Math 2002 Number Systems
Homework Set 4

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Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 255, Telephone: 2-7717, e-mail: markus.pflaum@colorado.edu.

Problem 1: Let $f : X \to Y$ be a function for which there exist functions $g_1 : Y \to X$ and $g_2 : Y \to X$ such that $g_1 \circ f = \mathrm{id}_X$ and $f \circ g_2 = \mathrm{id}_Y$. Show that then f is invertible and that $g_1 = g_2$. (4P)

Problem 2: Show that for all $x, y \in \mathbb{R}$

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|) \quad \text{and} \quad \min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$$

(4P)

Problem 3: Consider the triple $F = (\mathbb{R}, \mathbb{R}, \Gamma)$ with

a) $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$,

b) $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2 + 1\}$,

c) $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 1\}$.

d) $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \sin y = \cos x\}$.

In which of these cases is F a function? Explain! (4P)

Problem 4:

a) Let $n \in \mathbb{N}_{>0}$. Find and prove by induction a formula for $\sum_{k=1}^{n} \frac{1}{k(k+1}$. (2P)

b) Prove by induction the following formula for positive natural n:

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n} \ .$$

(2P)

Problem 5: Prove the following statements for all positive natural numbers:

a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, (2P)

b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. (2P)