathbb{C}, \mathcal{E}_\mathbb{C})$. The sheaf $\mathcal{D}_{\bar{d}}$ is defined locally in coordinates. First let $\bar{d} : X \times \mathbb{C} \to \mathbb{C}$ be the canonical projection and consider the pull-back bundle $\bar{d}^*\, T^*X = T^*X \times \mathbb{C}$. Then we have the sheaf $\mathcal{E}(\bar{d}^*\, T^*X)$ of smooth resp. holomorphic functions on $\bar{d}^*\, T^*X$ and the sheaves $\mathcal{E}_{\bar{d}^*\, T^*X}(\nu) = \mathcal{E}(\nu)$ on $X \times \mathbb{C}$ of in the fibers ν-homogeneous functions on $\bar{d}^*\, T^*X$. For open O in $\mathbb{C}$ and coordinate patches U in X, $\mathcal{D}_{\bar{d}}(z, O)$ shall now be the $\mathbb{C}$-algebra of sums $p = \sum\limits_\nu p_\nu\,,\;\; p_\nu \in \mathcal{E}(T^*U \times O)$ such that the following conditions hold:

(i) For every fixed $\hbar \in O$ and every $x \in U$ the restriction of $p_\nu(-, \hbar)$ to T_x^*X is ν-homogeneous. In other words $p_\nu \in \mathcal{E}(\nu)(U \times O)$ for all $\nu \in \mathbb{N}$.

(ii) There exists an integer ν_0 such that $p_\nu = 0$ for all $\nu \geq \nu_0$.

(*iii*) The multiplication is given by

$$\left(\sum_\nu p_\nu\right) \cdot \left(\sum_\mu q_\mu\right) = \left(\sum_\rho r_\rho\right)$$

with

$$r_\rho(-,\hbar) = \sum_{\nu+\mu-|\alpha|=\rho} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} p_\nu(-,\hbar) \frac{\partial^{|\alpha|}}{\partial z^\alpha} q_\mu(-,\hbar), \quad \hbar \in O.$$

The associativity of the product in (iii) is shown by equation (2.2). Reasoning like for $\mathcal{D}_\hbar$ we can construct transition homomorphisms $\mathcal{D}_{\bar{d}}(\tilde{z}, z, O) : \mathcal{D}_{\bar{d}}(z, O) \to \mathcal{D}_{\bar{d}}(\tilde{z}, O)$ which give rise to a congruence relation between elements of $\mathcal{D}_{\bar{d}}(z, O)$ and $\mathcal{D}_{\bar{d}}(\tilde{z}, O)$. The algebra $\mathcal{D}_{\bar{d}}(U \times O)$ of equivalence classes is isomorphic to $\mathcal{D}_{\bar{d}}(z, O)$. If we vary U over all coordinate patches in X and O over all open sets in $\mathbb{C}$, we finally receive a sheaf $\mathcal{D}_{\bar{d}}$ of noncommutative $\mathbb{C}$-algebras on $X \times \mathbb{C}$, called the sheaf of **universally deformed polynomial observables** on X.

Denoting by $\bar{d}$ the canonical projection $\bar{d} : X \times \mathbb{C} \to \mathbb{C}$ we want to associate to any smooth resp. holomorphic vector field $\mathcal{V}$ over $U \subset X$ an element $\mathcal{V}^{\bar{d}}$ in $\mathcal{D}_{\bar{d}}(U \times \mathbb{C})$ such that for any second vector field $\mathcal{W}$ over U the equation

$$\mathcal{V}^{\bar{d}} \cdot \mathcal{W}^{\bar{d}} - \mathcal{W}^{\bar{d}} \cdot \mathcal{V}^{\bar{d}} = -i\bar{d}\, [\mathcal{V}\,,\, \mathcal{W}]^{\bar{d}} \tag{2.11}$$

holds in $\mathcal{D}_{\bar{d}}(U \times \mathbb{C})$. But this is accomplished exactly like in the definition of $\mathcal{V}^\hbar \in \mathcal{D}_\hbar(U)$. Express $\mathcal{V} = \sum v_k \frac{\partial}{\partial z_k}$ in terms of local coordinates and let $\mathcal{V}^{\bar{d}}$ be the equivalence class of $\sum i\, v_k\, (\xi_k \circ q) \in \mathcal{D}_{\bar{d}}(z, \mathbb{C})$ in $\mathcal{D}_{\bar{d}}(U \times \mathbb{C})$, where $q : X \times \mathbb{C} \to X$ is the projection on the first coordinate. This $\mathcal{V}^{\bar{d}}$ fulfills (2.11).

As $-i\frac{\partial^{\bar{d}}}{\partial z_k}$ is the equivalence class of ξ_k in $\mathcal{D}_{\bar{d}}(U \times O)$, we denote by $p\left(-i\frac{\partial^{\bar{d}}}{\partial z}\right)$ the equivalence class of any sum $p = \sum p_\nu \in \mathcal{D}_{\bar{d}}(z, O)$ of fiberwise ν-homogeneous holomorphic functions p_ν.

The commutative ringed space $(\mathbb{C}, \mathcal{E})$ can be embedded into the center of $\mathcal{D}_{\bar{d}}$ by $\mathcal{E}(O) \ni b \mapsto \Delta(b) = b \circ \bar{d} \in \mathcal{D}_{\bar{d}}(X \times O)$, $O \subset \mathbb{C}$. Together with the canonical projection $\bar{d} : X \times \mathbb{C} \to \mathbb{C}$ this defines a morphism $D = (\bar{d}, \Delta) : (X \times \mathbb{C}, \mathcal{D}_{\bar{d}}) \to (\mathbb{C}, \mathcal{E}_\mathbb{C})$.

Theorem 2.1.6 *Let X be a smooth resp. complex manifold and $\mathcal{E}_\mathbb{C}$ the sheaf of smooth resp. holomorphic complex valued functions on $\mathbb{C}$. Then the morphism $D = (\bar{d}, \Delta) : (X \times \mathbb{C}, \mathcal{D}_{\bar{d}}) \to (\mathbb{C}, \mathcal{E}_\mathbb{C})$ defined on the space of universally deformed polynomial observables on X is fibered and for any $\hbar \in \mathbb{C}$ a flatly filtered deformation of $(X, \mathcal{D}_\hbar)$ over $(\mathbb{C}, \mathcal{E}_\mathbb{C})$ with distinguished point $\hbar$. In the complex case D even gives rise to a flat deformation.*

PROOF: By definition it is clear that (D, Δ) is fibered. To show that it is a deformation, let us first regard the complex case. By the definition of the product in $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$ and the product in $(\mathcal{D}_\hbar)_x$ it is obvious that $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} / \mathfrak{m}_\hbar (\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} \cong (\mathcal{D}_\hbar)_x$ for all $x \in X$. It remains to prove the flatness of $\mathcal{E}_\hbar \to (\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$. Now as $\mathcal{E}_\hbar$-module, $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$ is isomorphic to $\mathcal{E}_{n+1} \otimes \mathbb{C}[\xi_1, ..., \xi_n]$, where the last variable of $\mathcal{E}_{n+1}$ is $\hbar$. But $\mathcal{E}_{n+1} \otimes \mathbb{C}[\xi_1, ..., \xi_n]$ is flat as an $\mathcal{E}_\hbar$-module, hence also $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$. This proves (D, Δ) being a flat deformation.

Next we have to show for complex and differentiable X that (D, Δ) is a flatly filtered deformation. We only prove this claim in the differentiable case, as the considerations for complex X are analogous. Now let $(\mathcal{D}_{\hbar\!\!\!/})^m_{(x,\hbar)}$, $m \in \mathbb{N}$ be the right ideal in $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$ consisting of all germs of sums $\sum_{|\alpha| \leq \nu} p_\alpha \xi^\alpha$, $\nu \in \mathbb{N}$ such that p_α is a smooth function defined on a neighborhood of $(x, \hbar)$ and $\left[\frac{\partial^\mu \partial^{|\beta|} p_\alpha}{\partial \hbar^\mu \partial z^\beta}\right](x, \hbar) = 0$ for all α and all $\beta \in \mathbb{N}^n$, $\mu \in \mathbb{N}$ with $\mu + |\beta| \leq m$. Then $\left((\mathcal{D}_{\hbar\!\!\!/})^m_{(x,\hbar)}\right)_{m \in \mathbb{N}}$ gives a filtration of $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$. Furthermore $\Delta(\mathfrak{m}_\hbar^m) \subset (\mathcal{D}_{\hbar\!\!\!/})^m_{(x,\hbar)}$, where $\mathfrak{m}_\hbar$ is the maximal ideal in $\mathcal{S}_\hbar$. The right ideal $(\mathcal{D}_{\hbar\!\!\!/})^\infty_{(x,\hbar)} = \bigcap_{m \in \mathbb{N}} (\mathcal{D}_{\hbar\!\!\!/})^m_{(x,\hbar)}$ is also a left ideal in $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)}$ and consists of all germs of sums $\sum_\alpha p_\alpha \, \xi^\alpha$ such that $\left[\frac{\partial^\mu \partial^{|\beta|} p_\alpha}{\partial \hbar^\mu \partial z^\beta}\right](x, \hbar) = 0$ for all α, all $\mu \in \mathbb{N}$ and all $\beta \in \mathbb{N}^n$. By BOREL's Theorem $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} / (\mathcal{D}_{\hbar\!\!\!/})^\infty_{(x,\hbar)}$ is isomorphic to $\mathbb{C}[[x_1, ..., x_n, \hbar]][\xi_1, ..., \xi_n]$ which is obviously flat over $\mathcal{S}_\hbar / \mathfrak{m}_\hbar^\infty \cong \mathbb{C}[[\hbar]]$. Hence Theorem 1. in BOURBAKI [9], *Commutative Algebra*, Chapter II §5.2 implies that $(\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} / \mathfrak{m}_\hbar^m (\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} \cong \left((\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} / (\mathcal{D}_{\hbar\!\!\!/})^\infty_{(x,\hbar)}\right) \Big/ \left(\mathfrak{m}_\hbar^m (\mathcal{D}_{\hbar\!\!\!/})_{(x,\hbar)} / (\mathcal{D}_{\hbar\!\!\!/})^\infty_{(x,\hbar)}\right)$ is a flat $\mathcal{S}_\hbar / \mathfrak{m}_\hbar^m \cong (\mathcal{S}_\hbar / \mathfrak{m}_\hbar^\infty) / (\mathfrak{m}_\hbar^m / \mathfrak{m}_\hbar^\infty)$ module. The claim follows. $\square$

2.2 Quantized polynomial observables

After having defined a deformation of $(X, \mathcal{D}_0)$ we will now construct a quantization of $(X, \mathcal{D}_0)$, the so called normal order quantization. This method works for every smooth paracompact X and for every complex manifold for which a connection with holomorphic CHRISTOFFEL symbols exists. It will be obvious by construction that the quantization is dependent on the choice of a connection on X.

Choose a smooth connection ∇ (resp. a connection ∇ with holomorphic CHRISTOFFEL symbols) on X and around any $x \in X$ normal coordinates $\exp_x : K_x \to B_x$, where K_x is an open neighborhood of 0 in $T_x X$ and B_x one of x in X. The exponential map induces charts $T\exp_x : K_x \times T_x X \cong TK_x \to TB_x \subset TX$ and $T^*\exp_x : K_x \times T_x^* X \cong T^* K_x \to T^* B_x \subset T^* X$.

Now let a be any sum $a = \sum_\nu a_\nu \in \mathcal{D}_0(U)$, where $U \subset X$ is open. Let $\tilde{x} \in U$ and $z : V_{\tilde{x}} \to \mathbb{K}^n$ be a coordinate system on the open neighborhood $V_{\tilde{x}}$ of $\tilde{x}$ in U. After possibly shrinking $V_{\tilde{x}}$ we can assume that the mappings $\exp_x^{-1} : V_{\tilde{x}} \to T_x^*X$ are well-defined for every $x \in V_{\tilde{x}}$. Furthermore we can assume that on $V_{\tilde{x}}$ there exists a smooth resp. holomorphic covariant frame $e_1, ..., e_n$ on TX, i.e. $\nabla e_k = 0$ for $k = 1, ..., n$. Then we can define the normal coordinates $z_x : V_{\tilde{x}} \to \mathbb{K}^n$ for $x \in V_{\tilde{x}}$ by $\exp_x^{-1} = \sum_{k=1}^n z_{x,k} \cdot e_k(x)$. Now we have the means to define the function $a^{\mathrm{n},z} : (V_{\tilde{x}} \times T^*V_{\tilde{x}} \times \mathbb{C}) \to \mathbb{C}$ by

$$a^{\mathrm{n},z}(x, \xi, \hbar) \;=\; \mathcal{D}_\hbar(z, z_x)\,(a) \tag{2.12}$$

for $x \in V_{\tilde{x}}$, $\xi \in T^*V_{\tilde{x}}$ and $\hbar \in \mathbb{C}$. $a^{\mathrm{n},z}$ obviously does not depend on the specially chosen covariant frame $e_1, ..., e_n$ over $V_{\tilde{x}}$. Furthermore by Lemma 2.1.3, LEIBNIZ rule and the chain rule $a^{\mathrm{n},z}$ is a smooth resp. holomorphic function, because $V_{\tilde{x}} \times V_{\tilde{x}} \ni (x, y) \mapsto \exp_x^{-1}(y) \in T^*X$ is smooth resp. holomorphic both in x and y. Finally set

$$a^z : T^*V_{\tilde{x}} \times \mathbb{C} \to \mathbb{C}, \quad (\xi, \hbar) \mapsto a^{\mathrm{n},z}(\pi(\xi), \xi, \hbar). \tag{2.13}$$

Then by running $\tilde{x}$ through U and choosing appropriate coordinate systems $z : V_{\tilde{x}} \to \mathbb{K}^n$ in the above sense, we can now define the **operator of normal order quantization** $\mathfrak{q}_U(a) \in \mathcal{D}_{\bar{d}}(U)$ of a by

$$\mathfrak{q}_U(a)\Big|_{T^*V_{\tilde{x}}} \;=\; a^z\left(-i\frac{\partial^{\bar{d}}}{\partial z}\right). \tag{2.14}$$

Confer 2.2.6 for a justification of the term normal order quantization. To show that $\mathfrak{q}_U(a)$ is well-defined, we have to prove that $\mathcal{D}_{\bar{d}}(\tilde{z}, z, O)\, a^z = a^{\tilde{z}}$ for any second coordinate system $\tilde{z} : V_{\tilde{x}} \to \mathbb{K}^n$. But by Eq. (2.13) we have

$$a^z|_{T_x^*U\times\mathbb{C}} \;=\; a^{\mathrm{n},z}(x, -, \cdot)|_{T_x^*U\times\mathbb{C}} \;=\; \mathcal{D}_{\bar{d}}(z, z_x, \mathbb{C})\,(a \circ q)|_{T_x^*U\times\mathbb{C}}, \tag{2.15}$$

where $q : T^*X \times \mathbb{C} \to T^*X$ is the projection on the first coordinate. Therefore the equation

$$\begin{aligned} &\mathcal{D}_{\bar{d}}(\tilde{z}, z, \mathbb{C}) a^z|_{T_x^*U\times\mathbb{C}} \\ &= \mathcal{D}_{\bar{d}}(\tilde{z}, z, \mathbb{C})\,\mathcal{D}_{\bar{d}}(z, z_x, \mathbb{C})\,(a \circ q)|_{T_x^*U\times\mathbb{C}} \;=\; \mathcal{D}_{\bar{d}}(\tilde{z}, z_x, \mathbb{C})\,(a \circ q)|_{T_x^*U\times\mathbb{C}} \\ &= a^{\tilde{z}}|_{T_x^*U\times\mathbb{C}} \end{aligned} \tag{2.16}$$

holds. This proves the claim.

We need a further lemma for later considerations about the quantization map $\mathfrak{q}_U$.

Lemma 2.2.1 *Let $a \in \mathcal{D}_0(U)$ be a monomial in the ξ-s of the form $a = (f \circ \pi) \cdot \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$. Then a^z can be written as a polynomial*

$$a^z(-, \hbar) = \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leq \alpha}} f_\beta(\pi(-))\, \hbar^{|\alpha-\beta|} \cdot \xi^\beta(-), \tag{2.17}$$

where the $f_\beta : U \to \mathbb{C}$ are smooth (resp. holomorphic) and $f_\alpha(\cdot) = f(\cdot)$.

PROOF: Use induction on the order $m = |\alpha| = \alpha_1 + \ldots + \alpha_n$.

If $a = f \circ \pi$ it is clear that $a^z(-, \hbar) = f \circ \pi(-)$, so a^z is smooth (resp. holomorphic) in all variables and a polynomial of order 0 in every fiber. Now assume that any monomial $b = (f \circ \pi) \cdot \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n} \in \mathcal{D}_0$ of order m is mapped to a b^z of the form $b^z(-, \hbar) = \sum_{\beta \in \mathbb{N}^n, \beta \leq \alpha} g_\beta(\pi(-)) \hbar^{|\alpha-\beta|} \cdot \xi_1^{\beta_1} \cdot \ldots \cdot \xi_n^{\beta_n}$ with smooth (resp. holomorphic) g_β and $g_\alpha(\cdot) = f(\cdot)$. Let $a = (f \circ \pi) \cdot \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n} \cdot \xi_i$ be of order $m+1$. Then for all $x \in U$

$$a = \sum_{1 \leq k \leq n} \frac{\partial z_{x,k}}{\partial z_i}\, b \cdot \xi_{z_{x,k}} \quad . \tag{2.18}$$

Applying $\mathcal{D}_{\bar{d}}(z, z_x, \mathbb{C})$ and evaluating at $(-, \hbar) \in T_x^*X \times \mathbb{C}$ gives

$$\begin{aligned} a^z(-, \hbar) \;&=\; \sum_{1 \leq k \leq n} \frac{\partial z_{\pi(-),k}}{\partial z_i}(\pi(-)) \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leq \alpha}} \sum_{\substack{\gamma \in \mathbb{N}^n \\ \gamma \leq \beta}} \sum_{1 \leq l \leq n} \frac{(-i\hbar)^{|\gamma|}}{\gamma!}\, g_\beta(\pi(-))\, \hbar^{|\alpha-\beta|} \cdot \\ &\quad \cdot \left(\frac{\partial^{|\gamma|}}{\partial z^\gamma} \frac{\partial z_l}{\partial z_{\pi(-),k}} \right)(\pi(-)) \cdot \xi^{\beta-\gamma}(-) \cdot \xi_l(-) \\ &=\; f(\pi(-))\, \xi^{(\alpha_1, \ldots, \alpha_i+1, \ldots, \alpha_n)}(-) + \sum_{|\delta| \leq |\alpha|} f_\delta(\pi(-))\, \hbar^{|\alpha-\delta|+1} \cdot \xi^\delta(-) \end{aligned} \tag{2.19}$$

So a^z is also smooth (resp. holomorphic) and a polynomial of order $\leq m+1$ with respect to the ξ-s. Furthermore equation (2.19) shows that the polynomial expansion of a^z has the required form. This proves the lemma. □

Now we have enough tools to formulate and prove the following theorem.

Theorem 2.2.2 *Let X be a smooth resp. complex manifold, ∇ a connection on X with holomorphic* CHRISTOFFEL *symbols in the complex case and $\mathcal{E}_\mathbb{C}$ the sheaf of smooth resp. holomorphic functions on $\mathbb{C}$. Associate to X the sheaves $\mathcal{D}_0$ and $\mathcal{D}_{\bar{d}}$ of classical and universally deformed polynomial observables on X. Then the operator $\mathfrak{q}_U : \mathcal{D}_0 \to \mathcal{D}_{\bar{d}}(U \times \mathbb{C})$ of normal order quantization is linear and gives rise to a linear sheaf morphism $\mathfrak{q} : \mathcal{D}_0 \to q_* \mathcal{D}_{\bar{d}}$ over $q = pr_1 : X \times \mathbb{C} \to X$. Furthermore $Q = (q, \mathfrak{q}) : (X \times \mathbb{C}, \mathcal{D}_{\bar{d}}) :\to (X, \mathcal{D}_0)$ is a quantization of $(X, \mathcal{D}_0)$ over $D = (\bar{d}, \Delta) : (X \times \mathbb{C}, \mathcal{D}_{\bar{d}}) \to (\mathbb{C}, \mathcal{E}_\mathbb{C})$ and is called the* **normal order quantization** *of $(X, \mathcal{D}_0)$ with respect to ∇.*

PROOF: The $\mathfrak{q}_U$ are linear by definition and coordinate independent according to our above considerations. As they obviously commute with restriction morphisms, the $\mathfrak{q}_U$ uniquely define a sheaf morphism $\mathfrak{q} : \mathcal{D}_0 \to q_* \mathcal{D}_{\bar{d}}$.

To verify that $Q = (q, \mathfrak{q})$ indeed is a quantization over D, we have to show $\mathfrak{j} \circ \mathfrak{q} = id_{\mathcal{D}_0}$. Here $\mathfrak{j} : \mathcal{D}_{\bar{d}} \to j_* \mathcal{D}$ is the sheaf morphism over $j : X \to X \times \mathbb{C}$, $x \mapsto (x, 0)$ mapping $(\mathcal{D}_{\bar{d}})_{(x,0)}$ to the fiber $(\mathcal{D}_{\bar{d}})_{(x,0)} / \mathfrak{m}_0 \, (\mathcal{D}_{\bar{d}})_{(x,0)} \cong (\mathcal{D}_0)_x$ for every $x \in X$. We prove the relation $\mathfrak{j} \circ \mathfrak{q} = id_{\mathcal{D}_0}$ by inductively showing $\mathfrak{j} \circ \mathfrak{q}\,(a) = a$ for every monomial in the ξ-s. If $a = f \circ \pi$, we obviously have $\mathfrak{j} \circ \mathfrak{q}\,(f \circ \pi) = \mathfrak{j}(f \circ q) = f \circ q \circ j = f \circ \pi$. In case $\mathfrak{j} \circ \mathfrak{q}(b) = b$ for $b = (f \circ \pi) \cdot \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n} \in \mathcal{D}_0$ of order m, equation (2.19) entails for $a = b \cdot \xi_i$ on $T_x^* X$

$$
\begin{aligned}
a^z(-, 0) &= \sum_{1 \le k \le n} \frac{\partial z_{x,k}}{\partial z_i}(\pi(-)) \cdot \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \le m}} \sum_{1 \le l \le n} g_\beta(\pi(-), 0) \cdot \frac{\partial z_l}{\partial z_{x,k}}(\pi(-)) \cdot \xi^\beta(-) \cdot \xi_l(-) \\
&= b(-) \cdot \xi_i(-) \; = \; F(-) \,, \qquad (2.20)
\end{aligned}
$$

so the relation $\mathfrak{j} \circ \mathfrak{q}\,(a) = a$ is shown for all monomials of order $m + 1$. $\square$

Corollary 2.2.3 *Associate to the cotangent bundle $T^* X$ with its canonical symplectic structure the* POISSON *space $(X, \mathcal{D}_0)$ with* POISSON *bracket $\{\,,\,\}$ like in example* 1.2.4 *ii). Then the normal order quantization Q is even a differentiable (resp. holomorphic)* DIRAC *quantization, that is for any pair $a, b \in \mathcal{D}_0(U)$, $U \subset X$ open there exists an element $r_{a,b} \in \mathcal{D}_{\bar{d}}(U \times \mathbb{C})$ such that*

$$
[\mathfrak{q}(a), \mathfrak{q}(b)] = -i \, \bar{d} \, \mathfrak{q}(\{a, b\}) \; + \; \bar{d}^2 \, r_{a,b}. \qquad (2.21)
$$

PROOF: Let $a = (f \circ \pi) \, \xi^\alpha$, $b = (g \circ \pi) \, \xi^\beta$ be monomials of order l respectively m. Then

$$
\begin{aligned}
a^z(-, \hbar) &= (f \circ \pi)(-) \, \xi^\alpha(-) + \sum_{\substack{\tilde{\alpha} \le \alpha \\ \tilde{\alpha} \ne \alpha}} f_{\tilde{\alpha}}(\pi(-)) \, \hbar^{|\alpha - \tilde{\alpha}|} \, \xi^{\tilde{\alpha}}(-), \qquad (2.22) \\
b^z(-, \hbar) &= (g \circ \pi)(-) \, \xi^\beta(-) + \sum_{\substack{\tilde{\beta} \le \beta \\ \tilde{\beta} \ne \beta}} g_{\tilde{\beta}}(\pi(-)) \, \hbar^{|\beta - \tilde{\beta}|} \, \xi^{\tilde{\beta}}(-). \qquad (2.23)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathfrak{q}(a) &= (f \circ \pi) \, (-i)^{|\alpha|} \, \frac{\partial^{\bar{d}\,|\alpha|}}{\partial z^\alpha} + \bar{d} \cdot P(-i \frac{\partial^{\bar{d}}}{\partial z}), \qquad (2.24) \\
\mathfrak{q}(b) &= (g \circ \pi) \, (-i)^{|\beta|} \, \frac{\partial^{\bar{d}\,|\beta|}}{\partial z^\beta} + \bar{d} \cdot Q(-i \frac{\partial^{\bar{d}}}{\partial z}) \qquad (2.25)
\end{aligned}
$$

with P, Q fiberwise polynomial functions on $\pi^{-1}(U)$. But now in $\mathcal{D}_{\hbar}(U \times \mathbb{C})$

$$\begin{aligned}
\left[(f \circ \pi)\,(-i)^{|\alpha|}\,\frac{\partial^{\hbar|\alpha|}}{\partial z^{\alpha}}\,,\,(g \circ \pi)\,(-i)^{|\beta|}\,\frac{\partial^{\hbar|\beta|}}{\partial z^{\beta}}\right] &= -i\,\hbar\,\left\{(f \circ \pi)\,\xi^{\alpha}\,,\,(g \circ \pi)\,\xi^{\beta}\right\}\left(-i\frac{\partial^{\hbar}}{\partial z}\right) \\
&\quad + \hbar^2 \cdot r_{f,g}, \qquad (2.26)\\
\left[\hbar \cdot P\left(-i\frac{\partial^{\hbar}}{\partial z}\right)\,,\,(g \circ \pi)\,(-i)^{|\beta|}\,\frac{\partial^{\hbar|\beta|}}{\partial z^{\beta}}\right] &= \hbar^2 \cdot r_{P,g}, \qquad (2.27)\\
\left[(f \circ \pi)\,(-i)^{|\alpha|}\,\frac{\partial^{\hbar|\alpha|}}{\partial z^{\alpha}}\,,\,\hbar \cdot Q\left(-i\frac{\partial^{\hbar}}{\partial z}\right)\right] &= \hbar^2 \cdot r_{f,Q}, \qquad (2.28)\\
\left[\hbar \cdot P\left(-i\frac{\partial^{\hbar}}{\partial z}\right)\,,\,\hbar \cdot Q\left(-i\frac{\partial^{\hbar}}{\partial z}\right)\right] &= \hbar^2 \cdot r_{P,Q}, \qquad (2.29)
\end{aligned}$$

where $r_{f,g}$, $r_{P,g}$, $r_{f,Q}$, $r_{P,Q}$ are appropriate elements of $\mathcal{D}_{\hbar}(U \times \mathbb{C})$. As furthermore

$$\mathfrak{q}(\{a, b\}) = \left\{(f \circ \pi)\,\xi^{\alpha}\,,\,(g \circ \pi)\,\xi^{\beta}\right\}\left(-i\frac{\partial^{\hbar}}{\partial z}\right) + \hbar \cdot s_{a,b} \qquad (2.30)$$

with $s_{a,b} \in \mathcal{D}_{\hbar}(U \times \mathbb{C})$, equations (2.26) to (2.29) entail the expansion in (2.21). □

The normal order quantization has the nice feature of being functorial. This is expressed in the following proposition.

Theorem 2.2.4 *Associate to any* RIEMANNIAN *manifold X its normal order quantization $Q = (q, \mathfrak{q}) : (X \times \mathbb{C}, \mathcal{D}_{\hbar}) \to (X, \mathcal{D}_0)$ of the* POISSON *space $(X, \mathcal{D}_0)$ with respect to the* LEVI-CIVITA *connection ∇. This gives rise to a functor Q_{pol} from the category $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ of* RIEMANNIAN *manifolds with isometric embeddings as morphisms to the category $\underline{\mathsf{Quant}}^{diff.}_{Dirac}$ of differentiable* DIRAC *quantizations. Q_{pol} has range in $\underline{\mathcal{C}^{\infty}\text{-}\mathsf{Quant}}^{filt.}_{0,(\mathbb{R},\mathcal{C}^{\infty}_{\mathbb{R}})}$ and preserves the monoidal structures by mapping products in $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to (tensor) products in $\underline{\mathcal{C}^{\infty}\text{-}\mathsf{Quant}}^{filt.}_{0,(\mathbb{R},\mathcal{C}^{\infty}_{\mathbb{R}})}$.*

PROOF: We postpone the proof till the end of the next section, where we will provide for some mathematical tools giving a short proof of the theorem. □

2.2.5 An element $P = p\left(-i\frac{\partial^{\hbar}}{\partial z}\right) \in \mathcal{D}_{\hbar}(U \times O)$ is already characterized by all its projections

$$P_{\hbar} := p_{\hbar}\left(-i\frac{\partial^{\hbar}}{\partial z}\right) := p\left(-i\frac{\partial^{\hbar}}{\partial z}, \hbar\right) \in \mathcal{D}_{\hbar}(U \times \{\hbar\})/\mathfrak{m}_{\hbar}\mathcal{D}_{\hbar}(U \times \{\hbar\}) \cong \mathcal{D}_{\hbar}(U) \qquad (2.31)$$

with $\hbar \in O$ and $p_\hbar(-) = p(-,\hbar) : T^*U \to \mathbb{C}$. We will write $\mathfrak{q}_\hbar$ for the composition of $\mathfrak{q}$ with the projection of $\mathcal{D}_{\dj}$ on $\mathcal{D}_\hbar$. In that notation equation (2.21) then reads

$$[\mathfrak{q}_\hbar(a), \mathfrak{q}_\hbar(b)] \; = \; -i\,\hbar\,\mathfrak{q}_\hbar(\{a,b\}) \; + \; \hbar^2\, r_{a,b,\hbar} \tag{2.32}$$

for every $\hbar \in \mathbb{C}$. Up to first order in $\hbar$ this is the standard DIRAC quantization condition.

2.2.6 Let us briefly motivate that the morphism $\mathfrak{q}$ corresponds to what physicists call a normal order quantization. In case $X \cong \mathbb{R}^n$, we have a global and flat coordinate system $(z,\xi) : T^*X \to \mathbb{R}^{2n}$ of T^*X and

$$\mathfrak{q}\left((f \circ \pi) \cdot \xi^\alpha\right) = (-i\,\dj)^{|\alpha|}\, f\, \frac{\partial^{\dj\,|\alpha|}}{\partial z^\alpha} \tag{2.33}$$

for $f \in \mathcal{E}(U)$, $U \in X$ open. In particular $\mathfrak{q}(z_k \cdot \xi_l) = (-i\dj)\, z_k\, \frac{\partial^{d}}{\partial z_l} = \mathfrak{q}(z_k) \cdot \mathfrak{q}(\xi_l)$, which shows that the quantization map $\mathfrak{q}$ normally orders the coordinate functions z_k and ξ_l. In other words this means that quantized momentum variables are written to the right of quantized space variables.

In case X is not flat, let us describe the operator $(\;)^z$ which was used to define the quantization map in terms of local coordinates. First $(\;)^z$ expands any $a \in \mathcal{D}(U)$ over a point $x \in U$ as a polynomial in the $\xi_{x,k}$, which correspond to a normal coordinate system $z_x : U \to \mathbb{C}^n$. This expansion induces the normal ordering. Afterwards $(\;)^z$ transforms it back over x to the given coordinate system z by means of the operator $\mathcal{D}_{\dj}(z, z_x, \mathbb{C})$.

2.3 Action on smooth, holomorphic and integrable functions

Let X be an arbitrary smooth or complex manifold and denote by $\mathcal{E}_X$ the sheaf of smooth resp. holomorphic functions on X. In this section we will now first introduce the natural action of the sheaf $\mathcal{D}_\hbar$, $\hbar \in \mathbb{C}$ of $\hbar$-deformed polynomial observables on X on the sheaf $\mathcal{E}_X$. Likewise we will define a natural action of the sheaf $\mathcal{D}_{\dj}$ of universally deformed polynomial observables on X on the sheaf $\mathcal{E}_{X\times\mathbb{C}}$. Then later on in this section we will show how one can use this for calculational purposes in the case, where X is a RIEMANNIAN manifold. Additionally we give some facts concerning a HILBERT space representation of quantized observables and show that we can thus construct a functor acting on a category of RIEMANNIAN manifolds and with range in the category of HILBERT space representations.

Now let us define the first sheaf morphism $\mathcal{D}_\hbar \times \mathcal{E}_X \to \mathcal{E}_X$ in local coordinates $z : U \to \mathbb{K}^n$ by

$$\left(a\left(-i\frac{\partial^\hbar}{\partial z}\right), f\right) \longmapsto a\left(-i\frac{\partial^\hbar}{\partial z}\right) \cdot f \;=\; \sum_{\alpha\in\mathbb{N}^n} (-i\hbar)^{|\alpha|}\, a_\alpha\, \frac{\partial^{|\alpha|} f}{\partial z^\alpha}, \tag{2.34}$$

where $a = \sum (a_\alpha \circ \pi)\, \xi^\alpha \in \mathcal{E}(T^*U)$, $a_\alpha, f \in \mathcal{E}(U)$ and $a_\alpha = 0$ for all but finitely many $\alpha \in \mathbb{N}^n$. Define a second sheaf morphism $\mathcal{D}_{đ} \times \mathcal{E}_{X\times\mathbb{C}} \to \mathcal{E}_{X\times\mathbb{C}}$ in the same way by

$$\left(p\left(-i\frac{\partial^{đ}}{\partial z}\right), r\right) \longmapsto p\left(-i\frac{\partial^{đ}}{\partial z}\right) \cdot r \;=\; \sum_{\alpha\in\mathbb{N}^n} (-ið)^{|\alpha|}\, p_\alpha\, \frac{\partial^{|\alpha|} r}{\partial z^\alpha}, \tag{2.35}$$

where $p = \sum (p_\alpha \circ \pi \circ pr_1)\,(\xi^\alpha \circ pr_1) \in \mathcal{E}(T^*U \times O)$, $O \subset \mathbb{C}$ open, $pr_1 : T^*X \times \mathbb{C} \to T^*X$ is the projection on the first coordinate, $p_\alpha, r \in \mathcal{E}(U \times O)$, and $p_\alpha = 0$ for all but finitely many $\alpha \in \mathbb{N}^n$. The reader will easily verify that these morphisms are well-defined and give rise to actions $\mathcal{D}_\hbar \mapsto \mathcal{H}om(\mathcal{E}_X, \mathcal{E}_X)$ and $\mathcal{D}_{đ} \mapsto \mathcal{H}om(\mathcal{E}_{X\times\mathbb{C}}, \mathcal{E}_{X\times\mathbb{C}})$. Because the sheaf $\mathcal{D}_Y$ of differential operators on a smooth or complex manifold Y acts faithfully on $\mathcal{E}_Y$, the above definitions show that the action of $\mathcal{D}_\hbar$ on $\mathcal{E}_X$ is faithfull in the case $\hbar \neq 0$ and has kernel $\mathcal{K}$ with $\mathcal{K}(U) = \{a = \sum_{\nu\in\mathbb{N}} a_\nu \in \mathcal{D}_0(U) : a_\nu \in \mathcal{E}(\nu)(U),\, a_0 = 0\}$ for $\hbar = 0$. The sheaf $\mathcal{D}_{đ}$ acts faithfully on $\mathcal{E}_{X\times\mathbb{C}}$.

Note 2.3.1 The above considerations show that the element $\frac{\partial^\hbar}{\partial z}$ can be interpreted as the differential operator $\hbar\frac{\partial}{\partial z}$ for $\hbar \neq 0$. The reason why we have chosen the separate notation $\frac{\partial^\hbar}{\partial z}$ is that we want to include the case $\hbar = 0$: then $\frac{\partial^\hbar}{\partial z} \neq 0$, whereas $\hbar\frac{\partial}{\partial z} = 0$.

The representation of deformed observables on spaces of smooth functions enables us to write down easier formulas for quantized observables. Note that from now on we have to assume X to be a RIEMANNIAN manifold.

Proposition 2.3.2 *Let $a \in \mathcal{D}_0(U)$ be a polynomial symbol on a* RIEMANNIAN *manifold X, and $U \subset X$ open. Express a locally for every $x \in U$ in the form*

$$a\Big|_{T^*_x U} = \sum_\alpha (a_{x,\alpha} \circ \pi)\, \xi_x^\alpha, \tag{2.36}$$

where $a_{x,\alpha} \in \mathcal{C}^\infty(U_x)$ and $(z_x, \xi_x) : T^*U_x \to \mathbb{R}^{2n}$ is induced by a normal coordinate system $z_x : U_x \to \mathbb{R}^n$ on an open neighborhood $U_x \subset U$ of x. Then the action of $\mathfrak{q}_\hbar(a)$ on $\mathcal{C}^\infty(U)$ is given by

$$\mathcal{C}^\infty(U) \ni f \mapsto \mathfrak{q}_\hbar(a) f \;=\; \left(U \ni x \mapsto \sum_\alpha (-i\hbar)^\alpha a_{x,\alpha}(x) \frac{\partial^{|\alpha|} f}{\partial z_x^\alpha}(x) \in \mathbb{C}\right) \in \mathcal{C}^\infty(U), \tag{2.37}$$

and the action of $\mathfrak{q}(a)$ *on* $\mathcal{C}^\infty(U \times \mathbb{R})$ *by*

$$\mathcal{C}^\infty(U \times \mathbb{R}) \ni r \mapsto \mathfrak{q}(a)r = \\ = \left(U \times \mathbb{R} \ni (x,\hbar) \mapsto \sum_\alpha (-i\hbar)^\alpha a_{x,\alpha}(x) \frac{\partial^{|\alpha|} r}{\partial z_x^\alpha}(x) \in \mathbb{C} \right) \in \mathcal{C}^\infty(U \times \mathbb{R}). \tag{2.38}$$

PROOF: By Eq. (2.12) and relation (2.13) we can write

$$a^{z_x}\Big|_{T_x^* X \times \mathbb{C}} = \sum_\alpha a_{x,\alpha}(x)\, \xi_x^\alpha, \tag{2.39}$$

hence the claim follows by (2.34) and (2.35). □

Note 2.3.3 Proposition 2.3.2 gives an easier description of the quantization operator $\mathfrak{q}_\hbar$ with $\hbar \neq 0$ than the original definition in section 2.2. The reason why we have chosen the more complicated definition as the basic one is that it also holds in the case $\hbar = 0$.

By the above considerations the quantized observables can be interpreted as differential operators over X. As X is assumed to be a RIEMANNIAN manifold, we can form the HILBERT space $L^2(X)$ of square-integrable functions on X in case X is even orientable. The globally defined quantized observables $A \in \mathcal{D}_\hbar(X)$ then give rise to densely defined linear (in general unbounded) operators on $L^2(X)$. A common admissable dense domain for all quantized observables is the space $\mathcal{C}_0^\infty(X)$ of smooth functions on X with compact support. The action of $\mathcal{D}_\hbar(X)$ on $L^2(X)$ is faithful for $\hbar \neq 0$.

It is a well-known fact that $Obj\left(\underline{\mathsf{Diff}}^{emb.}_{or.Riem}\right) \ni X \to L^2(X) \in Obj\left(\underline{\mathbb{C}\text{-}\mathsf{Vec}}_{Hilb}\right)$ gives rise to a functor L^2 from the category of orientable RIEMANNIAN manifolds with isometric embeddings to the category $\underline{\mathbb{C}\text{-}\mathsf{Vec}}_{Hilb}$ of HILBERT spaces. The following proposition now states that L^2 and the functor of normal order quantization fit together in a certain sense.

Theorem 2.3.4 *Assume* $\hbar \neq 0$. *Then the functors of normal order quantization and of square integrable functions give rise to a monoidal functor* $\mathsf{R}_\hbar$ *going from the category* $\underline{\mathsf{Diff}}^{emb.}_{or.Riem}$ *of orientable* RIEMANNIAN *manifolds with isometric embeddings as morphisms to the category of* HILBERT *space representations* $\underline{\mathsf{Rep}}_{Hilb}$ *(with dense common admissable domain). In particular for every isometric embedding* $f : X \to Y$ *there is an isometric intertwining operator* $\mathsf{R}_\hbar(f) = (f^*, f^*) : (\mathcal{D}_\hbar(Y), L^2(Y)) \to (\mathcal{D}_\hbar(X), L^2(X))$ *given by the pull-back with* f. *Furthermore the equation*

$$f^*\Big(\mathfrak{q}_\hbar(b)\, h\Big) = \mathfrak{q}_\hbar(f^*\, b)\,(f^*\, h) \tag{2.40}$$

holds for every $b \in \mathcal{D}_0(Y)$ *and every* $h \in \mathcal{C}_0^\infty(Y)$.

Let us first give the proof of Theorem 2.2.4.

PROOF (of Theorem 2.2.4): In the same way like associating to a manifold its sheaf of differential operators is functorial, one can show that associating $\mathcal{D}_\hbar$ resp. $\mathcal{D}_{\bar{d}}$ to X is functorial from $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to the category of sheaves. Thus we have a functor from $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to deformations. Now the construction of the exponential function exp and its corresponding normal coordinates is natural on the category $\underline{\mathsf{Diff}}^{emb.}_{Riem}$, so even the process of normal order quantization is functorial. Let us show this in more detail, and assume $f : X \to Y$ to be an isometric embedding. Then we have

$$f \circ \exp = \exp \circ Tf. \tag{2.41}$$

To prove that quantization is natural, it now suffices to show that

$$f^* \left(\mathfrak{q}_\hbar(b)\, g \right) \;=\; \mathfrak{q}_\hbar(f^* b)\, f^* g \tag{2.42}$$

holds for every $b \in \mathcal{D}_0(U)$ and $g \in \mathcal{C}^\infty(U)$ with $U \subset Y$ open. So first choose $x \in f^{-1}(U)$ and expand locally $b = \sum_{\alpha \in \mathbb{N}^n} b_{f(x),\alpha}\, \xi^\alpha_{f(x)}$ with $b_{f(x),\alpha} \in \mathcal{C}^\infty_Y$ and $b \circ f(x) = \sum_{\alpha \in \mathbb{N}^n} (b \circ f)_{x,\alpha}\, \xi^\alpha_x$ with $(b \circ f)_{x,\alpha} \in \mathcal{C}^\infty_X$. Eq. (2.41) then entails $f_* \xi_x = \xi_{f(x)}$, so we have $b_{f(x),\alpha} = (b \circ f)_{x,\alpha}$. But by Proposition 2.3.2 this just entails $\left(\mathfrak{q}_\hbar(b)\, g \right)(f(x)) \;=\; \left(\mathfrak{q}_\hbar(f^* b)\, f^* g \right)(x)$, so Eq. (2.42) holds. This proves the first part of claim. The second follows directly from the definition of $\underline{\mathcal{C}^\infty\text{-}\mathsf{Quant}}^{filt.}_{0,(\mathbb{R},\mathcal{C}^\infty_{\mathbb{R}})}$ and its monoidal structure in section 1.3. □

PROOF (of Theorem 2.3.4): As f is an isometric embedding, the pull-back f^* induces an isometric embedding from $L^2(Y)$ to $L^2(X)$. The rest is now a direct consequence of the preceding proof, in particular of Eq. (2.42), and the fact that $L^2(X \times Y) = L^2(X) \otimes_{Hilb} L^2(Y)$, where $\otimes_{Hilb}$ is the tensor product in the category of HILBERT spaces. □

2.4 Quantization of some examples from physics

The results of the last section give us the means to calculate the quantization of certain observables important for physics. Note that we still assume X to be a RIEMANNIAN manifold. In the following we are going to calculate the quantization of observables which are of order 0,1 or 2 in the fibers of T^*X or in other words which are of order 0, 1 or 2 in momentum.

(i) Obviously the quantization of a smooth function $a = f \circ \pi \in \mathcal{C}^\infty_{T^*X}(0)(U)$ with

$U \subset X$ open and $f \in \mathcal{C}^\infty(U)$ is trivial in the sense that

$$\mathfrak{q}(a) \;=\; f \circ q \quad \text{ and } \quad \mathfrak{q}_\hbar(a) \;=\; f. \tag{2.43}$$

$\mathfrak{q}_\hbar(a)$ acts on $\mathcal{C}^\infty(U)$ as a multiplication operator:

$$\mathfrak{q}_\hbar(a) \cdot g = fg, \quad g \in \mathcal{C}^\infty(U). \tag{2.44}$$

(*ii*) Now assume $\mathcal{V} : T^*U \to \mathbb{C}$ to be smooth and $\mathbb{R}$-linear in the fibers. Then we can regard $\mathcal{V}$ as a vector field over U and write in local coordinates $\mathcal{V}\big|_V = \sum v_k \frac{\partial}{\partial z_k}$ where $V \subset U$ is open, $z : V \to \mathbb{R}^n$ a coordinate system and $v_k \in \mathcal{C}^\infty(V)$ for $k = 1, ..., n$. Hence he quantization of $\mathcal{V}$ is

$$\mathfrak{q}_{V \times \mathbb{C}}(\mathcal{V}) \;=\; -i \sum v_k \frac{\partial^{đ}}{\partial z_k} \;=\; \mathcal{V}^{đ} \quad \text{ resp. } \quad (\mathfrak{q}_\hbar)_V\,(\mathcal{V}) \;=\; -i \sum v_k \frac{\partial^\hbar}{\partial z_k} \;=\; \mathcal{V}^\hbar \tag{2.45}$$

and its action on smooth functions given by

$$(\mathfrak{q}_\hbar)_V\,(\mathcal{V}) \cdot g \;=\; \mathcal{V}^\hbar \cdot g \;=\; -i\hbar\,(\mathcal{V}g), \quad g \in \mathcal{C}^\infty(U). \tag{2.46}$$

(*iii*) Finally choose a smooth $T : T^*U \to \mathbb{C}$ which is quadratic in the fibers. Then there exists a unique symmetric tensor field $\mathbf{T} : U \to T^2X \otimes \mathbb{C}$ such that $T(\omega) = (\omega \otimes \omega) \circ \mathbf{T}_x$ for $\omega \in T^*X$, $x \in U$. Expand $\mathbf{T}$ in normal coordinates over x:

$$\mathbf{T} = \sum_{k,l} t_{x,kl} \frac{\partial}{\partial z_{x,k}} \frac{\partial}{\partial z_{x,l}} \tag{2.47}$$

with $t_{x,kl} \in \mathcal{C}^\infty(U)$. This gives $T = \sum_{kl} (t_{x,kl} \circ \pi)\, \xi_{x,k}\, \xi_{x,l}$ and

$$T^\hbar \cdot f(x) = -\hbar^2 \sum t_{x,kl}(x) \left(\frac{\partial}{\partial z_{x,k}} \frac{\partial}{\partial z_{x,l}} f \right)(x). \tag{2.48}$$

Recalling that the HESSIAN of f is the symmetric tensor field $f^{\mathrm{H}} : U \to T^{*2}X$ with $f^{\mathrm{H}}(x) = \sum_{kl} \left(\frac{\partial}{\partial z_{x,k}} \frac{\partial}{\partial z_{x,l}} f \right)(x)\, dz_{x,k}\big|_{T_xX} \odot dz_{x,l}\big|_{T_xX}$, where $\odot$ denotes the symmetric tensor product, the relation

$$T^\hbar \cdot f \;=\; -\hbar^2 \left(\mathbf{T} \lrcorner\, f^{\mathrm{H}} \right) \tag{2.49}$$

now follows. In the same way

$$T^{đ} \cdot f \;=\; -đ^2 \left(\mathbf{T} \lrcorner\, f^{\mathrm{H}} \right). \tag{2.50}$$

Let us now calculate the quantization of some specific observables in three important mechanical systems. For further information on these classical systems see for example Abraham, Marsden [1], Schottenloher [46] or Thirring [51, 52]. The last two references also treat their quantization.

2.4.1 Harmonic oscillator. The classical configuration space of the harmonic oscillator is given by $X = \mathbb{R}^n$ with its canonical Euclidean metric. Then the phase space $M = T^*X$ is isomorphic to $\mathbb{R}^{2n}$ with canonical coordinates $(x, \xi) = (x_1, ..., x_n, \xi_1, ..., \xi_n)$. Furthermore the Hamiltonian is given by $H = \frac{||\xi||^2}{2m} + \frac{1}{2}\, m\, \omega^2\, ||x||^2$, where $m > 0$ is the mass of the oscillator and ω its angular frequency. By the above considerations and the flatness of $X = \mathbb{R}^n$ we have $\mathfrak{q}_\hbar(\xi_k) = -i\,\hbar\, \frac{\partial}{\partial x_k}$ and $\mathfrak{q}_\hbar(x_k) = x_k$. Hence the the quantized Hamiltonian has the form

$$\mathfrak{q}_\hbar(H) \;=\; -\frac{\hbar^2}{2m} \sum_{k=1}^{n} \frac{\partial^2}{\partial {x_k}^2} + \frac{1}{2}\, m\, \omega^2\, ||x||^2, \tag{2.51}$$

which is just the quantized Hamiltonian of the harmonic oscillator in the Schrödinger picture. In geometric quantization (see Woodhouse [66]) this result can be derived only after a metaplectic correction.

2.4.2 Free particle moving on a Riemannian manifold. In the next example we let X be a Riemannian manifold with metric g and choose $M = T^*X$ with its canonical symplectic structure as the phase space of a particle moving on the Riemannian manifold X. The metric g on TX induces a metric g^* on T^*X by the canonical isomorphism $TX \to T^*X$, $v \to g(-, v)$. The dynamics of a free particle with mass $m > 0$ is then described by the Hamiltonian $H = \frac{1}{2m} g^*$. Now H is an observable on M which is quadratic in momentum, so paragraph (iii) above entails $\mathfrak{q}_\hbar(H) \cdot f = -\frac{\hbar^2}{2m} \left(g^* \lrcorner f^{\mathrm{H}} \right)$ for any $f \in \mathcal{C}^\infty(M)$. Hence

$$\mathfrak{q}_\hbar(H) \;=\; -\frac{\hbar^2}{2m}\, \Delta_g \tag{2.52}$$

follows, where Δ_g is the metric Laplacian on X. Compare this derivation again with the corresponding one in Woodhouse, *Geometric Quantization* [66].

2.4.3 Kepler problem and Hydrogen atom. The Kepler problem describes a particle of mass m moving in $\mathbb{R}^3 \setminus \{0\}$ under the influence of a central field force $F = -k\, \frac{x}{||x||^3}$, where $k > 0$ is the constant of force. Thus the phase space for the Kepler problem is the cotangent bundle T^*X with canonical coordinates $(x,\xi) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ of T^*X. The potential for the central field is $V(x) = -\frac{k}{x}$. This leads to the classical HAMILTONIAN

$$H = \frac{1}{2m} ||\xi||^2 + V(x) = \frac{1}{2m} ||\xi||^2 - \frac{k}{||x||}. \tag{2.53}$$

In case we consider an electron moving around a proton, i.e. we consider the Hydrogen atom, the constant k is given by $k = e^2$, where e is the elementary electric charge and m is the mass m_e of the electron.

The quantization of H now obviously is

$$\mathfrak{q}_\hbar(H) = -\frac{\hbar^2}{2m} \sum_{k=1}^{3} \frac{\partial^2}{\partial {x_k}^2} + V(x). \tag{2.54}$$

We are not only interested in the quantization of the HAMILTONIAN but also of the angular momentum $L = (L_1, L_2, L_3)$ and the LENZ-RUNGE vector $A = (A_1, A_2, A_3)$. Recall that classically $L_j = x_k \xi_l - x_l \xi_k$ and $A_j = \frac{1}{m}(L_k \xi_l - L_l \xi_k) + k \frac{x}{|x|}$, where (j,k,l) is a cyclic permutation of $(1,2,3)$. The normal order quantization of the angular momentum now produces the correct quantum observables, namely

$$\mathfrak{q}_\hbar(L_j) = -i\hbar \left(x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} \right). \tag{2.55}$$

But the normal order quantization of A_1 gives

$$\mathfrak{q}_\hbar(A_1) = -\frac{\hbar^2}{m} \left(x_3 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \right) + k \frac{x}{|x|}, \tag{2.56}$$

which is not a formally self-adjoint operator on $L^2(\mathbb{R}^3 \setminus \{0\}$ and thus not the "correct" quantization of A_1. An analog result holds for A_2 and A_3. The correct quantization of A_k would be

$$\mathfrak{q}_\hbar(A_k) = \frac{1}{2m} \Big(\{\mathfrak{q}_\hbar(L_k), \mathfrak{q}_\hbar(\xi_l)\} - \{\mathfrak{q}_\hbar(L_l), \mathfrak{q}_\hbar(\xi_k)\} \Big) + k \frac{x}{|x|}, \tag{2.57}$$

where $\{\,,\,\}$ is the anticommutator in the algebra of quantized observables. See BOHM [8] and SCHOTTENLOHER [46] for more information on the quantization of the LENZ-RUNGE vector.

Remark 2.4.4 As the quantization of the LENZ-RUNGE vector in the last example shows, normal order quantization does not always give the "correct" quantizations for higher order polynomials in momentum. This suggests that one should try to generalize normal order quantization to a quantization scheme which maps self-adjoint classical observables to (formally) self-adjoint quantized observables. A WEYL quantization scheme for arbitrary RIEMANNIAN manifolds or a mixed quantization between normal and antinormal quantizations would probably be an appropriate Ansatz for this.

Finally let us explain the importance of the HAMILTONIAN in physics or in other words the reason why we have quantized the HAMILTONIAN in all three examples above. To put it shortly, the HAMILTONIAN gives a classical or quantum mechanical system its dynamics. In the classical arena the HAMILTONIAN vector field X_H corresponding to H generates a one parameter family of local symplectomorphism of the phase space, or canonical transformations in physical terms, and thus induces the time evolution of the system. In the quantum setting a self-adjoint quantized HAMILTONIAN $\mathfrak{q}_\hbar(H)$ generates a one parameter family of unitary transformations of the HILBERT space of states. This family is responsible for the time evolution of the quantum mechanical system and is given by $\mathbb{R} \ni t \mapsto \exp\left(-i\mathfrak{q}_\hbar(H)t\right) \in U\left(L^2(X)\right)$.

2.5 Observables of infinite order in momentum

Let X be a complex manifold. Then it is possible to generalize the deformation of the sheaf $\mathcal{D}_0$ over X to a deformation of a larger sheaf $\mathcal{D}_0^\infty$. This sheaf consists of holomorphic functions over T^*X, whose TAYLOR coefficients fulfill a certain growth condition, and will be called the sheaf of **classical observables of infinite order in momentum**. In the following we will give its definition and a deformation.

Let $z : U \to \mathbb{C}^n$ be a local coordinate system of X and define the linear space $\mathcal{D}^\infty(z)$ of holomorphic functions on T^*U with **bounded growth** in the fibers by

$$\begin{aligned}\mathcal{D}^\infty(z) \;\; = \;\; \Big\{ & F \in \mathcal{O}(T^*U) : F = \sum_{\nu\in\mathbb{N}} f_\nu \text{ where } f_\nu \in \mathcal{O}(\nu)(U) \text{ and} \\ & \forall\varepsilon > 0\, \forall K \subset T^*U \text{ compact } \exists C_{K,\varepsilon} > 0 \text{ such that } |f_\nu|_K \leq C_{K,\varepsilon}\frac{\varepsilon^\nu}{\nu!},\ \nu \in \mathbb{N}\Big\}.\end{aligned} \tag{2.58}$$

Before proceeding let us first prove the little

Lemma 2.5.1 *A formal sum $F = \sum_{\nu\in\mathbb{N}} f_\nu$ with $f_\nu = \sum_{|\alpha|=\nu} (f_\alpha \circ \pi)\, \xi^\alpha$ lies in $\mathcal{D}^\infty(z)$, iff for any $\varepsilon > 0$ and compact $K \subset U$ there exists a $D_{K,\varepsilon} > 0$ such that*

$$|f_\alpha|_K \leq D_{K,\varepsilon}\frac{\varepsilon^{|\alpha|}}{\alpha!} \tag{2.59}$$

for all $\alpha \in \mathbb{N}^n$.

PROOF: First let F be an element of $\mathcal{D}^\infty(z)$, $K \subset U$ be compact and $\varepsilon > 0$. Then consider in local coordinates $(z, \xi) : T^*U \to \mathbb{C}^{2n}$ the following Cauchy-integral:

$$f_\alpha(z) = \frac{1}{(2\pi i)^n}\int\limits_{T^n_z} \frac{f_\nu(z,\xi)}{\xi^{\alpha+1}}d\xi, \quad |\alpha| = \nu, \tag{2.60}$$

where $z \in K$, T^n_z is the n-torus in $T^*_zU \cong \mathbb{C}^n$ around 0 and $\alpha + 1 = (\alpha_1 + 1, ..., \alpha_n + 1)$. This integral entails

$$|f_\alpha|_K \leq |f_\nu|_{P(K,1)} \leq C_{P(K,1),\varepsilon}\frac{\varepsilon^\nu}{\alpha!}, \tag{2.61}$$

where $P(K, r)$ for $r > 0$ is the polycylinder $\{(z, \xi) \in T^*U : z \in K, |\xi_i| \leq r\}$. Hence Eq. (2.61) shows the condition being necessary.

The condition is also sufficient, because if F is a formal sum fullfilling Eq. (2.59), then

$$|f_\nu|_{P(K,r)} \leq \sum_{|\alpha|=\nu} |f_\alpha|_K \, r^\nu \leq \frac{1}{\nu!} D_{K,\varepsilon} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} (\varepsilon r)^\nu = \frac{1}{\nu!} (n \varepsilon r)^\nu D_{K,\varepsilon} \tag{2.62}$$

is true. □

Now let us furnish $\mathcal{D}^\infty(z)$ with parameter dependant products

$$m_\hbar : \mathcal{D}^\infty(z) \times \mathcal{D}^\infty(z) \to \mathcal{D}^\infty(z), \quad (F, G) \; = \; \left(\sum f_\nu, \sum g_\mu\right) \mapsto H = \sum h_\rho \tag{2.63}$$

by

$$h_\rho = \sum_{\nu+\mu-|\alpha|=\rho} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} f_\nu \, \frac{\partial^{|\alpha|}}{\partial z^\alpha} g_\mu \, . \tag{2.64}$$

H is indeed well-defined. To see that choose for $K \subset T^*U$ compact and $\varepsilon > 0$ constants $\tilde{\varepsilon} > 0$, $\delta > 0$ being small enough and constants $C_{K,\varepsilon} > 0$, $D_{K,\varepsilon} > 0$ being big enough such that

(*i*) $K_\delta := \{\omega \in T^*U : |(z,\xi)(\omega) - (z,\xi)(\tilde{\omega})| \leq \delta$ for all $\tilde{\omega} \in K\}$ is compact in T^*U,

(*ii*) $2\tilde{\varepsilon} < \varepsilon$ and $\frac{2\hbar\tilde{\varepsilon}}{\delta^2} < \frac{1}{2}$,

(*iii*) $|f_\nu|_{K_\delta} \; \leq \; \frac{1}{\nu!} C_{K,\varepsilon} \, \tilde{\varepsilon}^\nu$ and $|g_\mu|_{K_\delta} \; \leq \; \frac{1}{\mu!} D_{K,\varepsilon} \, \tilde{\varepsilon}^\mu$ for all $\nu, \mu \in \mathbb{N}$.

Applying the Cauchy inequalities then gives:

$$\begin{aligned}
|h_\rho|_K \;&\leq\; \sum_{\substack{\nu,\mu\in\mathbb{N},\,\alpha\in\mathbb{N}^n\\ \rho=\nu+\mu-|\alpha|}} \frac{\hbar^{|\alpha|}}{\alpha!} \left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha} f_\nu\right|_K \left|\frac{\partial^{|\alpha|}}{\partial z^\alpha} g_\mu\right|_K \\
&\leq\; \sum_{\substack{\nu,\mu\in\mathbb{N},\,\alpha\in\mathbb{N}^n\\ \rho=\nu+\mu-|\alpha|}} \left(\frac{\hbar}{\delta^2}\right)^{|\alpha|} \alpha! \; |f_\nu|_{K_\delta} \; |g_\mu|_{K_\delta} \\
&\leq\; \sum_{\substack{\nu,\mu\in\mathbb{N},\,\alpha\in\mathbb{N}^n\\ \rho=\nu+\mu-|\alpha|}} \left(\frac{\hbar}{\delta^2}\right)^{|\alpha|} \frac{\alpha!}{\nu!\,\mu!} C_{K,\varepsilon} \, \tilde{\varepsilon}^\nu \, D_{K,\varepsilon} \, \tilde{\varepsilon}^\mu \\
&\leq\; \frac{1}{\rho!} C_{K,\varepsilon} \, D_{K,\varepsilon} \sum_{\alpha\in\mathbb{N}^n} \left(\frac{\hbar}{\delta^2}\right)^{|\alpha|} \sum_{\substack{\nu,\mu\in\mathbb{N}\\ \rho=\nu+\mu-|\alpha|}} \frac{(\rho+|\alpha|)!}{\nu!\,\mu!} \tilde{\varepsilon}^\nu \, \tilde{\varepsilon}^\mu \\
&=\; \frac{1}{\rho!} C_{K,\varepsilon} \, D_{K,\varepsilon} \sum_{\alpha\in\mathbb{N}^n} \left(\frac{\hbar}{\delta^2}\right)^{|\alpha|} (2\,\tilde{\varepsilon})^{\rho+|\alpha|} \;\leq\; \frac{1}{\rho!} \left(C_{K,\varepsilon} \, D_{K,\varepsilon} \, 2^n\right) \varepsilon^\rho,
\end{aligned} \tag{2.65}$$

so $H = \sum h_\rho$ is well-defined and lies in $\mathcal{D}^\infty(z)$. Using equation (2.2), one can prove that the bilinear map $m_\hbar$ associative. Altogether we receive a scale of algebras $\mathcal{D}^\infty_\hbar(z)$, $\hbar \in \mathbb{C}$ such that $\mathcal{D}_\hbar$ is a subalgebra of $\mathcal{D}^\infty_\hbar(z)$.

By the above considerations it becomes also clear that $\mathcal{D}^\infty_\hbar(z)$ has the structure of a FRÉCHET space with seminorms

$$||F||_{K,\varepsilon} = \sup_{\nu \in \mathbb{N}} \left\{ \frac{\nu!\, |f_\nu|_K}{\varepsilon^\nu} \right\},$$

and that the $m_\hbar$ are continuous with respect to that topology. Furthermore $\mathcal{O}(U)$ can be continuously embedded into $\mathcal{D}^\infty_\hbar(z)$ by $\mathcal{O}(U) \ni h \to H = h \circ \pi \in \mathcal{D}^\infty_\hbar(z)$.

Now we can define an action $\mathcal{D}^\infty_\hbar(z) \times \mathcal{O}(U) \to \mathcal{O}(U)$ on $\mathcal{O}(U)$ by

$$(F, h) \longmapsto F \cdot h \;=\; m_\hbar(F, H) \circ j|_U \;=\; \sum_{\alpha \in \mathbb{N}^n} (-i\hbar)^{|\alpha|} f_\alpha \frac{\partial^{|\alpha|} h}{\partial z^\alpha},$$

where $F = \sum_{\nu \in \mathbb{N}} f_\nu \in \mathcal{D}^\infty_\hbar(z)$, $f_\nu = \sum_{|\alpha| = \nu} (f_\alpha \circ \pi)\, \xi^\alpha$, $f_\alpha, h \in \mathcal{O}(U)$ and $j : X \to T^*X$ is the zero section. The action is continuous by definition and the fact that $m_\hbar$ and the pull-back $(j|_U)^* : \mathcal{D}^\infty_\hbar(z) \to \mathcal{O}(U)$ are continuous.

The question which now arises is, whether the algebras $\mathcal{D}^\infty_\hbar(z)$ can be patched together for different coordinate sytems. The answer is given in the following proposition.

Proposition 2.5.2 *Let $\tilde{z} : U \to \mathbb{C}^n$ be another chart on U. Then there exists a unique algebra morphism $\mathcal{D}^\infty_\hbar(\tilde{z}, z) : \mathcal{D}^\infty_\hbar(z) \to \mathcal{D}^\infty_\hbar(\tilde{z})$ such that*

$$\begin{tikzcd}
\mathcal{D}^\infty_\hbar(z) \arrow[r, "\mathcal{D}^\infty_\hbar(\tilde{z}, z)"] & \mathcal{D}^\infty_\hbar(\tilde{z}) \\
\mathcal{D}_\hbar(z) \arrow[u] \arrow[r, "\mathcal{D}_\hbar(\tilde{z}, z)"'] & \mathcal{D}_\hbar(\tilde{z}) \arrow[u]
\end{tikzcd}$$

commutes. Furthermore

$$\mathcal{D}^\infty_\hbar(\hat{z}, z) \;=\; \mathcal{D}^\infty_\hbar(\hat{z}, \tilde{z}) \circ \mathcal{D}^\infty_\hbar(\tilde{z}, z)$$

for any third chart $\hat{z} : U \to \mathbb{C}^n$.

PROOF: Let $F = \sum\limits_{\alpha\in\mathbb{N}^n} (f_\alpha \circ \pi)\, \xi^\alpha$ be an element of $\mathcal{D}^\infty_\hbar(z)$ and $F_\alpha = \sum\limits_{\beta\leq\alpha} (f_\beta \circ \pi)\, \xi^\beta$ for $\alpha \in \mathbb{N}^n$. By Eq. (2.62) there now exists for $\varepsilon > 0$ and $K \subset T^*U$ compact an $r_K > 0$ such that

$$\begin{aligned} ||F - F_\alpha||_{K,\varepsilon} &= ||\sum_{\beta\geq\alpha} (f_\beta \circ \pi)\xi^\beta||_{K,\varepsilon} \leq \sup_{\beta\geq\alpha} \left\{ \frac{\beta!\, |f_\beta|_{\pi(K)}}{(r_K\varepsilon)^{|\beta|}} \right\} \\ &= \left(\frac{1}{2}\right)^\alpha \sup_{\beta\geq\alpha} \left\{ \left(\frac{1}{2}\right)^{\beta-\alpha} \frac{\beta!\, |f_\beta|_{\pi(K)}}{\left(\frac{r_K\varepsilon}{2}\right)^{|\beta|}} \right\} \leq \left(\frac{1}{2}\right)^\alpha ||F||_{P(\pi(K),1),\frac{r_K\varepsilon}{2}} \end{aligned} \tag{2.66}$$

holds. This gives $\lim\limits_{\alpha\in\mathbb{N}^n} ||F - F_\alpha||_{K,\varepsilon} = 0$, so $\mathcal{D}_\hbar(z)$ is dense in $\mathcal{D}^\infty_\hbar(z)$ and uniqueness of $\mathcal{D}^\infty_\hbar(\tilde{z}, z)$ follows. Now we already know that

$$\mathcal{D}_\hbar(\tilde{z}, z)\ (\xi^\alpha) = \left(\sum_{\beta\leq\alpha} \varphi_{\beta,\alpha} \circ \pi \right) \tilde{\xi}^\beta \tag{2.67}$$

with unique $\varphi_{\beta,\alpha} \in \mathcal{O}(U)$, which leads us to the definition

$$\mathcal{D}^\infty_\hbar(\tilde{z}, z)\,(F) := \tilde{F} := \sum_{\beta\in\mathbb{N}^n} \tilde{f}_\beta, \quad \text{with} \quad \tilde{f}_\beta = \sum_{\alpha\geq\beta} \varphi_{\beta,\alpha} f_\alpha. \tag{2.68}$$

To prove that this indeed gives a well-defined homomorphism $\mathcal{D}^\infty_\hbar(\tilde{z}, z) : \mathcal{D}^\infty_\hbar(z) \to \mathcal{D}^\infty_\hbar(\tilde{z})$, let us first show that the series $\tilde{f}_\beta = \sum\limits_{\alpha\geq\beta} \varphi_{\beta,\alpha} f_\alpha$ converges normally. We only consider the nontrivial case $\hbar \neq 0$. Let $h_\alpha \in \mathcal{O}(U)$, $a \in U$ be the function $x \mapsto h_a(x) = \frac{1}{(-i\hbar)^{|\beta|}\beta!} \left(\tilde{z}(x) - \tilde{z}(a)\right)^\beta$. Then by continuity of the action $\mathcal{D}^\infty_\hbar(z) \times \mathcal{O}(U) \to \mathcal{O}(U)$ the series $\sum\limits_{\alpha\in\mathbb{N}} F_\alpha \cdot h_a$ converges normally with respect to x and a to the in x and a holomorphic function $F \cdot h_a$. But for $\alpha \geq \beta$

$$\begin{aligned} F_\alpha \cdot h_a &= \left[(\mathcal{D}^\infty_\hbar(\tilde{z}, z) F_a) \cdot h_a \right](a) \\ &= \left[\left(\sum_{\substack{\gamma\leq\alpha \\ \delta\leq\gamma}} (f_\gamma \circ \pi)\, (\varphi_{\delta,\gamma} \circ \pi)\, \tilde{\xi}^\delta \right) \cdot h_a \right](a) = \sum_{\beta\leq\gamma\leq\alpha} f_\gamma\, \varphi_{\beta,\gamma}(a), \end{aligned} \tag{2.69}$$

which gives the claimed convergence.

Next we want to show $\tilde{F}$ being of bounded growth in the fibers, or in other words that it is an element of $\mathcal{D}^\infty_\hbar(\tilde{z})$ indeed. Let $\varepsilon_0 > 0$ be given and $a \in U$. Choose an open neighborhood $V \subset\subset U$ of a such that the series

$$g_{\varepsilon,y}(x) = \sum_{\beta\in\mathbb{N}^n} \left(\frac{\varepsilon}{-i\hbar}\right)^{|\beta|} \left(\tilde{z}(x) - \tilde{z}(y)\right)^\beta \tag{2.70}$$

converges uniformly with respect to $(\varepsilon, x, y) \in [0, \varepsilon_0] \times \bar{V} \times \bar{V}$. Then the partial sums $g^{\beta}_{\varepsilon,y}(x) = \sum_{\delta \leq \beta} \left(\frac{\varepsilon}{-i\hbar}\right)^{|\delta|} (\tilde{z}(x) - \tilde{z}(y))^{\delta}$ fulfill the relation

$$
\begin{aligned}
\left[F \cdot g^{\beta}_{\varepsilon,x}\right](x) &= \lim_{\alpha \in \mathbb{N}^n} \left[F_\alpha \cdot g^{\beta}_{\varepsilon,x}\right](x) = \lim_{\alpha \in \mathbb{N}^n} \left[(\mathcal{D}^{\infty}_{\hbar}(\tilde{z}, z) F_\alpha) \cdot g^{\beta}_{\varepsilon,x}\right](x) \\
&= \lim_{\alpha \in \mathbb{N}^n} \left[\left(\sum_{\substack{\gamma \leq \alpha \\ \delta \leq \gamma}} (f_\gamma \circ \pi)(\varphi_{\delta,\gamma} \circ \pi)\, \tilde{\xi}^{\delta}\right) \cdot g^{\beta}_{\varepsilon,x}\right](x) \\
&= \sum_{\delta \leq \beta} \delta! \varepsilon^{|\delta|} \tilde{f}_\delta(x). \qquad (2.71)
\end{aligned}
$$

Because $\mathcal{D}^{\infty}_{\hbar}(z)$ acts continuously on $\mathcal{O}(U)$ and therefore on $\mathcal{O}(V)$, we have uniformly with respect to y and ε:

$$
\lim_{\beta \in \mathbb{N}^n} F \cdot g^{\beta}_{\varepsilon,y} = F \cdot g_{\varepsilon,y}. \qquad (2.72)
$$

So for any compact $K \subset V$ the set $\left\{F \cdot g^{\beta}_{\varepsilon,x}(x) : 0 \leq \varepsilon \leq \varepsilon_0, x \in K\right\}$ is bounded. By Abel's Lemma we can find for every $\varepsilon < \varepsilon_0$ an upper bound $D_{K,\frac{1}{\varepsilon}} > 0$ such that

$$
\sum_{\gamma \in \mathbb{N}^n} \gamma!\, \varepsilon^{|\gamma|} \left|\tilde{f}_\gamma\right|_K \leq D_{K,\frac{1}{\varepsilon}}. \qquad (2.73)
$$

Therefore $\tilde{F} = \sum_{\gamma \in \mathbb{N}^n} (\tilde{f}_\gamma \circ \pi)\, \tilde{\xi}^{\gamma} \in \mathcal{D}^{\infty}_{\hbar}(\tilde{z})$, and $\mathcal{D}^{\infty}_{\hbar}(\tilde{z}, z)$ is well-defined. By definition it extends $\mathcal{D}_{\hbar}(\tilde{z}, z)$. Only the continuity of $\mathcal{D}^{\infty}_{\hbar}(\tilde{z}, z)$ remains to be shown. If $(F_k)_{k \in \mathbb{N}}$ is a convergent sequence in $\mathcal{D}^{\infty}_{\hbar}(\tilde{z}, z)$ with limit F, then we have uniformly for $x \in U$:

$$
0 = \left[\lim_{k \to \infty} (F - F_\nu) \cdot g_{\varepsilon_0,x}\right](x) = \lim_{k \to \infty} \sum \sum_{\gamma \in \mathbb{N}^n} \gamma!\, \varepsilon_0^{|\gamma|} \left(\tilde{f}_\gamma - \tilde{f}_{k,\gamma}\right)(x). \qquad (2.74)
$$

Abel's Lemma again shows $\sum_{\gamma \in \mathbb{N}^n} \gamma!\, \varepsilon^{|\gamma|} \left|\tilde{f}_\gamma - \tilde{f}_{k,\gamma}\right|_K = 0$ for $\varepsilon < \varepsilon_0$. □

By Proposition 2.5.2 it is possible to construct a sheaf $\mathcal{D}^{\infty}_{\hbar}$ of algebras over X such that the spaces $\mathcal{D}^{\infty}_{\hbar}(U)$ of sections over a coordinate patch U with coordinates $z : U \to \mathbb{C}^n$ are naturally isomorphic to $\mathcal{D}^{\infty}_{\hbar}(z)$. Then $\mathcal{D}_{\hbar}$ is a subsheaf of $\mathcal{D}^{\infty}_{\hbar}$ called the sheaf of $\hbar$-**deformed observables of infinite order in momentum**. $\mathcal{D}^{\infty}_{0}$ is the above mentioned sheaf of **classical observables of infinite order in momentum**.

In the same way it is also possible to extend the sheaf $\mathcal{D}_{\text{đ}}$ on $X \times \mathbb{C}$ to a sheaf $\mathcal{D}^{\infty}_{\text{đ}}$ having the following properties:

(*i*) There exists an algebra monomorphism $\Delta^{\infty} : \mathcal{O}_{\mathbb{C}} \to \mathcal{D}^{\infty}_{\text{đ}}$ of $\mathcal{O}_{\mathbb{C}}$ into the center of $\mathcal{D}^{\infty}_{\text{đ}}$ such that the diagram

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{C}} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta^\infty"] & \\ \mathcal{D}_{\bar{d}} \arrow[rr] & & \mathcal{D}_{\bar{d}}^\infty \end{tikzcd}$$

commutes.

(*ii*) $(\mathcal{D}_{\bar{d}}^\infty)_\hbar = \mathcal{D}_{\bar{d}}^\infty|_{X\times\{\hbar\}}/\mathfrak{m}_\hbar \mathcal{D}_{\bar{d}}^\infty|_{X\times\{\hbar\}}$ is isomorphic to $\mathcal{D}_\hbar^\infty$ for all $\hbar \in \mathbb{C}$. Here $\mathfrak{m}_\hbar$ is the maximal ideal of $\mathcal{O}_\hbar$.

We do not work out the details, as they are analogous to the ones in section 2.1. Let us only point out that for the coordinate patch U and an open $O \subset \mathbb{C}$ the space $\mathcal{D}_{\bar{d}}^\infty(U \times O)$ of sections over $U \times O$ is isomorphic to

$$\begin{aligned} \mathcal{D}_{\bar{d}}^\infty(z, O) \quad = \quad & \Big\{ P = \sum_{\nu\in\mathbb{N}} p_\nu \in \mathcal{O}(T^*U \times O) : p_\nu \in \mathcal{O}(\nu)(U \times O) \text{ and} \\ & \forall\varepsilon > 0\, \forall K \subset (T^*U \times O) \text{ compact } \exists C_{K,\varepsilon} > 0 \text{ such that} \\ & |p_\nu|_K \leq C_{K,\varepsilon} \frac{\varepsilon^\nu}{\nu!},\ \nu \in \mathbb{N} \Big\}. \end{aligned}$$

In the following we refer to $\mathcal{D}_{\bar{d}}^\infty$ as the sheaf of **universally deformed observables of infinite order in momentum** on the complex manifold X.

By the above conditions (*i*) and (*ii*) the morphism $D^\infty = (\bar{d}, \Delta^\infty) : (X \times \mathbb{C}, \mathcal{D}_{\bar{d}}^\infty) \to (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ over $\bar{d} = pr_2 : X \times \mathbb{C} \to \mathbb{C}$ is fibered. It is even a deformation of $(X, \mathcal{D}_\hbar^\infty)$ if we can yet show the flatness of $\mathcal{O}_\hbar \to (\mathcal{D}_{\bar{d}}^\infty)_{(x,\hbar)}$ for every $x \in X$.

To prove its flatness for an arbitrary $\hbar \in \mathbb{C}$ it is enough to show it for $\hbar = i$, because the diagram

$$\begin{tikzcd} \mathcal{O}_i \arrow[r] \arrow[d] & \mathcal{O}_\hbar \arrow[d] \\ (\mathcal{D}_{\bar{d}}^\infty)_{(x,i)} \arrow[r] & (\mathcal{D}_{\bar{d}}^\infty)_{(x,\hbar)} \end{tikzcd}$$

commutes. Here the isomorphism $\mathcal{O}_i \to \mathcal{O}_\hbar$ is given by $f_i \mapsto [f \circ (\_ + i - \hbar)]_\hbar$ and similarly for $(\mathcal{D}_{\bar{d}}^\infty)_{(x,i)} \to (\mathcal{D}_{\bar{d}}^\infty)_{(x,\hbar)}$. Now $\mathcal{D}_i$ (resp. $\mathcal{D}_i^\infty$) is isomorphic to the sheaf $\mathcal{D}_X$ (resp. $\mathcal{D}_X^\infty$) of differential operators of finite (resp. infinite) order over X. By results of microlocal analysis (see Björk [6], Theorem 3.4.4) we already know that $\mathcal{D}_X \to \mathcal{D}_X^\infty$ is faithfully flat as well as $\mathcal{D}_{X\times\mathbb{C}} \to \mathcal{D}^{\infty}_{X\times\mathbb{C}}$. Furthermore we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_i \arrow[r, "\Delta_i^{\infty}"] & (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)} \arrow[r, "\kappa^{\infty}"] & \left(\mathcal{D}^{\infty}_{X\times\mathbb{C}}\right)_{(x,i)} \\
\mathcal{O}_i \arrow[u] \arrow[r, "\Delta_i"'] & (\mathcal{D}_{\bar{d}})_{(x,i)} \arrow[u, "\beta"'] \arrow[r, "\kappa"'] & (\mathcal{D}_{X\times\mathbb{C}})_{(x,i)} \arrow[u]
\end{tikzcd}$$

where κ (resp. κ^{∞}) are the natural embeddings. We want to show Δ_i^{∞} being faithfully flat. For that it suffices to prove the faithful flatness of β because we already know Δ_i has this property. Now as $\mathcal{O}_i$-module $(\mathcal{D}_{X\times\mathbb{C}})_{(x,i)}$ is isomorphic to the direct sum $(\mathcal{D}_{\bar{d}})^{\mathbb{N}}_{(x,i)}$, so κ is faithfully flat. By the above result from microlocal analysis $\kappa^{\infty}\circ\beta$ then is faithfully flat. By MALGRANGE [33], Prop. 4.7 β is faithfully flat iff for every ideal $J \subset (\mathcal{D}_{\bar{d}})_{(x,i)}$ the identity

$$J\ (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)} \ = \ J\,\mathcal{D}^{\infty}_{(x,i)} \cap (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)} \tag{2.75}$$

holds. "$\subset$" is clear, so we prove "$\supset$". Let $d = \sum j_k\, d_k \in J\,\mathcal{D}^{\infty}_{(x,i)}$ such that $d \in (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$, $d_k \in \mathcal{D}^{\infty}_{(x,i)}$, $j_k \in J$. Then d_k can be written uniquely as a direct sum $d_k = d_{k0} + \tilde{d}_k$, where $d_{k0} \in (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$ and $\tilde{d}_k = \sum\limits_{\nu\geq 1} d_{k1} \left(\frac{\partial}{\partial\hbar}\right)^{\nu}$ with $d_{k\nu} \in (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$. This gives $d = d_0 + \tilde{d}$, where $d_0 = \sum j_k d_{k0}$, $\tilde{d} = \sum j_k\, d_{k1} = \sum\limits_{\nu\geq 1} \left(\sum d_{k\nu}\right) \left(\frac{\partial}{\partial\hbar}\right)^{\nu}$. As $d_0, d \in (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$ and $\tilde{d} = 0$ or $\tilde{d} \notin (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$, the expansion $d = d_0 + \tilde{d}$ gives $\tilde{d} = 0$, so $d = d_0 \in J\ (\mathcal{D}^{\infty}_{\bar{d}})_{(x,i)}$. This proves the faithful flatness of β and $\beta\circ\Delta_i$. Furthermore we receive the following theorem.

Theorem 2.5.3 *The sheaf morphism $D^{\infty} = (\bar{d}, \Delta^{\infty}) : (X\times\mathbb{C}, \mathcal{D}^{\infty}_{\bar{d}}) \to (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is fibered and a flat deformation of $(X, \mathcal{D}^{\infty}_{\hbar})$ for every distinguished point $\hbar \in \mathbb{C}$.*

Furthermore associating the deformation D^{∞} of $(X, \mathcal{D}^{\infty}_{\hbar})$ to a complex manifold $(X, \mathcal{O}_X)$ gives rise to a functor from the category of complex manifolds to the category $\underline{\mathsf{Def}}^{flat}_{0,(\mathbb{C},\mathcal{O})}$ of flat deformations over $(\mathbb{C}, \mathcal{O})$ with distinguished point 0.

PROOF: It only remains to show functoriality. But as the construction of the sheaf of infinite order differential operators on a complex manifold is natural, the claim follows.

□

Corollary 2.5.4 *The sheaf $\mathcal{D}^{\infty}_0$ on a complex manifold X of holomorphic functions of bounded growth on T^*X can be be flatly deformed to the sheaf $\mathcal{D}^{\infty} = \mathcal{D}^{\infty}_i$ of infinite order differential operators on X.*

Chapter 3

Symbol Calculus on Manifolds

Before we proceed further with our quantization program, we need an important tool, namely a complete symbol calculus on manifolds. Such a complete symbol calculus is also of interest by its own, for example because it provides a construction for inverses of elliptic differential operators in the algebra of pseudo-differential operators over a manifold, and was introduced in WIDOM [63, 64]. With the help of the complete symbol calculus we will be able to write down a more concrete formula for the product of two quantized observables and can extend the deformation quantization of polynomial observables in the last chapter. We use WIDOM'S papers [63, 64] only from section 3.3 on and depart in several points from his original ideas. Therefore we will describe the constructions in rather detail.

3.1 Symbols

The symbol spaces will be defined over base spaces of smooth vector bundles or more generally over base spaces of so-called conic bundles which we introduce now.

Definition 3.1.1 *A* **conic manifold** *is a smooth $\mathbb{R}^+$ principal fiber bundle $p : P \to M$. The smooth manifold $M = P/\mathbb{R}^+$ is the* **orbit space** *of the conic manifold. A map $f : P \to Q$ between conic manifolds is called* **conic**, *if it is smooth and invariant with respect to the $\mathbb{R}^+$-action.*

A fiber bundle $E \xrightarrow{\pi} X$ is called a **conic bundle**, *if its typical fiber P and E itself are conic manifolds such that the fiber bundle E has an atlas of trivializations consisting of conic maps.*

Lemma 3.1.2 *Every conic manifold $p : P \to M$ is trivial as an $\mathbb{R}^+$ fiber bundle.*

PROOF: Let (φ_i) be a partition of unity of M subordinate to a covering (U_i) of M by coordinate patches with trivializations $\psi_i : P|_{U_i} \to U_i \times \mathbb{R}^+$. Then define $\psi : P \to M \times \mathbb{R}^+$ by

$$\psi(c) \;=\; \left(p(c), \sum_i (pr_2 \circ)\psi_i(c) \cdot \varphi_i(p(c))\right). \tag{3.1}$$

ψ is surjective: Let $\theta \in \mathbb{R}^+$, $m \in M$. Then there exists $c \in p^{-1}(m)$. We have $\sum_i (pr_2 \circ \psi)(c) \cdot \varphi_i(p(c)) = \zeta \in \mathbb{R}^+$ and $\psi\left(\frac{\theta}{\zeta}c\right) = (m, \theta)$.

ψ is injective: Let $\psi(c) = \psi(d)$. As $p(c) = p(d)$, there exists a $\theta \in \mathbb{R}^+$ with $c = \theta d$. Then $(\theta - 1) \sum_i (pr_2 \circ \psi)(d) \cdot \varphi_i(p(d)) = 0$, that means $\theta = 1$. □

Example 3.1.3 (*i*) Let V be a real HILBERT space. Then $\dot{V} = V - \{0\}$ is a conic manifold over the unit sphere SV of V.

(*ii*) Let $E \to X$ be an n-dimensional vector bundle with metric g. Denote by $\dot{E}$ the fiber bundle over X generated by taking out the zero section of E. Then $\dot{E}$ is a conic bundle over the sphere bundle SE.

Any $\theta \in \mathbb{R}^+$ gives rise to a diffeomorphism $\theta : P \to P$. A function $f : P \to \mathbb{C}$ is now called homogeneous of order $\mu \in \mathbb{R}$, if $\theta^* f = \theta^\mu f$ for all $\theta \in \mathbb{R}^+$. Similarly a smooth vector field V on P is called homogeneous of order $\nu \in \mathbb{R}$, if $\theta^* V = \theta^\nu f$ for all $\theta \in \mathbb{R}^+$, where $\theta^* V$ is the vector field on P with $\theta^* V(c) = (T\theta^*)^{-1} V(c\theta)$ for all $c \in P$. In case f is smooth, we have

$$\theta^*(Vf) \;=\; \theta^*(V)\theta^*(\theta) \;=\; \theta^{\mu+\nu}(Vf). \tag{3.2}$$

Remark 3.1.4 The fiber bundle $p : P \to M$ also gives rise to a fundamental vector field $\mathfrak{t}^\# : P \to TP$ for any $\mathfrak{t} \in \mathbb{R} = Lie(\mathbb{R}^+)$. In case $\mathfrak{t} = 1$ we write $\Theta : P \to TP$ for $\mathfrak{t}^\#$. Θ is 0-homogeneous, because $(T\zeta^*)^{-1} \frac{\partial}{\partial\theta}\Big|_{\theta=0} (c\zeta\theta) = \frac{\partial}{\partial\theta}\Big|_{\theta=0} (c\theta)$ for all $\zeta \in \mathbb{R}^+$ and $c \in P$.

Definition 3.1.5 *Let $E \to X$ be a conic bundle, $\mu \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$ and $U \subset X$ open. The space $\mathrm{S}^\mu_{\rho,\delta}(U, E)$ of* **symbols** *on E over U of* **order** *μ and* **type** *(ρ, δ) consists of all functions $a \in \mathcal{C}^\infty(E|_U)$ with the following property:*

Let $K \subset E|_U$ compact, $L_1, ..., L_k$ 0-homogeneous vector fields and $V_1, ..., V_l$ (-1)-homogeneous vertical vector fields on $E|_U$. Then there exists $C = C_{K,L_1,...,L_k,V_1,...,V_l}(a) > 0$ such that

$$\left[\theta^* \left(L_k \cdot ... \cdot L_1 \cdot V_l \cdot ... \cdot V_1\, a\right)\right]\big|_K \;\leq\; C(1+\theta)^{\mu+k\delta-l\rho}. \tag{3.3}$$

for all $\theta \in \mathbb{R}^+$.

The elements of $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ *are the sections of the sheaf* $\mathrm{S}^{\mu}_{\rho,\delta}(\cdot,E)$ *of symbols on* E *of order* μ *and type* (ρ,δ).

By $\mathrm{S}^{-\infty}(U,E) = \bigcap_{\mu\in\mathbb{R}} \mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ *and* $\mathrm{S}^{\infty}_{\rho,\delta}(U,E) = \bigcup_{\mu\in\mathbb{R}} \mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ *we denote the sheaf of smoothing symbols resp. the sheaf of symbols of type* (ρ,δ).

Remark 3.1.6 The above definition of a symbol is an extension of the one given in DUISTERMAAT [15].

We can attach seminorms $p = p_{K,L_1,\ldots,L_k,V_1,\ldots,V_l} : \mathrm{S}^{\mu}_{\rho,\delta}(U,E) \to \mathbb{R}^+ \cup \{0\}$ to the symbol spaces $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ by

$$p(a) \;=\; \sup\left\{ \frac{|\theta^*(L_k\cdot\ldots\cdot L_1\cdot V_l\cdot\ldots\cdot V_1\, a)(c)|}{(1+\theta)^{\mu+k\delta-l\rho}} \;:\; \theta\in\mathbb{R}^+,\; c\in K \right\}, \tag{3.4}$$

where K and the vector fields $L_1,\ldots,L_k,V_1,\ldots,V_l$ are like in Definition 3.1.5. $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ then becomes a FRÉCHET space, such that the restriction morphisms $\mathrm{S}^{\mu}_{\rho,\delta}(U,E) \to \mathrm{S}^{\mu}_{\rho,\delta}(V,E)$ for $V\subset U$ open are continuous. Additionally we have natural and continuous inclusions $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)\subset \mathrm{S}^{\tilde\mu}_{\tilde\rho,\tilde\delta}(U,E)$ for $\tilde\mu\geq\mu$, $\tilde\rho\leq\rho$ and $\tilde\delta\geq\delta$.

Remark 3.1.7 By the inductive system $\left(\mathrm{S}^{\mu}_{\rho,\delta}(U,E)\to\mathrm{S}^{\infty}_{\rho,\delta}(U,E)\right)_{\mu\in\mathbb{R}}$ of locally convex topological vector spaces $\mathrm{S}^{\infty}_{\rho,\delta}(U,E)$ can be given the structure of a locally convex topological vectors space. This l.c. topology is too coarse, as $\mathrm{S}^{-\infty}(U,E)$ would then be dense in $\mathrm{S}^{\infty}_{\rho,\delta}(U,E)$ (cf. GRIGIS, SJÖSTRAND [24], Proposition 1.7). We will use another l.c. topology on $\mathrm{S}^{\infty}_{\rho,\delta}(U,E)$ resp. $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(U,E)$. See Theorem 3.1.9.

An easy but somewhat technical consideration shows that pointwise multiplication of $a\in\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ and $b\in\mathrm{S}^{\tilde\mu}_{\rho,\delta}$ defines an element $ab\in\mathrm{S}^{\mu+\tilde\mu}_{\rho,\delta}(U,E)$. This map is bilinear and gives a continuous, commutative and associative product on $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,E)$ which even commutes with restrictions. So $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,E)$ becomes a sheaf of commutative algebras on X, and $\mathrm{S}^{-\infty}(\cdot,E)$ an ideal sheaf in $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,E)$. We denote by $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot,E)$ the quotient $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,E)/\mathrm{S}^{-\infty}(\cdot,E)$.

In the following we will only be concerned with symbols on vector bundles $\pi : E\to X$ with metric g according to Example 3.1.3 (ii). Then an element $a\in\mathcal{C}^{\infty}(E|_U)$ with $U\subset X$ open is a called a **symbol** on E over U of **order** μ and **type** (ρ,δ) if its restriction to $\dot{E}$ is a symbol of order μ and type (ρ,δ), i.e. iff the following condition is satisfied:

For every trivialization $(x,\xi) : E|_V \to \mathbb{R}^n \times \mathbb{R}^N$ with $V \subset U$ open, every $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$ and every $K \subset V$ compact there exists $C = C_{K,\alpha,\beta} > 0$ such that the inequality

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \xi^\beta} a(e) \right| = C\,(1 + |e|)^{\mu+\delta|\alpha|-\rho|\beta|} \tag{3.5}$$

holds for every $e \in E|_K$.

We write $\mathrm{S}^\mu_{\rho,\delta}(\cdot, E)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, E)$ resp. $\mathrm{S}^{-\infty}(\cdot, E)$ for the corresponding symbol sheaves and $\mathrm{S}^\mu_{\rho,\delta}(\cdot, X)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, X)$ resp. $\mathrm{S}^{-\infty}(\cdot, X)$ in case that E is the cotangent bundle T^*X of a Riemannian manifold X. Sometimes we even skip the X and write only $\mathrm{S}^\mu_{\rho,\delta}(\cdot)$ and so on, if by the context it is clear what we mean. Note that $\mathrm{S}^\infty_{\rho,\delta}(\cdot, E) \neq \mathrm{S}^\infty_{\rho,\delta}(\cdot, \dot{E})$, but $\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, E) \cong \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, \dot{E})$.

Example 3.1.8 *(i)* Let X be a smooth manifold. Then every smooth function $a : T^*U \to \mathbb{C}$ which is a polynomial function on every fiber of order $\leq \mu$ with $\mu \in \mathbb{N}$ lies in $\mathrm{S}^\mu_{\rho,\delta}(U, T^*X)$. In particular we have an embedding $\mathcal{D}_0 \to \mathrm{S}^\infty_{\rho,\delta}(\cdot, T^*X)$.

(ii) Let $X = \mathbb{R}^n$, $n \in \mathbb{N}$ with its canonical Euclidean structure. Then the mapping $l : T^*X = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $(x,\xi) \mapsto |\xi|^2 = \sum\limits_{k=1}^n \xi_k^2$ is a symbol of order 2 and type $(1,0)$ on $\mathbb{R}^n$, but not one of order $\mu < 2$. Next regard the function $a : T^*X = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $(x,\xi) \mapsto \frac{1}{1+|\xi|^2} = \sum\limits_{k=1}^n \frac{1}{1+\xi_k^2}$. This function is a symbol of order -2 and type $(1,0)$ on $\mathbb{R}^n$, but not one of order $\mu < -2$.

See Hörmander [29] or Grigis, Sjöstrand [24] for more material about and examples of symbols.

The following theorem is an essential tool for the use of symbols in the theory of partial differential equations. We will not give the proof, but refer the interested reader to Hörmander [29], Grigis, Sjöstrand [24], Shubin [47] or any other good book on pseudo-differential operators.

Theorem 3.1.9 *Let $a_j \in \mathrm{S}^{\mu_{m_j}}_{\rho,\delta}(U,E)$ be symbols such that $m_j \in \mathbb{N}$ for all $j \in \mathbb{N}$ and $\lim\limits_{j\mapsto\infty} m_j = -\infty$. Then there exists a symbol $a \in \mathrm{S}^{\mu_{m_j}}_{\rho,\delta}(U,E)$ unique up to smoothing symbols, such that $a - \sum a_j \in \mathrm{S}^{\mu_{m_j}}_{\rho,\delta}(U,E)$ for all $j \in \mathbb{N}$.*

This induces a locally convex Hausdorff *topology on the vector space $\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(U,E)$, which is called the* **topology of asymptotic convergence**.

We will later interpret symbol spaces - or better their quotient spaces modulo smoothing symbols - as spaces of classical observables on T^*X. The polynomial observables then correspond to a special subclass of observables. The following proposition gives the precise statement.

Proposition 3.1.10 *Let X be a* RIEMANNIAN *manifold, and $\mathcal{D}_0$ the sheaf of polynomial observables on X. Then by composition with the projection $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,X) \to \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot,X)$ the canonical embedding*

$$\begin{array}{rcl} \mathcal{D}_0 & \to & \mathrm{S}^{\infty}_{\rho,\delta}(\cdot,X) \\ \mathcal{D}_0(U) \ni f & \mapsto & f \in \mathrm{S}^{\infty}_{\rho,\delta}(U,X) \qquad \textit{for } U \subset X \textit{ open} \end{array} \tag{3.6}$$

gives rise to a monomorphism $\sigma_0 : \mathcal{D}_0 \to \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot,X)$ of sheaves of algebras.

PROOF: If $f, g \in \mathcal{D}_0(U)$ are polynomial observables such that $f - g \in \mathrm{S}^{-\infty}(U,X)$, then $f - g$ is a bounded polynomial function on each fiber of T^*U. Therefore $f - g$ is constant on each fiber, hence $f - g = 0$ by $f - g \in \mathrm{S}^{-\infty}(U,X)$. □

3.2 Fourier transform and pseudo-differential operators on manifolds

Let a be a polynomial observable defined on the (trivial) cotangent bundle $T^*U = U \times \mathbb{R}^n$ of an open set $U \subset \mathbb{R}^n$ of EUCLIDEAN space. According to the results of chapter 2 the quantization $\mathfrak{q}_\hbar(a)$ of $a = \sum_\alpha (a_\alpha \circ \pi)\xi^\alpha$, $a_\alpha \in \mathcal{C}^\infty(U)$ can be identified for $\hbar \neq 0$ with the differential operator

$$\mathcal{C}^\infty(U) \ni f \mapsto \left(x \mapsto \sum_\alpha a_\alpha \left(\frac{\hbar}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial z^\alpha} f \right) \in \mathcal{C}^\infty(U). \tag{3.7}$$

Using the FOURIER transform $\hat{f}$ the following integral representation of $\mathfrak{q}_\hbar(a)$ holds:

$$\mathfrak{q}_\hbar(a)\, f \;=\; \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}<\xi,\cdot>}\, a(\xi)\, \hat{f}(\xi)\, d\xi. \tag{3.8}$$

It would be very helpful for structural and calculational considerations to extend this formula to RIEMANNIAN manifolds. To achieve this it is necessary to have an appropriate notion of FOURIER transform on manifolds. In the following we are going to define such a FOURIER transform and will get back later to the problem of an integral representation for the quantization map $\mathfrak{q}_\hbar$.

3.2.1 Assume X to be a Riemannian manifold and consider the exponential function exp with respect to the Levi-Civita connection on X. Choose a neighborhood $W \subset TX$ of the zero section in TX such that $(\pi, \exp) : W \to X \times X$ maps W diffeomorphically onto an open neighborhood of the diagonal Δ of $X \times X$. Then there exists a smooth function $\psi : TX \to [0,1]$ called a **cut-off function** such that $\psi|_{\tilde{W}} = 1$ and $\mathrm{supp}(\psi) \subset W$ for an open neighborhood $\tilde{W} \subset W$ of the zero section in TX. Now the following microlocal lift $\mathcal{M}_\psi$ is well-defined:

$$\mathcal{M}_\psi : \mathcal{D}(X) \to \mathrm{S}^{-\infty}(X, TX), \quad f \mapsto \Big(v \mapsto {}^{\psi}f\,(v) = \psi(v) f(\exp v)\Big)\Big). \tag{3.9}$$

$\mathcal{M}_\psi$ is a linear but not multiplicative map between function spaces.

Over the tangent bundle TX we can define the **Fourier transform** as the following sheaf morphism:

$$\mathcal{F} : \mathrm{S}^{-\infty}(U, TX) \to \mathrm{S}^{-\infty}(U, T^*X), \quad a \mapsto \hat{a}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{T_{\pi(\xi)}X} e^{-i<\xi,v>}\, a(v)\, dv. \tag{3.10}$$

We also have a **reverse Fourier transform**:

$$\mathcal{F}^{-1} : \mathrm{S}^{-\infty}(U, T^*X) \to \mathrm{S}^{-\infty}(U, TX), \quad b \mapsto \check{b}(v) = \frac{1}{(2\pi)^{n/2}} \int_{T_{\pi(v)}X} e^{i<\xi,v>}\, b(\xi)\, d\xi. \tag{3.11}$$

It is easy to check that $\mathcal{F}^{-1}$ and $\mathcal{F}$ are inverse to each other, indeed.

Composing $\mathcal{M}_\psi$ and $\mathcal{F}$ gives rise to the **Fourier transform** $\mathcal{F}_\psi = \mathcal{F} \circ \mathcal{M}_\psi$ on the Riemannian manifold X. $\mathcal{F}_\psi$ has the left inverse

$$\mathrm{S}^{-\infty}(U, T^*X) \to \mathcal{C}^\infty(U), \quad a \mapsto \mathcal{F}^{-1}(a)|_{0_U}, \tag{3.12}$$

where $|_{0_U}$ means the restriction to the natural embedding of U into T^*X as zero section.

By definition $\mathcal{M}_\psi$ and $\mathcal{F}_\psi$ do depend on the smooth cut-off function ψ, but this arbitrariness will only have minor effects on the study of pseudo-differential operators defined through $\mathcal{F}_\psi$. We will get back to this point later on in this chapter.

3.2.2 Let a be a smooth function defined on T^*U and polynomial in the fibers. Then locally $a = \sum_\alpha (a_{x,\alpha} \circ \pi)\, \xi_x$ with respect to a normal coordinate system at $x \in U$ and functions $a_{x,\alpha} \in \mathcal{C}^\infty(U)$. Define the operator $A : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ by

$$f \mapsto Af \;=\; \left(U \ni x \mapsto \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, \hat{{}^{\psi}f}(\xi)\, d\xi \in \mathbb{C}\right) \tag{3.13}$$

and check that

$$Af\,(x) \;=\; \sum_\alpha a_{x,\alpha}(x)\,\frac{\partial^{|\alpha|}}{\partial {\xi_x}^\alpha}\left[\psi\,(f\circ\exp)\right]\,(0_x) \;=\; \sum_\alpha a_{x,\alpha}(x)\,\frac{\partial^{|\alpha|}}{\partial {z_x}^\alpha} f\,(x) \tag{3.14}$$

for all $x \in U$. Therefore Af is a differential operator independent of the choice of ψ.

However, if a is an arbitrary element of the symbol space $\mathrm{S}^\infty_{\rho,\delta}(U)$, Eq. (3.13) defines a continuous operator $A = A_\psi : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$, which is a pseudo-differential operator and independent of the cut-off function ψ only up to smoothing operators. Let us show this in more detail.

First assume $0 \leq \rho, \delta \leq 1$, $\rho > \delta$ and $\rho + \delta \geq 1$ for the rest of this chapter. Then choose $\mu \in \mathbb{R}$ such that $a \in \mathrm{S}^\mu_{\rho,\delta}(U)$, and consider the kernel distribution $\mathcal{D}(U \times U) \to \mathbb{C}$, $(f \otimes g) \mapsto \, < g, A_\psi f > \, = \, ∫_X \, g(x)\, A_\psi f(x)\, dx$ of A_ψ. It can be written as an oscillatory integral:

$$\begin{aligned}
&< g, A_\psi f > = \\
&= \frac{1}{(2\pi)^n} \int_{TX \oplus T^*X} \psi(v)\, g(\pi(v))\, f(\exp v)\, e^{-i<\xi,v>}\, a(\xi)\, d(\xi, v) \\
&= \frac{1}{(2\pi)^n} \int_X \int_{T_xX} \int_{T_x^*X} \psi(v)\, g(x)\, f(\exp v)\, e^{-i<\xi,v>}\, a(\xi)\, d\xi\, dv\, dx,
\end{aligned} \tag{3.15}$$

To check this use a density argument or in other words approximate the symbol a by elements $a_k \in \mathrm{S}^{-\infty}(U)$ and prove the statement for the a_k. The claim then follows from continuity of both sides of Eq. (3.15) with respect to a. Let $\tilde\psi : TX \to [0,1]$ be another cut-off function such that $\tilde\psi|_{\tilde O} = 1$ on a neighborhood $\tilde O \subset W$ of the zero section of T^*X. We can assume $O = \tilde O$ and claim the operator $A_\psi - A_{\tilde\psi}$ to be smoothing. For the proof it suffices to show that the (weak oscillatory) integral

$$K_{A_\psi - A_{\tilde\psi}}(v) \;=\; \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} (\psi(v) - \tilde\psi(v))\, e^{-i<\xi,v>}\, a(\xi)\, d\xi, \quad v \in TU \tag{3.16}$$

which comes from the integral representation for the kernel of $A_\psi - A_{\tilde\psi}$ defines a smooth function $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$. The phase function $T^*_{\pi(v)}X \ni \xi \mapsto -\xi(v) \in \mathbb{R}$ has only critical points for $v = 0$. Hence for $v \neq 0$ there exists a (-1)-homogeneous vertical vector field L on T^*X such that $L\, e^{-i<\cdot,v>} = e^{-i<\cdot,v>}$. The adjoint $L^\dagger$ of L satisfies $\left(L^\dagger\right)^k a \in \mathrm{S}^{\mu-\rho k}_{\rho,\delta}(U)$, where $k \in \mathbb{N}$. As the amplitude $(v,\xi) \mapsto (\psi(v) - \tilde\psi(v))\, a(\xi)$ vanishes on $O \times_U T^*U$, the equation

$$K_{A_\psi - A_{\tilde\psi}}(v) = \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} e^{-i<\xi,v>} \, \left(L^\dagger\right)^k a(\xi)\, d\xi \tag{3.17}$$

holds for any integer k fulfilling $\mu+\rho k < -1$. Note that the integral in Eq. (3.17) unlike the one in Eq. (3.16) is to be understood in the sense of LEBESGUE. Hence $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$ and $A_\psi - A_{\tilde\psi}$ is smoothing.

Let us now prove that A_ψ lies in the space $\Psi^\mu_{\rho,\delta}(U)$ of pseudo-differential operators of order μ and type (ρ,δ) over U, if $\rho+\delta \geq 1$ and $\rho > \delta$. We have to check that A_ψ is pseudolocal and looks with respect to some local coordinates like a pseudo-differential operator on an open set of $\mathbb{R}^n$.

(i) A_ψ is pseudolocal: Let $u, v \in \mathcal{C}_0^\infty(U)$ and $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Then the integral kernel K of $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ has the form

$$K(v) \;=\; \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} \psi(v)\, a(\xi)\, v(\pi(v))\, u(\exp v)\, e^{-i<\xi,v>}\, d\xi. \tag{3.18}$$

There exists an open neighborhood O of the zero section in TX such that the amplitude $\psi\,(v \circ \pi)\,(u \circ \exp)\, a$ vanishes on $O \times_U T^*X$. As for all nonzero $v \in TX$ the phase function $\xi \mapsto -<\xi, v>$ is noncritical, an argument similar to the one above for $K_{A_\psi - A_{\tilde\psi}}$ shows that K is a smooth function on TU, so $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ is smoothing.

(ii) A_ψ is a pseudo-differential operator in appropriate coordinates: Choose for $x \in U$ an open neighborhood $U_x \subset U$ so small that any two points in U_x can be connected by a unique geodesic. In particular we assume that $U_x \times U_x$ lies in the range of $(\pi, \exp): W \to X \times X$. Furthermore let us suppose that $\psi \circ z_y(\tilde y) = 1$ for all $y, \tilde y \in U_x$ and $z_y : U_x \to T_yX$, $\tilde y \mapsto \exp_y^{-1}(\tilde y)$. Then we have for $f \in \mathcal{C}^\infty(U_x)$, $y \in U_x$ and with $z_y = \exp_y^{-1} : U_x \to T_yX$:

$$\begin{aligned}
Af\,(y) \;&=\\
&=\; \frac{1}{(2\pi)^{n/2}} \int_{T^*_yX} a(\xi)\, {}^{\psi}\!\hat f(\xi)\, d\xi\\
&=\; \frac{1}{(2\pi)^{n/2}} \int_{T^*_xX} a\Big(\underbrace{T_{z_x(y)} \exp{}_x^*}_{=\,d_x(y)}(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \mathcal{F}(f \circ \exp)(d_x(y)\,(\xi))\, d\xi\\
&=\; \frac{1}{(2\pi)^n} \int_{T^*_xX} \int_{T_yX} a\Big(d_x(y)\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, (f \circ \exp{}_y)(v)\, e^{-i<d_x(y)\,(\xi),v>}\, dv\, d\xi\\
&=\; \frac{1}{(2\pi)^n} \int_{T^*_xX} \int_{T_xX} a\Big(d_x(y)\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \Big|\underbrace{\det\,(T\, z_y \circ z_x^{-1})}_{=\,Z_{y,x}}(v)\Big|\\
&\qquad (f \circ z_x^{-1})(v)\, e^{-i<d_x(y)\,(\xi), z_y \circ z_x^{-1}(v)>}\, dv\, d\xi
\end{aligned} \tag{3.19}$$

Now the phase function $(v,\xi) \mapsto -< d_x(y)(\xi), z_y \circ z_x^{-1}(v) >$ vanishes in case $\xi \neq 0$ iff $v = z_x(Y)$. Hence after possibly shrinking U_x the KURANISHI trick gives a smooth function

$G_x : U_x \times T_xX \to \mathrm{Iso}\left(T_x^*X, T_y^*X\right) \subset \mathrm{Lin}\left(T_x^*X, T_y^*X\right)$ such that $< d_x(y)(\xi), z_y \circ z_x^{-1}(v) >$ $=< G_x(y,v)(\xi), v - z_x(y) >$ for all $(y,v) \in U_x \times T_xX$ and $\xi \in T_x^*X$. A change of variables in Eq. (3.19) then gives:

$$
\begin{aligned}
Af(y) &= \\
&= \frac{1}{(2\pi)^n} \int_{T_x^*X} \int_{T_xX} a\Big(\big[d_x(y)\, G_x(y,v)\big]\,(\xi)\Big) \,\big|\det d_x(y)\big| \,\big|Z_{y,x}(v)\big| \,\big|\det G_x(y,v)\big| \\
&\quad (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{T_x^*X} \int_{T_xX} \tilde{a}_x(y,v,\xi)\, (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi, \qquad (3.20)
\end{aligned}
$$

where $(y,v,\xi) \mapsto \tilde{a}_x(y,v,\xi) = a\Big(\big[d_x(y)\, G_x(y,v)\big]\,(\xi)\Big) \,\big|\det d_x(y)\big| \,\big|Z_{y,x}(v)\big| \,\big|\det G_x(y,v)^{-1}\big|$ is a symbol lying in $\mathrm{S}^{\mu}_{\rho,\delta}(U_x \times U_x \times T_xX)$, if $\rho + \delta \geq 1$ and $\rho > \delta$. Hence A is a pseudo-differential operator with respect to normal coordinates over U_x of order μ and type (ρ, δ).

Our considerations now lead us to

Theorem 3.2.3 *Let X be a* RIEMANNIAN *manifold and* exp *the exponential function corresponding to the* LEVI-CIVITA *connection on X. Further assume that $0 \leq \rho,\ \delta \leq 1$, $\rho + \delta \geq 1$ and $\rho > \delta$. Then any smooth cut-off function $\psi : TX \to [0,1]$ gives rise to a linear sheaf morphism $\Psi_\psi : \mathrm{S}^{\infty}_{\rho,\delta}(\cdot, X) \to \Psi^{\infty}_{\rho,\delta}(\cdot, X)$, $a \mapsto A_\psi$ defined by*

$$
A_\psi f\,(x) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, {}^{\psi}\!\hat{f}(\xi)\, d\xi, \qquad (3.21)
$$

where $a \in \mathrm{S}^{\infty}_{\rho,\delta}(U, X)$, $f \in \mathcal{D}(U)$, $x \in U$ and $U \subset X$ open. This morphism preserves the natural filtrations of $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot, X)$ and $\Psi^{\infty}_{\rho,\delta}(\cdot, X)$. In particular it maps the subsheaf $\mathrm{S}^{-\infty}(\cdot, X)$ of smoothing symbols to the subsheaf $\Psi^{-\infty}(\cdot, X) \subset \Psi^{\infty}_{\rho,\delta}(\cdot, X)$ of smoothing pseudo-differential operators.

The quotient morphism $\Psi : (\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X) \to (\Psi^{\infty}_{\rho,\delta}/\Psi^{-\infty})(\cdot, X)$ is an isomorphism and independent of the choice of the cut-off function ψ.

PROOF: The first part of the theorem has been shown above. So it remains to prove that $\Psi : (\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty})(U, X) \to (\Psi^{\infty}_{\rho,\delta}/\Psi^{-\infty})(U, X)$ is bijective for all open $U \subset X$. Let us postpone this till Theorem 3.3.3, where we will show that an explicit inverse of Ψ is given by the complete symbol map introduced in the following section. $\square$

3.3 The symbol map

In the sequel denote by $\varphi : O \to \mathbb{C}$ the smooth phase function defined by

$$\tilde{W} \ni (x,\xi) \mapsto <\xi, \exp_{\pi(\xi)}^{-1}(x)> = <\xi, z_{\pi(\xi)}(x)> \in \mathbb{C}, \tag{3.22}$$

where $O \subset X \times T^*X$ is an open neighborhood of the "diagonal" $\Delta = \{(x,\xi) \in X \times T^*X : x = \pi(\xi)\}$, such that $\exp_y^{-1}(x)$ is well-defined for every $x \in X$ and $y = \pi(\xi) \in X$ with $(x,\xi) \in O$.

Theorem and Definition 3.3.1 *Let $A \in \Psi^\mu_{\rho,\delta}(U,X)$ be a pseudo-differential operator on a* Riemannian *manifold X, $0 \leq \rho, \delta \leq 1$, $\rho + \delta \geq 1$, $\rho > \delta$ and $\psi : T^*X \to [0,1]$ a cut-off function. Then the ψ-***cutted symbol*** *of A with respect to the* Levi-Civita *connection on X is the function*

$$\sigma_{\psi,A} : T^*U \to \mathbb{C}, \quad \xi \mapsto \left[A\left(\psi_{\pi(\xi)}(\cdot)\, e^{i\varphi(\cdot,\xi)}\right)\right](\pi(\xi)), \tag{3.23}$$

where $\psi_x = \psi \circ \exp_x^{-1} = \psi \circ z_x$ for all $x \in X$. $\sigma_{\psi,A}$ is an element of $\mathrm{S}^\mu_{\rho,\delta}(U,X)$. The corresponding element σ_A in the quotient $(\mathrm{S}^\mu_{\rho,\delta}/\mathrm{S}^{-\infty})(U,X)$ is called the **complete symbol** *of A. It is independent of the choice of ψ.*

Remark 3.3.2 Before starting with the proof of the theorem let us mention that in the above theorem and definition we depart from the consideration of Widom in [64] insofar, as we have chosen a different (notion of a) phase function. Therefore our symbols are not the same as the ones in Widom's article.

Proof: Let us first check that $\sigma_{\psi,A}$ lies in $\mathrm{S}^\mu_{\rho,\delta}(U)$. We can find a sequence (x_i) of points in U and coordinate patches $U_i \subset U$ with $x_i \in U_i$ such that there exist normal coordinates $z_i = z_{x_i} : U_i \to T_{x_i}X$ on the U_i. We can even assume that the operator $A_i : \mathcal{D}(U_i) \to \mathcal{C}^\infty(U_i)$, $u \mapsto Au|_{U_i}$ induces a pseudo-differential operator on $T_{x_i}X$. In other words there exist symbols $a_i \in \mathrm{S}^\mu_{\rho,\delta}(O_i \times O_i, T^*_{x_i}X)$ with $O_i = z_i(U_i)$ such that A_i is given by the following oscillatory integral:

$$A_i u(y) = \frac{1}{(2\pi)^n} \int_{T^*_{x_i}X} \int_{T_{x_i}X} a_i(z_i(y), v, \xi)\, (u \circ z_i^{-1})(v)\, e^{-i<\xi, v - z_i(y)>}\, dv d\xi. \tag{3.24}$$

Now let (ϕ_i) be a partition of unity subordinate to the covering (U_i) of U and for every index i $\phi_{i,1}, \phi_{i,2} \in \mathcal{C}^\infty(U)$ such that $\mathrm{supp}(\phi_{i,1}) \subset U_i$, $\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_{i,2}) = \emptyset$ and

$\phi_{i,1} + \phi_{i,2} = 1$. Then the symbol $\sigma_{\psi,A}$ can be written in the following form:

$$\begin{aligned}\sigma_{\psi,A}(\zeta) &= \Big[A\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\Big]\,(\pi(\zeta)) \\ &= \frac{1}{(2\pi)^n}\sum_i \Big[\phi_{i,1}\, A_i\Big(\phi_i(\cdot)\,\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\Big]\,(\pi(\zeta)) \\ &\quad + K\Big(\,\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)(\pi(\zeta)) \end{aligned} \tag{3.25}$$

where

$$K : \mathcal{D}(U) \to \mathcal{C}^\infty(U) \quad f \mapsto \sum_i \,\phi_{i,2}\, A_i(\phi_i\, f) \tag{3.26}$$

is a smoothing pseudo-differential operator and $\zeta \in T^*X_i$. By the KURANISHI trick we can assume that there exists a smooth function $G : O_i \times O_i \to \mathrm{Lin}\left(T^*_{x_i}X, T^*_yX\right)$ such that the image of G consists of isomorphisms and

$$< G\left(z_i(y), T(z_i \circ z_y^{-1})(v)\right)(\xi), v > = < \xi, z_i \circ z_y^{-1}(v) - z_i(y) > \tag{3.27}$$

for $y \in U_i$, $v \in T_yX$ appropriate and $\xi \in T^*_{x_i}X$. Several changes of variables then give the following chain of oscillatory integrals for $\zeta \in T^*U_i$:

$$\begin{aligned}
&\Big[A_i\Big(\phi_i(\cdot)\,\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\Big]\,(y) \\
&= \frac{1}{(2\pi)^n}\int_{T^*_{x_i}X}\int_{T_{x_i}X} \phi_{1,i}(y)\, a_i\Big(z_i^{-1}(y), v, \xi\Big)\ \phi_i(z_i^{-1}(v))\,\psi(z_y \circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta, z_y \circ z_i^{-1}(v)>}\, e^{-i<\xi, v - z_i(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n}\int_{T^*_{x_i}X}\int_{T_yX} \phi_{1,i}(y)\, a_i\Big(z_i^{-1}(y), z_i \circ z_y^{-1}(v), \xi\Big)\ \phi_i(z_y^{-1}(v))\,\psi(v) \\
&\qquad e^{i<\zeta,v>}\, e^{-i<\xi, z_i \circ z_y^{-1}(v) - z_i(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n}\int_{T^*_{x_i}X}\int_{T_yX} \phi_{1,i}(y)\, a_i\Big(z_i^{-1}(y), z_i \circ z_y^{-1}(v), \xi\Big)\ \phi_i(z_y^{-1}(v))\,\psi(v) \\
&\qquad e^{i<\zeta,v>}\, e^{-i<G(z_i(y), T(z_i \circ z_y^{-1})(v))(\xi), v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n}\int_{T^*_yX}\int_{T_yX} \phi_{1,i}(y)\, a_i\Big(z_i^{-1}(y), z_i \circ z_y^{-1}(v), G\left(z_i(y), T(z_i \circ z_y^{-1})(v)\right)^{-1}(\xi)\Big) \\
&\qquad \phi_i(z_y^{-1}(v))\,\psi(v)\,\Big|\det G\left(z_i(y), T(z_i \circ z_y^{-1})(v)\right)\Big|^{-1} e^{i<\zeta,v>}\, e^{-i<\xi,v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n}\int_{T^*_{x_i}X}\int_{T_{x_i}X} b_i(z_i(y), v, \xi)\, e^{-i<\xi - T^*(z_y \circ z_i^{-1})(\zeta), v - z_i(y)>}\, dv\, d\xi,
\end{aligned} \tag{3.28}$$

where the smooth function b_i on $O_i \times O_i \times T^*_{x_i}X$ is defined by

$b_i(z_i(y), v, \xi) \ =$

$$= \phi_{1,i}(y)\, a_i \left(z_i^{-1}(y), z_i \circ z_y^{-1} \circ T(z_y \circ z_i^{-1})\,(v), G\big(z_i(y), v\big)^{-1} \circ \left(T(z_i \circ z_y^{-1}\right)^{\dagger}(\xi)\right)$$
$$\phi_i(z_y^{-1}(T(z_y \circ z_i^{-1})\,(v)))\, \psi(T(z_y \circ z_i^{-1})\,(v)) \left|\det G\,(z_i(y), v)\right|^{-1} \tag{3.29}$$

and lies in $\mathrm{S}^{\mu}_{\rho,\delta}(O_i \times O_i, T^*_{x_i}X)$. Let $B_i \in \Psi^{\mu}_{\rho,\delta}(O_i, T_{x_i}X)$ be the pseudo-differential operator on O_i corresponding to the symbol b_i. The symbol σ_i defined by $\sigma_i(v,\xi) = e^{-i<\xi,v>} B_i \left(e^{i<\xi,\cdot>}\right)$, where $v \in O_i$ and $\xi \in T^*_{x_i}X$, now is an element of $\mathrm{S}^{\mu}_{\rho,\delta}(O_i, T^*_{x_i}X)$ as one knows by the general theory of pseudo-differential operators on real vector spaces (see for example Grigis and Sjöstrand [24]). But we have

$$\left[\phi_{i,1}\, A_i\big(\phi_i(\cdot)\, \psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\big)\right](\pi(\zeta)) \;=\; \sigma_i\left(z_i \circ \pi(\zeta), T^*\left(z_{\pi(\zeta)} \circ z_i^{-1}\right)(\zeta)\right), \tag{3.30}$$

hence Eq. (3.25) shows $\sigma_{\psi,A} \in \mathrm{S}^{\mu}_{\rho,\delta}(U,X)$ if we can prove that $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U,X)$ with $\sigma_{\psi,K}(\zeta) = K\big(\,\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\big)(\pi(\zeta))$. So let $k : U \times U \to \mathbb{C}$ be the kernel function of K and write $\sigma_{\psi,K}$ as an integral:

$$\sigma_{\psi,K}(\zeta) = \int_U k(\pi(\zeta), y)\, \psi_{\pi(\zeta)}(y)\, e^{-i\varphi(y,\zeta)}\, dy. \tag{3.31}$$

Let further $\Xi_1, ..., \Xi_k$ be differential forms over U and $\mathcal{V}_1, ..., \mathcal{V}_k$ the vertical vector fields on T^*X corresponding to $\frac{\partial}{\partial \Xi_1}, ..., \frac{\partial}{\partial \Xi_1}$. Differentiating Eq. (3.31) under the integral sign and using an adjointness relation then gives

$$\begin{aligned}
&|\zeta|^{2m}\, \mathcal{V}_1 \cdot ... \cdot \mathcal{V}_k\, \sigma_{\psi,K}\,(\zeta) \;= \\
&= \int_U k(\pi(\zeta), y)\, \psi_{\pi(\zeta)}(y)\; < \Xi_1, z_{\pi(\zeta)} > \;...\; < \Xi_k, z_{\pi(\zeta)} > \\
&\quad \frac{\partial^{(2m,...,2m)}}{\partial z^{2mn}_{\pi(\zeta)}} e^{-i<\zeta, z_{\pi(\zeta)}(\cdot)>}(y)\, dy \\
&= \int_U \left(\frac{\partial^{(2m,...,2m)}}{\partial z^{2mn}_{\pi(\zeta)}}\right)^{\dagger} \Big(k(\pi(\zeta),\cdot)\, \psi_{\pi(\zeta)}(\cdot)\Big)\,(y) < \Xi_1, z_{\pi(\zeta)} > \;...\; < \Xi_k, z_{\pi(\zeta)} > \\
&\quad e^{-i<\zeta, z_{\pi(\zeta)}(y)>}\, dy.
\end{aligned} \tag{3.32}$$

Therefore $|\zeta|^{2m}\, \mathcal{V}_1 \cdot ... \cdot \mathcal{V}_k\, \sigma_{\psi,K}\,(\zeta)$ is uniformly bounded as long as $\pi(\zeta)$ varies in a compact subset of U. A similar argument for horizonatl derivatives of $\sigma_{\psi,K}$ finally proves $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U,X)$.

Next we have to show that $\sigma_{\psi,A} - \sigma_{\tilde{\psi},A}$ is a smooothing symbol for any second cut-off function $\tilde{\psi} : T^*X \to [0,1]$. It suffices to prove that $\tilde{\sigma}_i$ with

$$\tilde{\sigma}_i(\zeta) = A_i\, \left(\phi_i\, (\psi - \tilde{\psi}) \circ z_{\pi(\zeta)}\, e^{-i\varphi(\cdot,\zeta)}\right) \tag{3.33}$$

is smoothing for every $\phi_i \in \mathcal{D}(U_i)$. We can find a (-1)-homogeneous first order differential operator L on the vector bundle $T^*_{x_i}X \times T_{x_i}X \to T_{x_i}X$ such that for all $v \neq z_i(\pi(\zeta))$ and $\xi \in T^*_{x_i}X$ the equation

$$Le^{-i<\cdot,v-z_i(\pi(\zeta))>}(\xi) \;=\; e^{-i<\xi,v-z_i(\pi(\zeta))>} \tag{3.34}$$

holds (see for example GRIGIS, SJÖSTRAND [24] Lemma 1.12 for a proof). Because $(\psi - \tilde{\psi}) \circ z_{\pi(\zeta)} \circ z_i^{-1}$ vanishes on a neighborhood of $z_i(\pi(\zeta))$, the following equational chain of iterated integrals is also true:

$$\begin{aligned}
\tilde{\sigma}_i(\zeta) \;&= \\
&= \frac{1}{(2\pi)^n} \int_{T^*_{x_i}X} \int_{T_{x_i}X} a_i\Big(z_i^{-1}(y), v, \xi\Big)\; \phi_i(z_i^{-1}(v))\; \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta, z_{\pi(\zeta)} \circ z_i^{-1}(v)>}\; e^{-i<\xi, v - z_i(\pi(\zeta))>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{T^*_{x_i}X} \int_{T_{x_i}X} a_i\Big(z_i^{-1}(y), v, \xi\Big)\; \phi_i(z_i^{-1}(v))\; \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta, z_{\pi(\zeta)} \circ z_i^{-1}(v)>}\; L^k\, e^{-i<\xi, v - z_i(\pi(\zeta))>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{T^*_{x_i}X} \int_{T_{x_i}X} \Big(\Big(L^\dagger\Big)^k a_i\Big)\Big(z_i^{-1}(y), v, \xi\Big)\; \phi_i(z_i^{-1}(v))\; \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta, z_{\pi(\zeta)} \circ z_i^{-1}(v)>}\; e^{-i<\xi, v - z_i(\pi(\zeta))>}\, dv\, d\xi.
\end{aligned} \tag{3.35}$$

Note that the last integral of this chain is to be understood in the sense of LEBESGUE if $k \in \mathbb{N}$ is large enough. The same argument which was used for proving that the symbol $\sigma_{\psi,K}$ from Eq. (3.31) is smoothing now shows $\sigma_i \in \mathrm{S}^{-\infty}(U_i, X)$. □

After having defined the notion of a complete symbol, we will now give its essential properties in the following theorem.

Theorem 3.3.3 *Let $0 \leq \rho, \delta \leq 1$, $\rho + \delta \geq 1$ and $\rho > \delta$. Furthermore let $A \in \Psi^\mu_{\rho,\delta}(U)$ be a pseudo-differential operator on a* RIEMANNIAN *manifold X and $a = \sigma_{\psi,A} \in \mathrm{S}^\mu_{\rho,\delta}(U)$ its ψ-cutted symbol with respect to the* LEVI-CIVITA *connection. Then A and the pseudo-differential operator A_ψ defined by Eq. (3.21) are equal modulo smoothing operators. Moreover the sheaf morphisms $\sigma : (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, X) \to (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X)$ and $\Psi : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, X)$ are inverse to each other.*

Before proving the theorem let us first state a lemma.

Lemma 3.3.4 *The operator*

$$K : \mathcal{D}(U) \to \mathcal{C}^\infty(U), \quad f \mapsto \Big(U \ni x \mapsto A[(1-\psi_x)f]\,(x) \in \mathbb{C}\Big) \tag{3.36}$$

is smoothing for any cut-off function $\psi : TX \to \mathbb{C}$.

PROOF: Consider the operators $K_{a,i} : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ where $K_{a,i}f\,(x) = \phi_i(x)A[(1-\psi_a)f](x)$ for $x \in U$, $a \in U_i$. The open covering (U_i) of U and a subordinate partition of unity (ϕ_i) are chosen such that for every point $a \in U_i$ there exist normal coordinates z_a on U_i. We can even assume that $\psi(z_a(y)) = 1$ for all $a, y \in U_i$. Hence $\mathrm{supp}(1-\psi_a) \cap \mathrm{supp}(\phi_i) = \emptyset$, and $K_{a,i}$ is smoothing, because A is pseudolocal. This yields a family of smooth functions $k_i : U \times U \to \mathbb{C}$ such that

$$K_{a,i}f\,(x) \;=\; \int_U k_i(x,y)\,(1-\psi_a(y))\,f(y)\,dy. \tag{3.37}$$

Now the smooth function $k \in \mathcal{C}^\infty(U \times U)$ with $k(x,y) = \sum_i k_i(x,y)\,(1-\psi_x(y))$ is the kernel function of K. □

PROOF (of Theorem 3.3.3): Check that for an arbitrary cut-off function $\tilde{\psi} : TX \to \mathbb{C}$ one can find a cut-off function $\psi : TX \to \mathbb{C}$ such that the equation

$$\psi_x(y)\,f(y) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} \psi_x(y)\, e^{i<\xi, z_x(y)>}\, \hat{\tilde{\psi}f}(\xi)\,d\xi \tag{3.38}$$

holds for $x, y \in U$. Approximating the integral in Eq. (3.38) by RIEMANN sums, we can find a sequence $(i_k)_{k\in\mathbb{N}}$ of natural numbers, a family $(\xi_{k,i})_{k\in\mathbb{N}, 1\le i\le i_k}$ of elements $\xi_{k,i} \in T_x^*X$ and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of positive numbers such that the following equations are satisfied:

$$\lim_{k\to\infty} \epsilon_k \;=\; 0 \tag{3.39}$$

$$[(\psi_x)f]\,(y) \;=\; \lim_{k\to\infty} \tfrac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} \psi_x(y)\, e^{i<\xi_{k,i}\,,\, z_x(y)>}\, \hat{\tilde{\psi}f}(\xi_{k,i}). \tag{3.40}$$

By an appropriate choice of the $\xi_{k,i}$ one can even assume that the series on the right hand side of the last equation converges as a function of $y \in U$ in the topology of $\mathcal{D}(U)$. As A is continuous on $\mathcal{D}(U)$, we now have:

$$\begin{aligned} A[(\psi_x)f](x) \;&= \\ = \;& \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} A\left[\psi_x(\cdot)\, e^{i<\xi_{k,i}\,,\, z_x(\cdot)>}\right](x)\, \hat{\tilde{\psi}f}(\xi_{k,i}) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} \sigma_A(\xi_{k,i})\, \hat{\tilde{\psi} f}(\xi_{k,i}) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} \sigma_A(\xi)\, \hat{\tilde{\psi} f}(\xi)\, d\xi \\
&= A_{\tilde{\psi}} f. \qquad (3.41)
\end{aligned}
$$

On the other hand Lemma 3.3.4 provides for a smoothing operator K such that for $f \in \mathcal{D}(U)$ and $x \in U$:

$$
Af\,(x) \;=\; A[(\psi_x)f]\,(x) + Kf\,(x) \;=\; A_{\tilde{\psi}} f\,(x) + Kf\,(x). \qquad (3.42)
$$

Thus the first part of the theorem follows.

Now let $a \in \mathrm{S}^{\mu}_{\rho,\delta}(U, T^*X)$ be a symbol, and regard the corresponding pseudo-differential operator $A = A_\psi \in \Psi^{\mu}_{\rho,\delta}(U, X)$. We can assume after possibly altering ψ that there exists a second cut-off function $\tilde{\psi} : T^*X \to \mathbb{C}$ such that $\tilde{\psi}|_{\mathrm{supp}(\psi)} = 1$. Then we have for $\zeta \in T_x^*X$:

$$
\begin{aligned}
&\sigma_{A,\psi}(\zeta) \;= \\
&\quad= \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, \mathcal{F}\left(\tilde{\psi}\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)}\right)\right)(\xi)\, d\xi \\
&\quad= \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, \mathcal{F}\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)}\right)(\xi)\, d\xi \\
&\quad= \frac{1}{(2\pi)^{n}} \int_{T_x^*X} \int_{T_xX} a(\xi)\, \psi(v)\, e^{-i<\xi,v>}\, e^{i<\zeta,v>}\, dv\, d\xi, \qquad (3.43)
\end{aligned}
$$

where the last integral is an iterated one. Next choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that $\mathrm{supp}(\phi) \subset [-2,2]$ and $\phi|_{[-1,1]} = 1$ and define $a_k \in \mathrm{S}^{-\infty}(U)$ by $a_k(\xi) \;=\; \phi\left(\frac{|\xi|}{k}\right)$. Then obviously $a_k \to a$ in $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U)$ for $\tilde{\mu} > \mu$. We can now write:

$$
\begin{aligned}
&\sigma_{A,\psi}(\zeta) \;= \\
&\quad= \lim_{k\to\infty} \frac{1}{(2\pi)^{n}} \int_{T_x^*X} \int_{T_xX} a_k(\xi)\, \psi(v)\, e^{-i<\xi-\zeta,v>}\, dv\, d\xi \\
&\quad= \lim_{k\to\infty} a_k(\zeta) - \frac{1}{(2\pi)^{n/2}} \int_{T_xX} (1-\psi(v))\, \left[\mathcal{F}^{-1} a_k\right](v)\, e^{i<\zeta,v>}\, dv \\
&\quad= a(\zeta) - \lim_{k\to\infty} \frac{1}{(2\pi)^{n/2}} \int_{T_xX} (1-\psi(v))\, \left[\mathcal{F}^{-1}\left(\left(L^\dagger\right)^m a_k\right)\right](v)\, e^{i<\zeta,v>}\, dv \\
&\quad= a(\zeta) + \lim_{k\to\infty} K_m(\zeta, a_k), \qquad (3.44)
\end{aligned}
$$

where $m \in \mathbb{N}$, L is a smooth (-1)-homogeneous first order differential operator on the vector bundle $T^*X \times_U \dot{T}U \to \dot{T}U$ such that $Le^{i<\cdot,v>}(\xi) = e^{i<\xi,v>}$, $L^\dagger$ is also (-1)-homogeneous and

$$K_m(\zeta, a_k) \;=\; -\frac{1}{(2\pi)^{n/2}} \int_{T_xX} (1-\psi(v)) \left[\mathcal{F}^{-1}\left(\left(L^\dagger\right)^m a_k\right)\right](-v)\, e^{i<\zeta,v>}\, dv. \tag{3.45}$$

Note that $\dot{T}U$ consists of all nonzero vectors $v \in TU$ and that $\left(L^\dagger\right)^m a \in \mathrm{S}^{\tilde{\mu}-\rho m}_{\rho,\delta}$ for every $\tilde{\mu} \geq \mu$. Now fixing $\tilde{\mu} \geq \mu$ and choosing m large enough that $\rho m > \tilde{\mu} + 2(n-1)$ yields

$$\begin{aligned} K_m(\zeta, a) \;&=:\; \lim_{k\to\infty} K_m(\zeta, a_k) \;= \\ &= \lim_{k\to\infty} -\frac{1}{(2\pi)^n} \int_{T_xX} \int_{T_x^*X} \left(\left(L^\dagger\right)^m a_k\right)(\xi)\,(1-\psi(v))\, e^{-i<\xi-\zeta,v>}\, d\xi\, dv \\ &= -\frac{1}{(2\pi)^n} \int_{T_xX} \int_{T_x^*X} \left(\left(L^\dagger\right)^m a\right)(\xi)\,(1-\psi(v))\, e^{i<\zeta,v>}\, e^{-i<\xi,v>}\, d\xi\, dv \end{aligned} \tag{3.46}$$

in the sense of LEBESGUE integrals. As the phase function $\xi \mapsto -<\xi, v>$ is noncritical for every nonzero v and $1-\psi(v)$ vanishes in an open neighborhood of the zero section of TU, a standard consideration already carried out in preceeding proofs entails that $K_m(\cdot, a)$ is a smooothing symbol. Henceforth $\sigma_{A,\tilde{\psi}}$ and a differ by the smoothing symbol $K_m(\cdot, a)$. This gives the second part of the theorem. □

Note 3.3.5 Because of our particular choice of a phase function we are able to explicitely construct an inverse to the symbol map by the integral representation in Theorem 3.2.3. This is an advantage over the phase function chosen by WIDOM [63, 64], where an explicit inverse to σ cannot be given.

Example 3.3.6 Consider the EUCLIDEAN space $X = \mathbb{R}^n$ and the symbols $l : T^X \to \mathbb{C}$, $(x, \xi) \mapsto |\xi|^2$ and $a : T^X \to \mathbb{C}$, $(x,\xi) \mapsto \frac{1}{1+|\xi|^2}$ of Example 3.1.8 (*ii*). Then the pseudo-differential operator corresponding to l is minus the LAPLACIAN: $\Psi(l) = -\Delta$. As one can now calculate directly, $\Psi(a)$ is an inverse in $\Psi^\infty_{1,0}(X)$ of the differential operator $1-\Delta$ having symbol $T^*X \ni (x,\xi) \mapsto 1 + |\xi|^2 \in \mathbb{C}$.

So with the help of a symbol calculus one can construct inverses to certain differential operators in the algebra of pseudo-differential operators. This is the main reason why a symbol calculus is so important for the theory of partial differential equations.

3.4 Product expansions

Most considerations of elliptic partial differential operators as well as our quantization program for symbol spaces introduced in the next chapter require to know how to express the symbol of the product of two pseudo-differential operators A, B in terms of the symbols of its components. In the flat case on $\mathbb{R}^n$ it is a well known fact (see for example SHUBIN [47] or GRIGIS, SJÖSTRAND [24]) that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} \sigma_a(x,\xi) \, \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \sigma_B(x,\xi). \tag{3.47}$$

Note that precisly speaking, the product AB of pseudo-differential operators is only well-defined, if at least one of them is properly supported. But this is only a minor set-back, as any pseudo-differential operator is properly supported modulo a smoothing operator and the above asymptotic expansion also describes a symbol only up to smoothing symbols.

In the following we want to derive a similar but more complicated formula for the case of pseudo-differential operators on manifolds. To achieve this let us first state a proposition.

Proposition 3.4.1 *Let $A \in \Psi^{\mu}_{\rho,\delta}(U,X)$ with $\rho+\delta \geq 1$, $\rho > \delta$ be a pseudo-differential operator on the* RIEMANNIAN *manifold X and $a = \sigma_A$ its symbol. Then for any $f \in \mathcal{C}^{\infty}(U)$ the function*

$$\sigma_{A,f} : T^*U \to \mathbb{C}, \quad \zeta \mapsto \left[A \left(\psi_{\pi(\zeta)} f \, e^{i\varphi(\cdot,\zeta)} \right) \right] (\pi(\zeta)) \tag{3.48}$$

is a symbol of order μ and type (ρ,δ) and has an asymptotic expansion of the form

$$\sigma_{A,f}(\zeta) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \left[\left. \frac{\partial^{|\alpha|}}{\partial {\xi_{\pi(\zeta)}}^{\alpha}} \right|_{\zeta} a \right] \left[\left. \frac{\partial^{|\alpha|}}{\partial {z_{\pi(\zeta)}}^{\alpha}} \right|_{\pi(\zeta)} f \right]. \tag{3.49}$$

PROOF: Let $x = \pi(\zeta)$, $N \in \mathbb{N}$ and $\psi : T^*X \to \mathbb{C}$ a cut-off function. Then the following chain of equations holds:

$$\begin{aligned}
\sigma_{A,f}(\zeta) &= \left[A \left(\psi_{\pi(\zeta)} f \, e^{i\varphi(\cdot,\zeta)} \right) \right] (x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T^*_x X} a(\xi) \left[\mathcal{F} \left({}^{\psi} f \, e^{i<\zeta,\cdot>} \right) \right] (\xi) \, d\xi \\
&= \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_x X} a(\xi) \, \psi(v) \, f(\exp v) \, e^{i<\zeta-\xi,v>} \, dv \, d\xi \\
&= \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_x X} a(\zeta+\xi) \, \psi(v) \, f(\exp v) \, e^{-i<\xi,v>} \, dv \, d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^n} \sum_{|\alpha|\le N} \frac{1}{\alpha!} \int_{T_x^*X} \int_{T_xX} \frac{\partial^{|\alpha|}}{\partial {\xi_x}^{\alpha}} a(\zeta)\, \xi^\alpha\, \psi(v)\, f(\exp v)\, e^{-i<\xi,v>}\, dv\, d\xi + r_N(\zeta) \\
&= \sum_{|\alpha|\le N} \frac{1}{i^{|\alpha|}\,\alpha!} \left[\frac{\partial^{|\alpha|}}{\partial {\xi_x}^{\alpha}}\bigg|_\zeta a \right] \cdot \left[\frac{\partial^{|\alpha|}}{\partial {z_x}^{\alpha}}\bigg|_x f \right] + r_N(\zeta). \qquad (3.50)
\end{aligned}
$$

Note that in this equation all integrals are iterated ones and check that with TAYLOR'S formula r_N is given by

$$
r_N(\zeta) = \frac{1}{(2\pi)^n} \sum_{|\beta|=N+1} \frac{N+1}{\beta!} \int_{T_x^*X} \int_0^1 (1-t)^N \frac{\partial^{|\beta|}}{\partial {\xi_x}^{\beta}} a(\zeta + t\xi)\, dt\, \xi^\beta \left[\mathcal{F}\left({}^{\psi}f\right)\right](\xi)\, d\xi. \quad (3.51)
$$

Choosing N large enough, the SCHWARZ inequality and PLANCHEREL'S formula now entail

$$
\begin{aligned}
&|r_N(\zeta)| \le \\
&\quad \tfrac{1}{(2\pi)^{n/2}} \sum_{|\beta|=N+1} \tfrac{N+1}{\beta!} \left\|\int_0^1 (1-t)^N \tfrac{\partial^{|\beta|}}{\partial {\xi_x}^{\beta}} a(\zeta + t(\cdot))dt\right\|_{L^2(T_x^*X)} \left\| \tfrac{\partial^{|\beta|}}{\partial {v_x}^{\beta}}\ {}^{\psi}f\Big|_{T_xX}\right\|_{L^2(T_xX)}. \quad (3.52)
\end{aligned}
$$

But for $K \subset U$ compact and $t \in [0,1]$ there exist boundaries $D_{N,K}, \tilde{D}_{N,K} > 0$ such that

$$
\left| \frac{\partial^{|\beta|}}{\partial {\xi_x}^{\beta}} a(\zeta + t\xi) \right| \le D_{N,K}(1 + |\zeta| + |\xi|)^{\mu - \rho(N+1)} \qquad (3.53)
$$

and

$$
\begin{aligned}
\left\| \int_0^1 (1-t)^k \frac{\partial^{|\beta|}}{\partial {\xi_x}^{\beta}} a(\zeta + t(\cdot))dt \right\|_{L^2(T_x^*X)} &\le D_{N,K} \left\|(1 + |\zeta| + |\cdot|)^{\mu-\rho(N+1)}\right\|_{L^2(T_x^*X)} \\
&= \tilde{D}_{N,K}\, (1 + |\zeta|)^{n+\mu-\rho(N+1)} \qquad (3.54)
\end{aligned}
$$

for all $x \in K$, $\zeta, \xi \in T_x^*X$ and all indices $\beta \in \mathbb{N}^n$ with $|\beta| = N+1$. Inserting this in (3.52) we can find a $C_{N,K} > 0$ such that for all $x \in K$ and $\zeta \in T_x^*X$

$$
|r_N(\zeta)| \le C_{N,K} \Big(1 + |\zeta|\Big)^{n+\mu-\rho(N+1)} \sum_{|\gamma|\le N+1} \sup \left\{ \left| \frac{\partial^{|\gamma|}}{\partial {z_x}^{\gamma}} f(y) \right| \,:\, y \in \operatorname{supp}\Big(\psi_{\pi(\zeta)}\Big) \right\}. \quad (3.55)
$$

Because of $\lim_{N\to\infty} n + \mu + \rho(N+1) = -\infty$ and a similar consideration for the derivatives of $\sigma_{A,f}$ the claim now follows. □

Next let us consider two pseudo-differential operators $A, B \in \Psi^{\infty}_{\rho,\delta}(U,X)$. Modulo smoothing operators we can assume one of them to be properly supported, so that

$AB \in \Psi^{\infty}_{\rho,\delta}(U,X)$ exists. Defining the extended symbol $b^{ext} = \sigma^{ext}_{B,\psi} : U \times T^*U \to \mathbb{C}$ of B by

$$b^{ext}(x,\zeta) \;=\; \psi_{\pi(\zeta)}(x)\, e^{-i\varphi(x,\zeta)} \left[B\left(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\right)\right](x) \tag{3.56}$$

the symbol σ_{AB} of the composition AB is now given modulo $\mathrm{S}^{-\infty}(U)$ by

$$\sigma_{AB}(\zeta) \;=\; \left[AB\left(\psi_{\pi(\zeta)} e^{i\varphi(\cdot,\zeta)}\right)\right](\pi(\zeta)) \;=\; \left[A\left(\psi_{\pi(\zeta)}\, e^{i\varphi(\cdot,\zeta)}\, \sigma^{ext}_{B,\psi}(\cdot,\zeta)\right)\right](\pi(\zeta)). \tag{3.57}$$

By Proposition 3.4.1 this entails that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(\zeta) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{i^{|\alpha|}\alpha!} \left[\left.\frac{\partial^{|\alpha|}}{\partial {\xi_{\pi(\zeta)}}^{\alpha}}\right|_{\zeta} \sigma_A\right] \left[\left.\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\zeta)}}^{\alpha}}\right|_{\pi(\zeta)} \sigma^{ext}_B(\cdot,\zeta)\right]. \tag{3.58}$$

As we want to find an even more detailed expansion for σ_{AB}, let us calculate the derivatives $\left.\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\zeta)}}^{\alpha}}\right|_{\pi(\zeta)} \sigma^{ext}_B(\cdot,\zeta)$ in the following proposition.

Proposition 3.4.2 *Let $A \in \Psi^{\mu}_{\rho,\delta}(U,X)$ with $\rho+\delta \geq 1$, $\rho > \delta$ be a pseudo-differential operator on the* Riemannian *manifold X and $a = \sigma_A$ its symbol. Then for any $f \in \mathcal{C}^{\infty}(U)$ the smooth function*

$$\sigma^{ext}_{A,f} : U \times T^*U \to \mathbb{C}, \quad (x,\zeta) \mapsto \psi_{\pi(\zeta)}(x)\, e^{-i\varphi(x,\zeta)} \left[A\left(\psi_{\pi(\zeta)} f\, e^{i\varphi(\cdot,\zeta)}\right)\right](x) \tag{3.59}$$

is a symbol of order μ and type (ρ,δ) on $U \times T^*U$ and has an asymptotic expansion of the form

$$\begin{aligned}\sigma^{ext}_{A,f}(x,\zeta) \;\sim\; &\sum_{k\in\mathbb{N}} \sum_{\substack{\beta,\beta_0,\dots,\beta_k\in\mathbb{N}^n\\ \beta_0+\dots+\beta_k=\beta\\ |\beta_1|,\dots,|\beta_k|\geq 2}} \frac{i^{k-|\beta|}}{k!\beta_0!\cdot\ldots\cdot\beta_k!}\, \psi_{\pi(\zeta)}(x) \left[\left.\frac{\partial^{|\beta|}}{\partial {\xi_x}^{\beta}}\right|_{d_x\varphi(\ ,\zeta)} a\right] \\ &\left[\left.\frac{\partial^{|\beta_0|}}{\partial {z_x}^{\beta_0}}\right|_{x} \psi_{\pi(\zeta)} f\right] \left[\left.\frac{\partial^{|\beta_1|}}{\partial {z_x}^{\beta_1}}\right|_{x} \varphi(\cdot,\zeta)\right]\cdot\ldots\cdot\left[\left.\frac{\partial^{|\beta_k|}}{\partial {z_x}^{\beta_k}}\right|_{x} \varphi(\cdot,\zeta)\right].\end{aligned} \tag{3.60}$$

Furthermore the derivative $T^*U \ni \zeta \mapsto \left[\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\zeta)}}^{\alpha}} \sigma^{ext}_{A,f}(\cdot,\zeta)\right](\pi(\zeta)) \in \mathbb{C}$ with $\alpha \in \mathbb{N}^n$ is a symbol of order $\mu+\delta|\alpha|$ and type (ρ,δ) on T^*U and has an asymptotic expansion of the form

$$\left[\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\zeta)}}^{\alpha}} \sigma^{ext}_{A,f}(\cdot,\zeta)\right](\pi(\zeta)) \sim \sum_{k\in\mathbb{N}} \sum_{\substack{\tilde\alpha,\alpha_0,\dots,\alpha_k\in\mathbb{N}^n\\ \tilde\alpha+\alpha_0+\dots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_0,\dots,\beta_k\in\mathbb{N}^n\\ \beta_0+\dots+\beta_k=\beta\\ |\beta_1|,\dots,|\beta_k|\geq 2}} \frac{i^{k-|\beta|}\alpha!}{k!\tilde\alpha!\alpha_0!\cdot\ldots\cdot\alpha_k!\beta_0!\cdot\ldots\cdot\beta_k!}$$

$$\left\{ \left.\frac{\partial^{|\tilde{\alpha}|}}{\partial z_{\pi(\zeta)}{}^{\tilde{\alpha}}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_{(-)}{}^{\beta}}\right|_{d_{(-)}\varphi(\ ,\zeta)} a \right] \right\} \left\{ \left.\frac{\partial^{|\alpha_0|}}{\partial z_{\pi(\zeta)}{}^{\alpha_0}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_0|}}{\partial z_{(-)}{}^{\beta_0}}\right|_{(-)} f \right] \right\} \cdot$$
$$\left\{ \left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\zeta)}{}^{\alpha_1}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\} \cdot \ldots \cdot \left\{ \left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\zeta)}{}^{\alpha_k}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\}. \tag{3.61}$$

Note 3.4.3 The differential operators of the form $\frac{\partial^{|\alpha|}}{\partial z_{\pi(\zeta)}{}^{\alpha}}$ act on the variables denoted by $(-)$, the differential operators of the form $\frac{\partial^{|\beta|}}{\partial z_{(-)}{}^{\beta}}$ on the variables $(\cdot)$.

Proof: Let $\eta \in \mathcal{C}^\infty(U \times U \times T^*U)$ be a smooth function such that

$$\eta(x,y,\zeta) \;=\; \varphi(y,\zeta) - \varphi(x,\zeta) - < d_x\varphi(\ ,\zeta), z_x(y) > \tag{3.62}$$

for $x, y \in \operatorname{supp}\big(\psi_{\pi(\zeta)}\big)$ and such that $T^*U \ni \zeta \mapsto \eta(x,y,\zeta) \in \mathbb{C}$ is linear for all $x, y \in U$. Furthermore denote by $F \in \mathcal{C}^\infty(U \times U \times T^*U)$ the function $(x,y,\zeta) \mapsto \psi_{\pi(\zeta)}(y) f(y) e^{i\eta(x,y,\zeta)}$. By the Leibniz rule and the definition of η we then have

$$\left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_y F(x,\cdot,\zeta) \right] \;=\; \sum_{0 \le k \le \frac{|\beta|}{2}} \sum_{\substack{\beta_0,\ldots,\beta_k \in \mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k| \ge 2}} \frac{i^k}{k!\beta_0! \cdot \ldots \cdot \beta_k!} \left[\left.\frac{\partial^{|\beta_0|}}{\partial z_x{}^{\beta_0}}\right|_y \psi_{\pi(\zeta)} f \right] \cdot$$
$$\left[\left.\frac{\partial^{|\beta_1|}}{\partial z_x{}^{\beta_1}}\right|_y \eta(x,\cdot,\zeta) \right] \cdot \ldots \cdot \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_x{}^{\beta_k}}\right|_y \eta(x,\cdot,\zeta) \right] e^{i\eta(x,y,\zeta)}. \tag{3.63}$$

As $(x,y,\zeta) \mapsto \left[\left.\frac{\partial^{|\beta_j|}}{\partial z_x{}^{\beta_j}}\right|_y \eta(x,\cdot,\zeta) \right]$, $0 \le j \le k$ is smooth and linear with respect to ζ, the preceeding equation entails that $(x,y,\zeta) \mapsto \left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_y F(x,\cdot,\zeta) \right]$ is absolutly bounded by a function $(x,y,\zeta) \mapsto C_\beta(x,y)\,|\zeta|^{\frac{|\beta|}{2}}$, where $C_\beta \in \mathcal{C}^\infty(U \times U)$ and $C_\beta \ge 0$. On the other hand we have

$$\sigma^{ext}_{A,f}(x,\zeta) \;=\; \psi_{\pi(\zeta)}(x) \left[A \left(\psi_{\pi(\zeta)}(\cdot) f(\cdot) e^{i\eta(\cdot,x,\zeta)}\, e^{i< d_x\varphi(\ ,\zeta), z_x(\cdot)>} \right) \right](x) \tag{3.64}$$

and by the proof of Proposition 3.4.1

$$\sigma^{ext}_{A,f}(x,\zeta) \;=\; \psi_{\pi(\zeta)}(x) \sum_{|\beta| \le N} \frac{i^{|\beta|}}{\beta!} \left[\left.\frac{\partial^{|\beta|}}{\partial \zeta_x{}^{\beta}}\right|_{d_x\varphi(\ ,\xi)} a \right] \left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_x F(x,\cdot,\zeta) \right] + r_N(x,\zeta)$$
$$=\; \psi_{\pi(\zeta)}(x) \sum_{|\beta| \le N} \sum_{0 \le k \le \frac{|\beta|}{2}} \sum_{\substack{\beta_0,\ldots,\beta_k \in \mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k| \ge 2}} \frac{i^{k-|\beta|}}{k!\beta_0! \cdot \ldots \cdot \beta_k!} \left[\left.\frac{\partial^{|\beta|}}{\partial \zeta_x{}^{\beta}}\right|_{d_x\varphi(\ ,\xi)} a \right] \cdot$$

$$\left[\frac{\partial^{|\beta_0|}}{\partial {z_x}^{\beta_0}}\bigg|_x \psi_{\pi(\zeta)} f\right] \cdot \left[\frac{\partial^{|\beta_1|}}{\partial {z_x}^{\beta_1}}\bigg|_x \varphi(\cdot,\zeta)\right] \cdot \ldots \cdot \left[\frac{\partial^{|\beta_k|}}{\partial {z_x}^{\beta_k}}\bigg|_x \varphi(\cdot,\zeta)\right] + r_N(x,\zeta), \quad (3.65)$$

where

$$\begin{aligned} &|r_n(x,\zeta)| \\ \leq \;& C_{N,K}(1+|d_x\varphi(\;\,,\zeta)|)^{n+\mu-\rho(N+1)} \sum_{|\gamma|\leq N+1} \sup\left\{\left|\left[\frac{\partial^{|\gamma|}}{\partial {z_x}^{\gamma}}\bigg|_y F(x,\cdot,\zeta)\right]\right| \;:\; y\in \operatorname{supp}(\psi_x)\right\} \\ \leq \;& D_{N,K}(1+|\zeta|)^{n+\mu-(\rho-1/2)(N+1)} \end{aligned} \quad (3.66)$$

holds for compact $K \subset U$, $x \in K$, $\zeta \in T^*X|_K$ and constants $C_{N,K}, D_{N,K} > 0$. Because $\rho > \frac{1}{2}$, we have $\lim_{N\to\infty} n+\mu-(\rho-\frac{1}{2})(N+1) = -\infty$. By Eqs. (3.65) and (3.66) and similar relations for the derivatives of $\sigma^{ext}_{A,f}$ we therefore conclude that Eq. (3.60) is true. The rest of the claim easily follows from LEIBNIZ rule. □

The last proposition and Eq. (3.58) entail the next theorem.

Theorem 3.4.4 *Let $A, B \in \Psi^\infty_{\rho,\delta}(U,X)$ with $\rho+\delta \geq 1$ and $\rho > \delta$ be two pseudo-differential operators on the* RIEMANNIAN *manifold X, one of them properly supported. Then the symbol σ_{AB} of the product AB has the following asymptotic expansion.*

$$\begin{aligned} \sigma_{AB}(\zeta) \sim \sum_{k\in\mathbb{N}} \sum_{\substack{\alpha,\tilde\alpha,\alpha_1,\ldots,\alpha_k\in\mathbb{N}^n \\ \tilde\alpha+\alpha_1+\ldots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_1,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_1+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\geq 2}} & \frac{i^{k-|\alpha|-|\beta|}}{k!\cdot\tilde\alpha!\cdot\alpha_1!\cdot\ldots\cdot\alpha_k!\beta_1!\cdot\ldots\cdot\beta_k!} \\ \left[\frac{\partial^{|\alpha|}}{\partial {\xi_{\pi(\zeta)}}^{\alpha}}\bigg|_\zeta \sigma_A\right] \left\{\frac{\partial^{|\tilde\alpha|}}{\partial {z_{\pi(\zeta)}}^{\tilde\alpha}}\bigg|_{\pi(\zeta)} \left[\frac{\partial^{|\beta|}}{\partial {\xi_{(-)}}^{\beta}}\bigg|_{d_{(-)}\varphi(\;\,,\zeta)} \sigma_B\right]\right\} & \cdot \\ \left\{\frac{\partial^{|\alpha_1|}}{\partial {z_{\pi(\zeta)}}^{\alpha_1}}\bigg|_{\pi(\zeta)} \left[\frac{\partial^{|\beta_1|}}{\partial {z_{(-)}}^{\beta_1}}\bigg|_{(-)} \varphi(\cdot,\zeta)\right]\right\} \cdot \ldots \cdot \left\{\frac{\partial^{|\alpha_k|}}{\partial {z_{\pi(\zeta)}}^{\alpha_k}}\bigg|_{\pi(\zeta)} \left[\frac{\partial^{|\beta_k|}}{\partial {z_{(-)}}^{\beta_k}}\bigg|_{(-)} \varphi(\cdot,\zeta)\right]\right\} & . \end{aligned} \quad (3.67)$$

Chapter 4

Quantization of Spaces of Symbols

In section 2.5 we found a deformation of the sheaf $\mathcal{D}_0^\infty$ of observables with infinite order in momentum on a complex cotangent bundle. Unfortunately it is not possible to directly carry over the methods used for the construction of $\mathcal{D}_0^\infty$ and its deformation to the real case. The reason is that we used complex techniques like the CAUCHY integral which are not applicable in the smooth and real arena. But, as ALAN WEINSTEIN [59] pointed out to me, the complete symbol calculus of chapter 3 should be the tool to find a quantization of a rather general class of observables on cotangent bundles. Roughly speaking this observable class will be a space of symbols on a smooth cotangent bundle. In the following those ideas will be worked out in detail.

4.1 Noncommutative algebra structures on symbol spaces

First let us define the POISSON space to be quantized. The standard POISSON brackets $\{\ ,\ \}$ over smooth function algebras on a cotangent bundle obviously give $\left(X, \mathrm{S}^\infty_{\rho,\delta}\right)$ the structure of a POISSON space. Next let a be an element of $\mathrm{S}^{-\infty}(U)$ and U a coordinate patch of X. Then take the derivatives $\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial \xi^\beta} a$ with respect to local coordinates over T^*U and realize that $\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial \xi^\beta} a$ again lies in $\mathrm{S}^{-\infty}(U)$. Hence $\mathrm{S}^{-\infty}$ is an ideal sheaf of $\mathrm{S}^\infty_{\rho,\delta}$ with respect to the POISSON brackets. This gives rise to the POISSON space $\left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$. By Proposition 3.1.10 we also have a natural morphism $\left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right) \to (X, \mathcal{D}_0)$ between POISSON spaces. Note that the arrow between the two POISSON spaces is inverse to the arrow between the corresponding sheaves of algebras.

Next let us introduce some linear mappings between symbol spaces. These mappings will be needed in the sequel to define the deformed products on $\left(X, \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$. Denote by $\iota_\hbar : \mathrm{S}^{\infty}_{\rho,\delta}(U,X) \to \mathrm{S}^{\infty}_{\rho,\delta}(U,X)$ for $\hbar \in \mathbb{R}$ the function

$$a \;\mapsto\; \left(T^*U \ni \xi \mapsto a(\hbar\xi) \in \mathbb{C}\right). \tag{4.1}$$

In case $\hbar \neq 0$ $\iota_\hbar$ is a linear isomorphism with inverse $\iota_{\hbar^{-1}}$. By a slight abuse of language we sometimes also write $\iota_\hbar$ for the mapping

$$\begin{array}{ccc} \mathrm{S}^{\infty}_{\rho,\delta}(U\times\mathbb{R}, pr_1^*(T^*X)) & \to & \mathrm{S}^{\infty}_{\rho,\delta}(U,X), \\ b & \mapsto & \left(T^*U \ni \zeta \mapsto b(\hbar\zeta,\hbar) \in \mathbb{C}\right). \end{array} \tag{4.2}$$

Here $pr_1 : X\times\mathbb{R} \to X$ is the projection on the first coordinate and $pr_1^*(T^*X)$ the pull-back bundle of T^*X via pr_1.

Now let $\hbar \in \mathbb{R}\setminus\{0\}$, $a,b \in \mathrm{S}^{\infty}_{\rho,\delta}(U,X)$ be symbols and $A_\hbar = \Psi_\psi(\iota_\hbar(a))$ (resp. $B_\hbar = \Psi_\psi(\iota_\hbar(b))$) the so-called $\hbar$-**deformed** pseudo-differential operator corresponding to a (resp. b). Then by Eq. (3.67) in Theorem 3.4.4 the symbol $c_\hbar = \iota_{\hbar^{-1}}\left(\sigma_{A_\hbar\, B_\hbar}\right)$ has asymptotic expansion

$$\begin{aligned} c_\hbar(\zeta) \;\sim\; & \sum_{k\in\mathbb{N}} \sum_{\substack{\alpha,\tilde\alpha,\alpha_1,\ldots,\alpha_k\in\mathbb{N}^n \\ \tilde\alpha+\alpha_1+\ldots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_1,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_1+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\geq 2}} \left(\frac{\hbar}{i}\right)^{|\alpha|+|\beta|-k} \frac{1}{k!\cdot\tilde\alpha!\cdot\alpha_1!\cdot\ldots\cdot\alpha_k!\beta_1!\cdot\ldots\cdot\beta_k!} \\ & \left[\left.\frac{\partial^{|\alpha|}}{\partial\xi_{\pi(\zeta)}{}^{\alpha}}\right|_{\zeta} a\right] \left\{\left.\frac{\partial^{|\tilde\alpha|}}{\partial z_{\pi(\zeta)}{}^{\tilde\alpha}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta|}}{\partial\xi_{(-)}{}^{\beta}}\right|_{d_{(-)}\varphi(\;,\zeta)} b\right]\right\}\cdot \\ & \left\{\left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\zeta)}{}^{\alpha_1}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\zeta)\right]\right\}\cdot\ldots\cdot \left\{\left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\zeta)}{}^{\alpha_k}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\zeta)\right]\right\}. \end{aligned} \tag{4.3}$$

In particular we have

$$\begin{aligned} c_\hbar(\zeta) \;=\; & \\ & a(\zeta)\cdot b(\zeta) + \left(\frac{\hbar}{i}\right)\sum_{l=1}^{n}\left[\frac{\partial a}{\partial\xi_{\pi(\zeta),l}}(\zeta)\right]\left\{\frac{\partial\left[b(d_{(-)}\varphi(\;,\zeta))\right]}{\partial z_{\pi(\zeta),l}}(\pi(\zeta))\right\} + \hbar^2 r(\zeta,\hbar), \end{aligned} \tag{4.4}$$

where $r \in \mathrm{S}^{\mu+\tilde\mu-\rho+\delta}_{\rho,\delta}(U\times\mathbb{R}, pr_1^*(T^*X))$, and μ (resp. $\tilde\mu$) is the order of the symbol a (resp. b). Note that the bilinear mapping $\mathrm{S}^{\infty}_{\rho,\delta}(U,X)\times\mathrm{S}^{\infty}_{\rho,\delta}(U,X) \ni (a,b) \to a\cdot_\hbar b = c_\hbar \in \mathrm{S}^{\infty}_{\rho,\delta}(U,X)$ is only associative modulo smoothing symbols. This follows easily from the fact that the symbol map σ and the map Ψ_ψ are inverse to each other up to smoothing symbols resp. smoothing pseudo-differential operators. Altogether we therefore receive a scale $\left(\mathrm{S}^\infty_{\rho,\delta;\hbar}(\cdot,X)\right)_{\hbar\in\mathbb{R}}$ of sheaves of (noncommutative) algebras, such that the underlying linear structure of $\mathrm{S}^\infty_{\rho,\delta;\hbar}(U,X)$ is that of $\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(U,X)$. The product of two equivalence classes $a+\mathrm{S}^{-\infty}, b+\mathrm{S}^{-\infty} \in \mathrm{S}^\infty_{\rho,\delta;\hbar}(U,X)$ is given by the equivalence class of $a\cdot_\hbar b = c_\hbar = \iota_{\hbar^{-1}}\left(\sigma_{A_\hbar B_\hbar}\right)$ in the case $\hbar \neq 0$ and by the equivalence class of the pointwise product $a\cdot b$ in the case $\hbar = 0$. The sheaf $\mathrm{S}^\infty_{\rho,\delta;\hbar}(\cdot,X)$ is called the sheaf of **$\hbar$-deformed symbols of type (ρ,δ)** on a smooth Riemannian manifold X.

4.1.1 Let us now for $\hbar \neq 0$ embed $\mathcal{D}_\hbar$ into $\mathrm{S}^\infty_{\rho,\delta;\hbar}$. Every $A \in \mathcal{D}_\hbar(U)$, $U \subset X$ open acts as a differential operator on $\mathcal{C}^\infty(U)$. Therefore the symbol $a = \iota_{\hbar^{-1}}\sigma_A$ defines a unique element $\sigma_{\hbar,A} = a + \mathrm{S}^{-\infty} \in \mathrm{S}^\infty_{\rho,\delta;\hbar}(U)$. If $B \in \mathcal{D}_\hbar$ is another $\hbar$-deformed polynomial symbol, we have for $b = \iota_{\hbar^{-1}}\sigma_B$

$$a \cdot_\hbar b \;=\; \iota_{\hbar^{-1}}\left(\sigma_{A_\hbar B_\hbar}\right) \;=\; \iota_{\hbar^{-1}}\left(\sigma_{AB}\right), \tag{4.5}$$

that means the mapping

$$\sigma_\hbar : \mathcal{D}_\hbar(U) \to \mathrm{S}^\infty_{\rho,\delta;\hbar}(U),\; A \mapsto \sigma_{\hbar,A} = \iota_{\hbar^{-1}}\sigma_A + \mathrm{S}^{-\infty} \tag{4.6}$$

is linear and multiplicative. It is also injective, as the symbol of a differential operator is smoothing if and only if the differential operator is zero. Let us also give a second definition for a. Assuming U to be sufficiently small choose an orthonormal frame $e_1, ..., e_n$ of TX over U and for every $x \in U$ normal coordinates $z_x : U \to \mathbb{R}^n$ such that $\left.\frac{\partial}{\partial z_{x,k}}\right|_x = e_k$ for $k = 1, ..., n$. Then express $A \in \mathcal{D}(U)$ for every $x \in U$ in the form $A = a_x\left(-i\frac{\partial^\hbar}{\partial z_x}\right)$, where a_x is a polynomial symbol over U, and check that

$$a(\xi) \;=\; \left[A\left(\psi_{\pi(\xi)}e^{i\varphi(\cdot,\frac{\xi}{\hbar})}\right)\right](\pi(\xi)) \;=\; a_x(\xi). \tag{4.7}$$

In the same spirit like above we want to define a noncommutative product on the symbol sheaf $\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, pr_1^*(T^*X))$. So let $a, b \in \mathrm{S}^\infty_{\rho,\delta}(U \times O, pr_1^*(T^*X))$ be symbols, where $U \subset X$ and $O \subset \mathbb{R}$ are open with O bounded, and denote for $\hbar \neq 0$ by $A_\hbar$ (resp. $B_\hbar$) the pseudo-differential operator $A_\hbar = \Psi_\psi(\iota_\hbar(a))$ (resp. $B_\hbar = \Psi_\psi(\iota_\hbar(b))$). Then define $c = a \cdot_{d} b$ by

$$c(\zeta,\hbar) \;=\; \begin{cases} a(\zeta,\hbar)\cdot b(\zeta,\hbar) & \text{for } \hbar = 0 \\ \sigma_{A_\hbar B_\hbar}\left(\frac{\zeta}{\hbar}\right) & \text{for } \hbar \neq 0 \end{cases} \tag{4.8}$$

We now want to show that c is a symbol on $pr_1^*(T^*X)$ of order $\mu+\tilde{\mu}$, where μ (resp. $\tilde{\mu}$) is the order of a (resp. b). Obviously c is smooth on $U \times (O \setminus \{0\})$. According to the asymptotic expansion in Theorem 3.4.4 we can find for every $N > 0$ a finite subset $I_N \subset J$ of the index set

$$\begin{aligned} J \;=\; & \Big\{(k,\alpha,\tilde{\alpha},\alpha_1,...,\alpha_k,\beta,\beta_1,...,\beta_k) \in \mathbb{N} \times (\mathbb{N}^n)^{2k+3} : \\ & \tilde{\alpha}+\alpha_1+...+\alpha_k = \alpha,\ \beta_1,...,\beta_k = \beta,\ |\beta_1|,...,|\beta_k| \geq 2\Big\} \end{aligned}$$

such that

$$\begin{aligned} & \sigma_{A_\hbar B_\hbar}(\zeta) - \sum_{k,\alpha,\tilde{\alpha},\alpha_1,...,\alpha_k,\beta,\beta_1,...,\beta_k \in I} \hbar^{|\alpha|} \frac{i^{k-|\alpha|-|\beta|}}{k! \cdot \tilde{\alpha}! \cdot \alpha_1! \cdot ... \cdot \alpha_k! \beta_1! \cdot ... \cdot \beta_k!} \\ & \left[\left.\frac{\partial^{|\alpha|}}{\partial \xi_{\pi(\zeta)}{}^{\alpha}}\right|_{(\zeta,\hbar)} a \right] \left\{ \left.\frac{\partial^{|\tilde{\alpha}|}}{\partial z_{\pi(\zeta)}{}^{\tilde{\alpha}}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_{(-)}{}^{\beta}}\right|_{(d_{(-)}\varphi(\ ,\zeta),\hbar)} b \right] \right\} \cdot \\ & \left\{ \left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\zeta)}{}^{\alpha_1}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\} \cdot ... \cdot \left\{ \left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\zeta)}{}^{\alpha_k}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\} \end{aligned}$$

lies in $\mathrm{S}^{-N}_{\rho,\delta}(U \times O, pr_1^*(T^*X))$. Hence we can find for $K \subset U$ compact a boundary $C_K > 0$ such that

$$\begin{aligned} & \Bigg| \sigma_{A_\hbar B_\hbar}\left(\frac{\zeta}{\hbar}\right) - \sum_{k,\alpha,\tilde{\alpha},\alpha_1,...,\alpha_k,\beta,\beta_1,...,\beta_k \in I} \left(\frac{\hbar}{i}\right)^{|\alpha|+|\beta|-k} \frac{1}{k! \cdot \tilde{\alpha}! \cdot \alpha_1! \cdot ... \cdot \alpha_k! \beta_1! \cdot ... \cdot \beta_k!} \\ & \left[\left.\frac{\partial^{|\alpha|}}{\partial \xi_{\pi(\zeta)}{}^{\alpha}}\right|_{(\zeta,\hbar)} a \right] \left\{ \left.\frac{\partial^{|\tilde{\alpha}|}}{\partial z_{\pi(\zeta)}{}^{\tilde{\alpha}}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_{(-)}{}^{\beta}}\right|_{(d_{(-)}\varphi(\ ,\zeta),\hbar)} b \right] \right\} \cdot \\ & \left\{ \left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\zeta)}{}^{\alpha_1}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\} \cdot ... \cdot \left\{ \left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\zeta)}{}^{\alpha_k}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\zeta) \right] \right\} \Bigg| \\ \leq \;& C_K \frac{|\hbar|^N}{(|\hbar|+|\zeta|)^N} \end{aligned} \tag{4.9}$$

for all $(\pi(\zeta),\hbar) \in K \times O$. As the sum $\sum ...$ in (4.9) is a continuous function of $\hbar \in O$, (4.9) tells us that $c(\zeta,\hbar) = \sigma_{A_\hbar B_\hbar}\left(\frac{\zeta}{\hbar}\right)$ can be continuously expanded to $\hbar = 0$ such that the required growth condition is fulfilled. Similar considerations for all partial derivatives of c now give the desired result, i. e. they show that c has the asymptotic expansion

$$c(\zeta,\hbar) \sim \sum_{k,\alpha,\tilde{\alpha},\alpha_1,...,\alpha_k,\beta,\beta_1,...,\beta_k \in I} \left(\frac{\hbar}{i}\right)^{|\alpha|+|\beta|-k} \frac{1}{k! \cdot \tilde{\alpha}! \cdot \alpha_1! \cdot ... \cdot \alpha_k! \beta_1! \cdot ... \cdot \beta_k!}$$

$$\left[\left.\frac{\partial^{|\alpha|}}{\partial \xi_{\pi(\zeta)}{}^{\alpha}}\right|_{(\zeta,\hbar)} a\right] \left\{\left.\frac{\partial^{|\tilde{\alpha}|}}{\partial z_{\pi(\zeta)}{}^{\tilde{\alpha}}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_{(-)}{}^{\beta}}\right|_{(d_{(-)}\varphi(\ ,\zeta),\hbar)} b\right]\right\} \cdot$$
$$\left\{\left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\zeta)}{}^{\alpha_1}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\zeta)\right]\right\} \cdot \ldots \cdot \left\{\left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\zeta)}{}^{\alpha_k}}\right|_{\pi(\zeta)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\zeta)\right]\right\} \tag{4.10}$$

and is an element of $\mathrm{S}^{\mu+\tilde{\mu}}_{\rho,\delta}(U \times O, pr_1^*(T^*X))$. For later purposes let us write down the first order approximation of $c = a \cdot_{\hbar\!\!\!/} b$ in powers of $\hbar$:

$$\begin{aligned} c(\zeta,\hbar) \;&=\; a(\zeta,\hbar)\cdot b(\zeta,\hbar) \\ &+ \left(\frac{\hbar}{i}\right) \sum_{l=1}^{n} \left[\frac{\partial a}{\partial \xi_{\pi(\zeta),l}}(\zeta,\hbar)\right] \left\{\frac{\partial\left[b(d_{(-)}\varphi(\ ,\zeta),\hbar)\right]}{\partial z_{\pi(\zeta),l}}(\pi(\zeta))\right\} + \hbar^2 r(\zeta,\hbar), \end{aligned} \tag{4.11}$$

where $r \in \mathrm{S}^{\mu+\tilde{\mu}-\rho+\delta}_{\rho,\delta}(U \times \mathbb{R}, pr_1^*(T^*X))$.

The symbol space $\mathrm{S}^{-\infty}(U \times O, pr_1^*(T^*X))$ forms an ideal with respect to the bilinear mapping $(a,b) \mapsto c = a \cdot_{\hbar\!\!\!/} b$. Furthermore the product $\cdot_{\hbar\!\!\!/}$ is associative up to smoothing symbols. This follows easily from the fact, that the maps Ψ_ψ and σ are inverse to each other modulo smoothing symbols resp. smoothing pseudo-differential operators. The quotient space

$$\mathrm{S}^{\infty}_{\rho,\delta;\hbar\!\!\!/}(U \times O, pr_1^*(T^*X)) \;=\; \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(U \times O, pr_1^*(T^*X))$$

now becomes a noncommutative algebra and $\mathrm{S}^{\infty}_{\rho,\delta;\hbar\!\!\!/}(\cdot, pr_1^*(T^*X))$ a sheaf of noncommutative algebras on $X \times \mathbb{R}$, the sheaf of **universally deformed symbols of type** (ρ,δ) on a smooth Riemannian manifold X.

4.1.2 The embedding $\sigma_\hbar : \mathcal{D}_\hbar \to \mathrm{S}^{\infty}_{\rho,\delta;\hbar}$ defined in 4.1.1 uniquely extends to an embedding $\sigma_{\hbar\!\!\!/} : \mathcal{D}_{\hbar\!\!\!/} \to \mathrm{S}^{\infty}_{\rho,\delta;\hbar\!\!\!/}$ such that the diagram

$$\begin{tikzcd} \mathcal{D}_{\hbar\!\!\!/} \arrow[r, "\sigma_{\hbar\!\!\!/}"] \arrow[d] & \mathrm{S}^{\infty}_{\rho,\delta;\hbar\!\!\!/} \arrow[d] \\ \mathcal{D}_\hbar \arrow[r, "\sigma_\hbar"'] & \mathrm{S}^{\infty}_{\rho,\delta;\hbar} \end{tikzcd}$$

commutes for every $\hbar \neq 0$. Just define $\sigma_{\hbar\!\!\!/}(P)$ for $P \in \mathcal{D}_{\hbar\!\!\!/}(U \times O)$, $U \subset X$ open and $O \subset \mathbb{R}$ open to be the function $p \in \mathrm{S}^{\infty}_{\rho,\delta}(U \times O, pr_1^*(T^*X))$ such that $p(\xi,\hbar) =$

$\left[P\left(\psi_{\pi(\xi)}(\cdot)e^{i\varphi(\cdot,\frac{\xi}{\hbar})}\right)\right](\pi(\xi))$ for $\hbar \in O \setminus \{0\}$ and $p(\xi,0) = \lim\limits_{\substack{\hbar\to 0\\ \hbar\in O\setminus\{0\}}} p(\xi,\hbar)$. To see that p is well-defined indeed, write for every $x \in U$ on an open neighborhood U_x of x

$$P|_{T^*U_x\times O} = p_x\left(-i\frac{\partial^{đ}}{\partial z_x}\right) \tag{4.12}$$

with a smooth function p_x on $T^*U_x \times O$, polynomial in the fibers. Then check that $p(\xi,\hbar) = p_{\pi(\xi)}(\xi,\hbar)$ for $\xi \in T^*U_x$ and $\hbar \in O$. As p_x is also smooth with respect to x, p is a smooth function on $T^*U \times O$ and polynomial in the fibers. The rest of the claim now follows easily.

4.2 Deformation and quantization

After having defined the sheaf $\mathrm{S}^{\infty}_{\rho,\delta;đ}(\cdot, pr_1^*(T^*X))$, we will now show that it gives rise to a deformation and quantization of the POISSON space $\left(X, \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$.

Denote by $D_{\rho,\delta} = (đ, \Delta_{\rho,\delta}) : (X\times\mathbb{R}, \mathrm{S}^{\infty}_{\rho,\delta;đ}) \to (\mathbb{R}, \mathcal{C}^{\infty})$ the fibered morphism

$$\begin{array}{rrcl} đ = pr_2 : & X\times\mathbb{R} & \to & \mathbb{R}, \\ & (\zeta,\hbar) & \mapsto & \hbar, \end{array} \tag{4.13}$$

$$\begin{array}{rrcll} (\Delta_{\rho,\delta})_O : & \mathcal{C}^{\infty}(O) & \to & \mathrm{S}^{\infty}_{\rho,\delta;đ}(X\times O, pr_1^*(T^*X)), & O\subset\mathbb{R} \text{ open}, \\ & f & \mapsto & f\circ pr_2 + \mathrm{S}^{-\infty}(X\times O, pr_1^*(T^*X)). & \end{array} \tag{4.14}$$

Furthermore let $(x,\hbar) \in (X\times\mathbb{R})$, and $\mathfrak{m}_\hbar \subset \mathcal{C}^{\infty}_\hbar$ be the maximal ideal in $\mathcal{C}^{\infty}_\hbar$. Then the stalk $\left(\mathrm{S}^{\infty}_{\rho,\delta;đ}\right)_{(x,\hbar)}$ is filtered by

$$\left(\mathrm{S}^{\infty}_{\rho,\delta;đ}\right)_{(x,\hbar)} = S^0_{(x,\hbar)} \supset S^1_{(x,\hbar)} \supset \ldots \supset S^m_{(x,\hbar)} \supset \ldots, \tag{4.15}$$

where $S^m_{(x,\hbar)}$ with $m\in\mathbb{N}$ consists of all (germs of) symbols $a \in \left(\mathrm{S}^{\infty}_{\rho,\delta;đ}\right)_{(x,\hbar)}$ such that $\left.\frac{\partial^m}{\partial\hbar^m}\right|_\hbar a = 0$. In other words $S^m_{(x,\hbar)} = \mathfrak{m}^m_\hbar \left(\mathrm{S}^{\infty}_{\rho,\delta;đ}\right)_{(x,\hbar)}$. As $\mathcal{C}^{\infty}/\mathfrak{m}_\hbar \cong \mathbb{C}$ is a field, $S^0_{(x,\hbar)}/S^1_{(x,\hbar)}$ is a flat $\mathcal{C}^{\infty}/\mathfrak{m}_\hbar$ module. Next define the graded algebras $gr\left(\mathcal{C}^{\infty}_\hbar\right)$ and $gr\left(S^0_{(x,\hbar)}\right)$ by

$$\begin{array}{rcl} gr\left(\mathcal{C}^{\infty}_\hbar\right) & = & \mathcal{C}^{\infty}_\hbar/\mathfrak{m}_\hbar \oplus \mathfrak{m}_\hbar/\mathfrak{m}^2_\hbar \oplus \ldots \oplus \mathfrak{m}^m_\hbar/\mathfrak{m}^{m-1}_\hbar \oplus \ldots \cong \mathbb{C}^{\mathbb{N}}, \\ gr\left(S^0_{(x,\hbar)}\right) & = & S^0_{(x,\hbar)}/S^1_{(x,\hbar)} \oplus \ldots \oplus S^m_{(x,\hbar)}/S^{m-1}_{(x,\hbar)} \oplus \ldots \cong \left[\left(\mathrm{S}^{\infty}_{\rho,\delta;\hbar}\right)_x\right]^{\mathbb{N}}, \end{array} \tag{4.16}$$

and check that the canonical morphism

$$\gamma : gr\left(\mathcal{C}^\infty_\hbar\right) \otimes_{gr_0\left(\mathcal{C}^\infty_\hbar\right)} gr_0\left(S^0_{(x,\hbar)}\right) \to gr\left(S^0_{(x,\hbar)}\right) \tag{4.17}$$

is bijective. By Theorem 1. in Bourbaki [9], *Commutative Algebra*, Chapter III §5.2 this gives the flatness of the $\mathcal{C}^\infty_\hbar/\mathfrak{m}^m_\hbar$ module $S^0_{(x,\hbar)}/S^m_{(x,\hbar)}$. Furthermore the fiber $\left(\mathrm{S}^\infty_{\rho,\delta;\bar{d}}\right)_\hbar = \mathrm{S}^\infty_{\rho,\delta;\bar{d}}\Big|_{X\times\{\hbar\}} \Big/ \mathfrak{m}_\hbar\, \mathrm{S}^\infty_{\rho,\delta;\bar{d}}\Big|_{X\times\{\hbar\}}$ over $\hbar \in \mathbb{R}$ is isomorphic to $\mathrm{S}^\infty_{\rho,\delta;\hbar}(\cdot, X)$, hence D provides a flatly filtered deformation of $(X, \mathrm{S}^\infty_{\rho,\delta;\hbar})$. Now we can formulate the main theorem.

Theorem 4.2.1 *Let X be a* Riemannian *manifold and $0 \le \rho, \delta \le 0$, $\rho + \delta \ge 1$, $\rho > \delta$. Then the fibered morphism $D_{\rho,\delta} = (\bar{d}, \Delta_{\rho,\delta}) : (X \times \mathbb{R}, \mathrm{S}^\infty_{\rho,\delta;\bar{d}}) \to (\mathbb{R}, \mathcal{C}^\infty)$ is for every $\hbar \in \mathbb{R}$ a flatly filtered deformation of $(X, \mathrm{S}^\infty_{\rho,\delta;\hbar})$ with distinguished point $\hbar$. In particular for $\hbar = 0$ it is a deformation of the ringed space $\left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$.*

Let $q = pr_1 : X \times \mathbb{R} \to X$ be the projection on the first coordinate, and $\mathfrak{q}_{\rho,\delta}$ the pull-back $\mathfrak{q}_{\rho,\delta} = pr_1^* : \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, X) \to \mathrm{S}^\infty_{\rho,\delta;\bar{d}}(\cdot, X)$. Then the morphism $Q_{\rho,\delta} = (q, \mathfrak{q}_{\rho,\delta}) : \left(X \times \mathbb{R}, \mathrm{S}^\infty_{\rho,\delta;\bar{d}}\right) \to \left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$ together with the deformation $D_{\rho,\delta} = (\bar{d}, \Delta_{\rho,\delta})$ gives rise to a differentiable Dirac *quantization of the* Poisson *space $\left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$, the so-called* **normal order quantization** *of $\left(X, \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$ with respect to the* Levi-Civita *connection.*

Proof: It remains to show the Dirac quantization condition

$$[\mathfrak{q}(a), \mathfrak{q}(b)] \;=\; \frac{\bar{d}}{i}\,\mathfrak{q}\left(\{a, b\}\right) + \bar{d}^2 r, \tag{4.18}$$

for $a, b \in \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(U, T^*X)$, $U \subset X$ open and a rest term $r \in \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(U, T^*X)$. By abuse of notation we also write a (resp. b) for a representative of a (resp. b) in $\mathrm{S}^\infty_{\rho,\delta}(U, T^*X)$. Then we have by Eq. (4.11)

$$\begin{aligned}
[\mathfrak{q}(a), \mathfrak{q}(b)](\zeta, \hbar) \;&=\; \frac{\hbar}{i}\sum_{k=1}^{n}\left[\frac{\partial a}{\partial \xi_{\pi(\zeta),k}}(\zeta,\hbar)\,\frac{\partial b}{\partial z_{\pi(\zeta),k}}(\zeta,\hbar) - \frac{\partial a}{\partial z_{\pi(\zeta),k}}(\zeta,\hbar)\,\frac{\partial b}{\partial \xi_{\pi(\zeta),k}}(\zeta,\hbar)\right] \\
&\quad + \hbar^2 r(\zeta,\hbar) \\
&=\; \frac{\hbar}{i}\,\mathfrak{q}\left(\{a,b\}\right)(\zeta,\hbar) + \hbar^2 r(\zeta,\hbar),
\end{aligned} \tag{4.19}$$

where $\zeta \in T^*U$ and $\hbar \in \mathbb{R}$. Hence the claim holds. □

Note 4.2.2 The deformation in the preceding theorem can be interpreted as a convergent deformation with respect to the topology of asymptotic convergence on symbol spaces.

4.3 Functorial aspects

The normal order quantization on symbol spaces induces a certain functor Q_{symb} introduced in the following proposition.

Proposition 4.3.1 *Associate to any* RIEMANNIAN *manifold X its normal order quantization of the* POISSON *space* $\left(X, \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}\right)$. *This gives rise to a functor Q_{symb} from the category $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to the category $\underline{\mathsf{Quant}}^{diff.}_{Dirac}$ of differentiable* DIRAC *quantizations.*

PROOF: It suffices to show that the symbol map is natural with respect to isometric embeddings between RIEMANNIAN manifolds. In other words we have to show that for an isometric embedding $f : X \to Y$ of RIEMANNIAN manifolds and every pseudo-differential operator $A \in \Psi^{\infty}_{\rho,\delta}(X)$ the relation

$$\sigma_{f_* A} = f_*(\sigma_A) \tag{4.20}$$

holds. So first choose a cut-off function $\psi : T^*Y \to \mathbb{R}$ like in 3.2.1 and note that $f^*\psi : T^*X \to \mathbb{R}$ is a cut-off function on X. Now using the relation $f \circ \exp = \exp \circ TF$ and Eq. (3.23) we get for $\xi \in T^*Y\Big|_{\operatorname{im} f}$

$$\begin{aligned}
\sigma_{f_* A}(\xi) &= f_* A\Big(\psi_{\pi(\xi)}(\cdot)\, e^{i <\xi, \exp^{-1}_{\pi(\xi)}(\cdot)>}\Big)(\pi(\xi)) \\
&= A\Big((f^*\psi)_{f^{-1}(\pi(\xi))}(\cdot)\, e^{i <\xi, \exp^{-1}_{\pi(\xi)}(f(\cdot))>}\Big)\left(f^{-1}(\pi(\xi))\right) \\
&= A\Big((f^*\psi)_{f^{-1}(\pi(\xi))}(\cdot)\, e^{i <f^*(\xi), \exp^{-1}_{f^{-1}(\pi(\xi))}(\cdot)>}\Big)\left(f^{-1}(\pi(\xi))\right) \\
&= A\Big((f^*\psi)_{\pi(f^*(\xi))}(\cdot)\, e^{i <f^*(\xi), \exp^{-1}_{(\pi(f^*(\xi))}(\cdot)>}\Big)\left((\pi(f^*(\xi))\right) \\
&= \sigma_A(f^*(\xi)) = f_*(\sigma_A)(\xi).
\end{aligned} \tag{4.21}$$

Hence the claim follows. □

By Theorem 4.2.1 and the preceeding proposition we know that the POISSON space $(X, \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty})$ can be quantized and that it gives rise to a functor Q_{symb} from $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to $\underline{\mathsf{Quant}}^{diff.}_{Dirac}$. But we do not yet know whether this normal order quantization quantization is consistent with the one of the POISSON space $(X, \mathcal{D}_0)$ of polynomial observables on X, resp. with the functor Q_{pol}. The following proposition shows that this is the case indeed.

Proposition 4.3.2 *The canonical embeddings $\sigma_0 : \mathcal{D}_0 \to \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}$ and $\sigma_{\hbar} : \mathcal{D}_{\hbar} \to \mathrm{S}^{\infty}_{\rho,\delta;\hbar}$ induce a morphism from the quantization map $\mathfrak{q} : \mathcal{D}_0 \to \mathcal{D}_{\hbar}$ to the quantization map $\mathfrak{q}_{\rho,\delta} : \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty} \to \mathrm{S}^{\infty}_{\rho,\delta;\hbar}$. In other words the diagrams*

$$\begin{tikzcd} & \mathcal{C}^\infty_{\mathbb{R}} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta_{\rho,\delta}"] & \\ \mathcal{D}_{\text{đ}} \arrow[rr, "\sigma_{\text{đ}}"'] & & \mathrm{S}^\infty_{\rho,\delta;\text{đ}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}_{\text{đ}} \arrow[r, "\sigma_{\text{đ}}"] \arrow[d, "i^*"'] & \mathrm{S}^\infty_{\rho,\delta;\text{đ}} \arrow[d, "i^*"] \\ \mathcal{D}_0 \arrow[r, "\sigma_0"'] & \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{D}_0 \arrow[r, "\sigma_0"] \arrow[d, "\mathfrak{q}"'] & \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty} \arrow[d, "\mathfrak{q}_{\rho,\delta}"] \\ \mathcal{D}_{\text{đ}} \arrow[r, "\sigma_{\text{đ}}"'] & \mathrm{S}^\infty_{\rho,\delta;\text{đ}} \end{tikzcd}$$

commute, where $i : X \to X \times \mathbb{R}$ is the canonical injection $x \mapsto (x, 0)$. Thus there exists a natural transformation from the functor Q_{symb} to the functor Q_{pol}. We say that the two functors Q_{symb} and Q_{pol} resp. their quantization maps $\mathfrak{q}$ and $\mathfrak{q}_{\rho,\delta}$ are **compatible***.*

PROOF: To prove the proposition it suffices to show commutativity of the above diagrams. As the first two diagrams obviously commute, we show this only for the last one. Let $a \in \mathcal{D}_0(U)$. Then by Proposition 2.3.2 we have for $r \in \mathcal{C}^\infty(U \times \mathbb{R})$, $x \in U$ and $\hbar \in \mathbb{R}$

$$[\mathfrak{q}(f) \cdot r]\,(x, \hbar) \;=\; \sum_\alpha (-i\hbar)^{|\alpha|}\, a_{x,\alpha}(x)\, \frac{\partial^{|\alpha|} r}{\partial z_x^\alpha}(x, \hbar), \tag{4.22}$$

where $a|_{T^*U_x} = \sum\limits_\alpha (-i\hbar)^{|\alpha|}\,(a_{x,\alpha} \circ \pi)\,\xi_x^\alpha$, $a_{x,\alpha} \in \mathcal{C}^\infty(U_x)$, $U_x \subset U$ open neighborhood of x. Hence the equation

$$\begin{aligned} \sigma_{\text{đ},\mathfrak{q}(f)}(\zeta, \hbar) \;&=\; \left[\mathfrak{q}(f)\left(\psi_{\pi(\zeta)}(\cdot)e^{i\varphi(\cdot,\frac{\zeta}{\hbar})}\right)\right](\pi(\zeta), \hbar) \;=\; \sum_\alpha a_{\pi(\zeta),\alpha}(\pi(\zeta))\xi_x^\alpha(\zeta) \;= \\ &=\; \mathfrak{q}_{\rho,\delta}(\sigma_0(a))\,(\zeta, \hbar) \end{aligned} \tag{4.23}$$

holds for $\zeta \in T^*U$ and $\hbar \in \mathbb{R}$. But this gives the claim. □

The above functorial properties of normal order quantization are not in contradiction to a theorem of VAN HOVE [56] saying that there is no plausible quantization functor going from the category $\underline{\mathsf{Diff}}_{symp}$ of symplectic manifolds and symplectomorphisms to the category $\underline{\mathbb{C}\text{-}\mathsf{Vec}}_{Hilb}$ of HILBERT spaces and isometric mappings. The reason why this is not the case lies in the fact that we have chosen different domain and range categories for our quantization functors. According to Theorem 2.3.4 the functors Q_{pol} and L^2 induce a functor $\mathsf{R}_\hbar$ from $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ to HILBERT space representations. Now all these three functors have domain $\underline{\mathsf{Diff}}^{emb.}_{Riem}$ which can be regarded as a subcategory of $\underline{\mathsf{Diff}}_{symp}$, but only as a proper one. Thus there is no contradiction to VAN HOVES'S observation.

We hope to extend the quantization functors defined in this work to a larger (proper) subcategory of the symplectic category $\underline{\mathsf{Diff}}_{symp}$ in a forthcoming project.

Conclusion

By reasons of completeness let us mention at the end of this paper some other important quantization schemes besides the already explained ones of geometric quantization and formal deformation quantization and let us give some references:

(*i*) Probably the first mathematically rigorous quantization scheme is the Weyl quantization due to Hermann Weyl [62]. Unfortunately it works well only for the quantization on Euclidean spaces. For attempts to generalize Weyl quantization to curved manifolds see Underhill [53] or Zhang, Min [67]. Moyal [36] used Weyl quantization to define a deformation of a certain commutative algebra of functions on Euclidean spaces (see Example 1.2.7).

(*ii*) Though it lacks of mathematical rigor, the path integral formalism by Feynman [18] is very popular in theoretical physics and far-reaching for calculational and conceptual purposes. In the theory of functional integration one tries to give Feynman's ideas a precise mathematical meaning and wants to apply them for quantum field theory. See for example Barry Simon [48] or Glimm, Jaffe [23].

(*iii*) The group quantization scheme together with a lot of topological and other aspects of quantization are extensively described in Isham [30].

(*iv*) Berezin's quantization is in some respect similar to our approach, as Berezin regards a quantization as a deformation of a commutative algebra by a real parameter $\hbar$. But then it departs from our work, as Berezin's quantization essentially uses complex methods and so-called coherent states. Berezin's quantization is appropriate for the quantization of symmetric complex spaces or Kähler manifolds. See for example Berezin's paper [4] and references therein or the more modern account by Rawnsley in [40, 41, 39]. Rawnsley does in fact construct a formal deformation for certain Kähler manifolds by means of Berezin's quantization and claims that in some good cases those deformations probably come from converging ones.

(v) MARC RIEFFEL has established the notion of a strict deformation quantization, that is a framework for the continuous deformation of C^*-algebras. There is a close connection to the noncommutative geometry by ALAIN CONNES [11]. See the monograph [44] by RIEFFEL as well as his articles [42, 43, 45] for more information on the topic of strict deformation quantization.

The deformation theoretical quantization program we have introduced in this work can be regarded as the starting point for further developments in quantization theory, in particular if combined with one or the other of the above methods. As already mentioned in the introduction neither of the existing quantization schemes gives a completely satisfactory answer to the quantization problem. But each one has its own advantages, so it should be the goal to constructively combine different methods, if that is possible. This imperative holds for our program of deformation quantization as well, and I think that some generalizations and improvements in this direction can be done.

First it seems possible to extend the MOYAL product to arbitrary RIEMANNIAN manifolds. In other words we regard it as very promising to establish a complete WEYL symbol for the whole category of RIEMANNIAN manifolds and then to define a functor of WEYL quantizations on RIEMANNIAN manifolds according to the methods introduced in chapter 4. The papers UNDERHILL [53] and ZHANG, MIN [67] could be very helpful for work in that direction.

The class of symplectic manifolds for which we have proven the existence of a concrete deformation quantization, namely the cotangent bundles with their canonial symplectic structure, should be enlarged. By using the general notions of conic manifolds resp. conic bundles defined in chapter 3 and by using the theory of FOURIER integral operators we hope to extend our method of quantizing symbol spaces to conic symplectic manifolds. Similarly it should be possible to find concrete deformation quantizations of symplectic manifolds with a real polarization.

Up to now we have only regarded the quantization of bosonic systems. But it is well-known that there exists also a quantization condition for fermionic systems: POISSON brackets for classical fermionic variables or in other words for GRASSMANN variables go over under quantization to anticommutators (see for example HENNEAUX, TEITELBOIM [28]). Now on the mathematical side we have a notion of a fermionic symbol map going from a Clifford algebra to an exterior algebra (cf. Berline, Getzler, Vergne [5]). It would then be very interesting to see whether it is possible to construct with such a fermionic symbol map a deformation quantization of Grassmann variables such that the above fermionic quantization condition is fulfilled.

Another important point which should be studied is the behaviour of deformation quantization under the action of symmetry groups. In this context one could think about the question whether first quantizing and then reducing by the symmetry group or first reducing and then quantizing gives the same result.

At the end I now hope to have convinced the reader that concrete deformation quantization gives new insights to the quantization process but also new questions for further research in this area. In forthcoming projects I intend to work on these.

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Erklärung

Ich versichere, die vorliegende Arbeit selbständig in Englisch angefertigt und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet zu haben.

München, im August 1995 Markus J. Pflaum