Research Statement

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My research is in discrete geometry and combinatorics, focusing on the symmetries of convex polytopes and tilings of Euclidean space by convex polytopes. I have classified all the convex polytopes with 2 or 3 flag orbits, and studied the possibility of polytopes or tilings which are highly symmetric combinatorially but not realizable with full geometric symmetry.

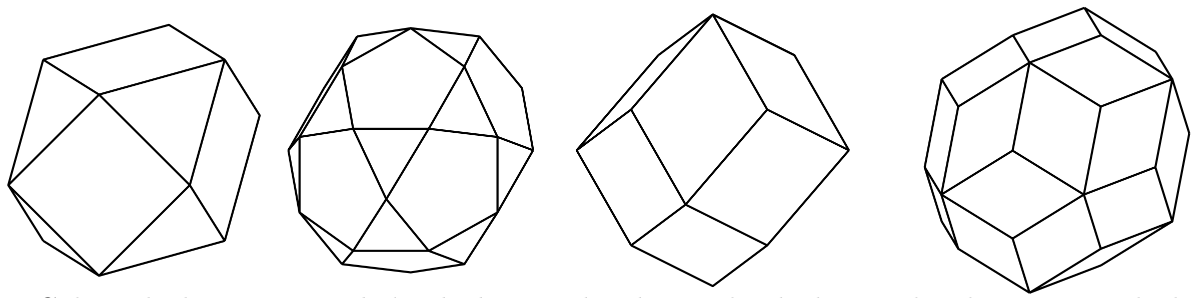
1. **Background.** Polytopes have been studied for millennia, and have been given many definitions. By *(convex) d-polytope*, I mean the convex hull of a finite collection of points in Euclidean space, whose affine hull is d-dimensional. Various other definitions don't require convexity, resulting in star-polytopes such as the Kepler-Poinsot polyhedra. A convex d-polytope may be regarded as a tiling of the $(d-1)$-sphere; generalizing this, we can regard tilings of other spaces as polytopes. The combinatorial aspects that most of these definitions have in common are captured by the definition of an *abstract polytope*. An abstract polytope is a ranked, partially ordered set of *faces*, from 0-faces (vertices), 1-faces (edges), up to $(d-1)$-faces (facets), satisfying certain axioms [7].

The *symmetries* of a convex polytope P are the Euclidean isometries which carry P to itself, and form a group, the *symmetry group* of P. Similarly, the automorphisms of an abstract polytope K are the order-preserving bijections from K to itself and form the *automorphism group* of K. A *flag* of a polytope is a maximal chain of its faces, ordered by inclusion: a vertex, an edge incident to the vertex, a 2-face containing the edge, and so on. The symmetry group of a convex polytope (and more generally, any group of automorphisms of an abstract polytope) acts freely on the set of its flags. A convex polytope whose symmetry group has k orbits on its flags is called a *k-orbit polytope*; a polytope whose automorphism group has k orbits on the flags is called a *combinatorially k-orbit polytope.*

The 1-orbit polytopes, or *regular* polytopes, are among the oldest objects of mathematical study. The construction of the 1-orbit 3-polytopes—the Platonic solids—and proof that there are no others is the subject of Euclid's *Elements.* The classification of the six 1-orbit 4-polytopes, and the three 1-orbit d-polytopes for every $d \geq 5$, was accomplished by Ludwig Schläfli, and independently by many others, in the 19th century [3].

2. **Dissertation work.** My research has been on highly symmetric, but not regular, polytopes. I examined the flag orbits resulting from standard operations used to construct polytopes, and classified all convex polytopes with 2 or 3 flag orbits. In the plane, there are infinitely many 2-orbit or 3-orbit polygons; indeed, there are infinitely many k-orbit polygons for any k. Any of these is combinatorially regular.

Besides these, there are only a few 2-orbit polytopes or tilings. We use the notation $(k.j\dots)$ to denote a vertex-transitive 3-polytope or plane tiling where the 2-faces containing each vertex are a k-gon, a j-gon, etc., in that cyclic order. The only 2-orbit 3-polytopes are the cuboctahedron (3.4.3.4) and the icosidodecahedron (3.5.3.5) (two

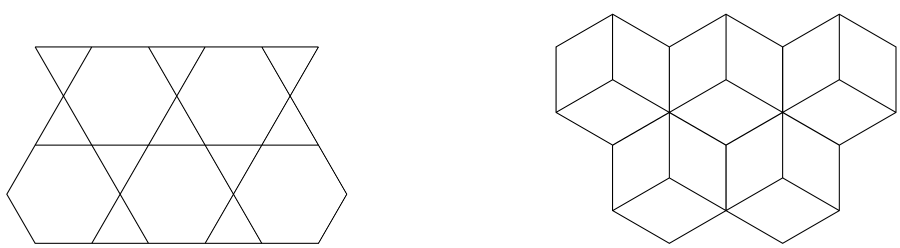


Cuboctahedron Icosidodecahedron Rhombic Dodecahedron Rhombic Triacontahedron

FIGURE 1. The two-orbit convex polyhedra

of the Archimedean solids) and their duals, the rhombic dodecahedron and the rhombic triacontahedron. There are no 2-orbit polytopes in dimension four or higher.

The 2-orbit plane tilings include a couple of distortions of the regular tiling by squares, into tilings by rhombi or by rectangles; these remain combinatorially regular. The only other 2-orbit plane tilings are the trihexagonal tiling (3.6.3.6) and its dual, the rhombille tiling (also known as "tumbling blocks" for the optical illusion it creates.) The only 2-orbit tilings of 3-space are the tetrahedral-octahedral (or "alternated cubic") honeycomb and its dual, the rhombic dodecahedral honeycomb. There are no 2-orbit tilings of dimension four or higher.



The Trihexagonal Tiling The Rhombille Tiling

FIGURE 2. The combinatorially irregular two-orbit plane tilings

Subsequently, I proved that any convex polytope which is combinatorially 2-orbit is isomorphic to one on the above list. The analogous fact was proved for combinatorially 1-orbit polytopes by McMullen [6], but is not true in general; there is known to be a combinatorially 84-orbit 4-polytope which is not isomorphic to any (geometrically) 84-orbit polytope P [1].

On the other hand, a combinatorially 2-orbit tiling by convex polytopes need not be isomorphic to one of the above list: indeed, there are infinitely many such plane tilings, of type $(3.n.3.n)$ for any $n \geq 7$, alternating triangles and n-gons (similar to the trihexagonal tiling.) However, we can disallow all these examples by requiring the tilings to be *normal*, which for tilings by convex tiles amounts to requiring that the size of the tiles is bounded (above and below.) It remains open whether combinatorially 2-orbit normal tilings must be isomorphic to one of the above listed ones. The only possible exception is a dual pair of tilings of 3-space. The analogous question for 1-orbit tilings also remains open, again with only a finite list of possible exceptions.

Similarly, it turns out that 3-orbit convex polytopes exist only up to dimension 8. There are infinitely many in 3 dimensions, including the prisms over regular polygons and their duals, the bipyramids. There are also infinitely many in 4 dimensions, including the so-called p, p-duoprisms, which can be formed as the Cartesian product of two regular p-gons. There are only four 3-orbit 5-polytopes, another four 6-polytopes, and just two each in dimensions 7 or 8. The ones in dimensions past five depend on the exceptional Coxeter (or Weyl) groups of type E_6, E_7, or E_8 for their existence.

Rather surprisingly, it turns out that for any $k > 1$, there are only finitely many dimensions where k-orbit polytopes exist (of course, there are 1-orbit polytopes in every dimension: the d-simplex, d-cube, and d-cross-polytope.)

3. **Earlier Research.** As a master's student, I collaborated with Paul Larson and Saharon Shelah on a topic in graph theory. Condorcet's paradox is that a population of voters with linear (totally ordered) preferences among a set of candidates can, under majority-wins voting, have cyclic outcomes (e.g. A beats B, B beats C, but C beats A.) In our paper, we generalized by allowing the voters to have arbitrary directed graphs among the candidates, and classified the types of voter graphs that result in different kinds of majority outcome.

Although I was a student in Mathematics, I was a research assistant in Electrical & Computer Engineering. During this time I analyzed models of higher-order error in the Global Positioning System due to ionospheric effects, using data from incoherent scatter radars around the world (in particular, spending one summer at the Jicamarca Radio Observatory in Peru) to characterize the typical error, which was previously poorly understood for latitudes beyond those of central North America.

4. **Future work.** In future work I would like to extend the classification of k-orbit polytopes to more values of k, and determine such things for general k as the set of dimensions containing k-orbit polytopes, or those containing infinite series of k-orbit polytopes.

I am also excited about the possible application of algebraic topology. A key tool in my research is the *orbit graph* of a polytope. This is a graph (allowing loops and multiple edges) consisting of a node for each flag orbit, with an edge labeled i joining two nodes if for each flag Φ in one of the corresponding orbits, the i-adjacent flag Φ^i (meaning the unique flag agreeing with Φ in all faces except for the i-face) is in the other. Such a graph describes the "symmetry type" of a polytope; for instance, every regular d-polytope has the same orbit graph, consisting of a single node and loops labeled $0, \dots, d-1$. The *chiral polytopes* are a notable family of two-orbit abstract polytopes [12, 9] which can be described by an orbit graph with two nodes joined by edges $0, \dots, d-1$. (These cannot be convex; any convex two-orbit d-polytope has an orbit graph with a single edge between the two nodes, either labeled 0 or $d-1$.)

By adding cells in an appropriate way, the orbit graph can be made into a CW complex whose fundamental group G is universal for polytopes with that symmetry type, in that the automorphism group of any such polytope is a quotient of G.

Given a particular polytope P with this orbit graph, we can add further cells determined by P to get a CW complex whose fundamental group is isomorphic to the automorphism group of P. In the case of regular polytopes, these extra cells correspond to the Coxeter relations encoded in Coxeter diagrams. For non-regular polytopes, the symmetry group need not be a Coxeter group, so this CW complex construction gives a generalization of the Coxeter diagram for other polytopes.

The 1-skeleton of the universal cover of these CW complexes can be treated as a flag graph of an abstract polytope (with a node for each flag, and an edge labeled i between i-adjacent flags). This gives an intriguing way of realizing polytopes with prescribed orbit graphs and group presentations. There has been much recent attention to issues such as the existence of universal chiral polytopes and chiral polytopes in arbitrary rank [11, 2, 8, 5, 4]. I would like to see whether these constructions and the methods of algebraic topology can shed light on these questions for general orbit graphs.

In another direction, I am curious about possible insights from applying the viewpoint of toric topology to the symmetries of convex or abstract polytopes. Robertson, Carter and Morton [10] use these sorts of topological methods to study finite sets of points in Euclidean space with a transitive group of isometries.

I am open to working with other faculty on different problems, especially anything involving discrete geometry, combinatorics, or topology. In addition, my research area is rich with potential topics for undergraduate research experience, which I would love to mentor. Many questions about tilings, particularly in 2 or 3 dimensions, are easy to grasp and invite investigation, and even if the students do not solve them completely, they can certainly make new observations and discover partial results. For instance: Is there a normal tiling of 3-space by convex polytopes isomorphic to cubes, with five around every edge? (That is, a realization of the regular hyperbolic tiling $\{4, 3, 5\}$.) There are many open questions like this.

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