Notions of reals

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We give a detailed exposition of the equivalence of several notions of real numbers used in set theory: the standard used in analysis, ${}^{\omega}2$, and ${}^{\omega}\omega$. We show in particular that these notions coincide for the cardinal characteristics add, cov, non, and cof (defined below) for the meager and null ideals.

The irrationals and ${}^{\omega}\omega$

Theorem 1. *${}^{\omega}\omega$ under the product topology is homeomorphic to the irrationals.*

Proof. Let $a = \langle a_0, a_1, \ldots\rangle$ be an infinite sequence of integers such that $a_i > 0$ for all $i > 0$. We want to give a precise definition of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

To start with, we assume that a is a sequence of positive real numbers with domain either ω or some positive integer. We define $[a_0, \ldots, a_l]$ for each $l < \mathrm{dmn}(a)$ by recursion:

$$[a_0] = a_0;$$
$$[a_0, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]}$$

We want to be very explicit as to how these approximations can be written as certain fractions. To this end we make the following recursive definitions:

$$p(a, 0) = a_0; \quad q(a, 0) = 1;$$
$$p(a, 1) = a_0 a_1 + 1; \quad q(a, 1) = a_1.$$

For $k \geq 2$:

$$\text{(1)} \qquad \begin{aligned} p(a, k) &= a_k p(a, k-1) + p(a, k-2); \\ q(a, k) &= a_k q(a, k-1) + q(a, k-2). \end{aligned}$$

Note that $p(a, k) > 0$ and $q(a, k) > 0$ for all $k \geq 0$. Also, let $a' = \langle a_1, a_2, \ldots\rangle$. Now we claim that for all $i \in \omega$,

$$p(a, i+1) = a_0 p(a', i) + q(a', i);$$
$$q(a, i+1) = p(a', i).$$

We prove these equations by induction on i. For $i = 0$ we have

$$\begin{aligned} p(a,1) &= a_0a_1 + 1 = a_0p(a',0) + q(a',0); \\ q(a,1) &= a_1 = p(a',0), \end{aligned}$$

as desired. For $i = 1$,

$$\begin{aligned} p(a,2) &= a_2p(a,1) + p(a,0) \\ &= a_0a_1a_2 + a_2 + a_0 \\ &= a_0(a_1a_2 + 1) + a_2 \\ &= a_0p(a',1) + q(a',1); \\ q(a,2) &= a_2q(a,1) + q(a,0) \\ &= a_1a_2 + 1 \\ &= p(a',1), \end{aligned}$$

as desired. Now we do the inductive step for $i \geq 2$:

$$\begin{aligned} p(a,i+1) &= a_{i+1}p(a,i) + p(a,i-1) \\ &= a_{i+1}(a_0p(a',i-1) + q(a',i-1)) + a_0p(a',i-2) + q(a',i-2) \\ &= a_0(a_{i+1}p(a',i-1) + p(a',i-2)) + a_{i+1}q(a',i-1) + q(a',i-2) \\ &= a_0p(a',i) + q(a',i); \\ q(a,i+1) &= a_{i+1}q(a,i) + q(a,i-1) \\ &= a_{i+1}p(a',i-1) + p(a',i-2) \\ &= p(a',i), \end{aligned}$$

as desired. So the above equations hold.

Note by an easy induction that $p(a,k), q(a,k) > 0$ for all k. Now we claim:

(2) $$[a_0,\ldots,a_k] = \frac{p(a,k)}{q(a,k)}$$

for every $k \in \omega$. We prove (2) by induction on k. For $k = 0$, we have

$$[a_0] = a_0 = \frac{p(a,0)}{q(a,0)},$$

as desired. For $k = 1$, we have

$$[a_0,a_1] = a_0 + \frac{1}{a_1} = \frac{a_0a_1 + 1}{a_1} = \frac{p(a,1)}{q(a,1)},$$

as desired. Inductively, for $k \geq 2$,

$$\begin{aligned}[a_0, \ldots, a_k] &= a_0 + \frac{1}{[a_1, \ldots, a_k]} \\ &= a_0 + \frac{q(a', k-1)}{p(a', k-1)} \\ &= \frac{a_0 p(a', k-1) + q(a', k-1)}{p(a', k-1)} \\ &= \frac{p(a,k)}{q(a,k)},\end{aligned}$$

as desired.

From now on we shall write p_k, q_k in place of $p(a,k), q(a,k)$ if a is understood. We also define $p_{-1} = 1$ and $q_{-1} = 0$. Then the equations (1) also hold for $k = 1$, since

$$\begin{aligned}a_1 p_0 + p_{-1} &= a_0 a_1 + 1 = p_1 \quad \text{and} \\ a_1 q_0 + q_{-1} &= a_1 = q_1.\end{aligned}$$

Next we claim that for $k \geq 1$,

(3) $$q_k p_{k-1} - p_k q_{k-1} = -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}).$$

In fact, multiply the equations (1) by q_{k-1} and p_{k-1} respectively:

$$\begin{aligned}p_k q_{k-1} &= a_k p_{k-1} q_{k-1} + p_{k-2} q_{k-1}; \\ q_k p_{k-1} &= a_k q_{k-1} p_{k-1} + q_{k-2} p_{k-1}.\end{aligned}$$

Subtracting the first of these equations from the second gives (3).

Now $q_0 p_{-1} - p_0 q_{-1} = 1$, so by (3) and induction we get, for $k \geq 0$,

(4) $$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

Hence for $k \geq 1$ we have

(5) $$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}} .$$

Next, for any $k \geq 1$,

(6) $$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k .$$

To see this, multiply the equations (1) by q_{k-2} and p_{k-2} respectively:

$$\begin{aligned}p_k q_{k-2} &= a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2}; \\ q_k p_{k-2} &= a_k q_{k-1} p_{k-2} + q_{k-2} p_{k-2}.\end{aligned}$$

Now subtract the first from the second and use (4): (6) follows.

From (6):

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1}a_k}{q_k q_{k-2}}. \tag{7}$$

Hence:

$$\left\langle \frac{p_{2k}}{q_{2k}} : k \in \omega \right\rangle \text{ is an increasing sequence;} \tag{8}$$

$$\left\langle \frac{p_{2k+1}}{q_{2k+1}} : k \in \omega \right\rangle \text{ is an decreasing sequence;} \tag{9}$$

Next we claim

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2l+1}}{q_{2l+1}} \text{ for all } k, l \in \omega \tag{10}$$

In fact, let $m = \max(k, l)$. Then

$$\begin{aligned} \frac{p_{2k}}{q_{2k}} &\le \frac{p_{2m}}{q_{2m}} \quad \text{by (8)} \\ &< \frac{p_{2m+1}}{q_{2m+1}} \quad \text{by (5)} \\ &\le \frac{p_{2l+1}}{q_{2l+1}} \quad \text{by (9)} \end{aligned}$$

So (10) holds. Next we claim:

$$p_k < p_{k+1} \text{ and } q_{k+1} < q_{k+2} \quad \text{for all } k \in \omega. \tag{11}$$

In fact, this is clear from the recursive definitions.

Now we assume that our sequence a is infinite, and all a_i are positive integers. It follows from (8), (9), (10), (11), and (5) that the approximations $\frac{p_k}{q_k}$ converge, and by definition the limit is the value of the infinite continued fraction described at the beginning. For a_0 a negative integer but all a_i positive integers for $i > 0$, we define $a' = \langle 1, a_1, a_2, \ldots \rangle$ and define the continued fraction to be

$$a_0 - 1 + \lim_{k\to\infty} \frac{p(a', k)}{q(a', k)}$$

Now we want to see how to represent any real number as a finite or infinite continued fraction. We make a recursive definition for any real number $\alpha > 1$. Let $r(\alpha, 0) = \alpha$. Suppose that we have defined $r(\alpha, i) > 1$. Write $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$ with $a(\alpha, i)$ a positive integer and $s(\alpha, i+1)$ a nonnegative real < 1. If $s(\alpha, i+1) = 0$, the construction stops. Otherwise we define $r(\alpha, i+1) = \frac{1}{s(\alpha, i+1)}$. This finishes the construction. Let $l(\alpha)$

be the index i such that $s(\alpha, i+1) = 0$, or $l(\alpha) = \omega$ if there is no such index. We need the following technical fact.

(12) If $\alpha > 1$ and $l(\alpha) > 1$, then $l(r(\alpha, 1)) = l(\alpha) - 1$, and for each $j \leq l(\alpha) - 1$ we have $r(r(\alpha, 1), j) = r(\alpha, j+1)$ and $a(r(\alpha, 1), j) = a(\alpha, j+1)$.

By induction on j we prove that $r(r(\alpha, 1), j)$ is defined and equals $r(\alpha, j+1)$ for each $j \leq l(\alpha) - 1$. For $j = 0$ we have $r(r(\alpha, 1), 0)$ defined and it equals $r(\alpha, 1)$, as desired. Now assume our result for j, with $j + 1 \leq l(\alpha) - 1$. Then

$$r(r(\alpha, 1), j) = r(\alpha, j+1) = a(\alpha, j+1) + s(\alpha, j+2).$$

Now $j + 2 \leq l(\alpha)$, so $s(\alpha, j+2) > 0$, and hence by definition, $r(\alpha, j+2) = \frac{1}{s(\alpha, j+2)} = r(r(\alpha, 1), j+1)$. This completes the inductive proof.

Now if $j \leq l(\alpha) - 1$, then

$$\begin{aligned} r(r(\alpha, 1), j) &= a(r(\alpha), 1), j) + s(r(\alpha, 1), j+1); \\ r(\alpha, j+1) &= a(\alpha, j+1) + s(\alpha, j+2); \end{aligned}$$

so $a(r(\alpha, 1), j) = a(\alpha, j+1)$. Finally, if $j = l(\alpha)$, then $r(\alpha, j) = a(\alpha, j)$, and hence $r(r(\alpha, 1), j-1) = r(\alpha, j) = a(\alpha, j)$ and so $l(r(\alpha, 1)) = j - 1$, as desired in (12).

(13) If $\alpha > 1$ and $n \leq l(\alpha)$, then $\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots, a(\alpha, n-1), r(\alpha, n)]$.

We prove this by induction on n. For $n = 0$, $[r(\alpha, 0)] = \alpha$. Assume that our condition is true for n, and $n + 1 \leq l(\alpha)$. Then

$$\begin{aligned} &[a(\alpha, 0), a(\alpha, 1), \ldots, a(\alpha, n), r(\alpha, n+1)] \\ &\qquad = a(\alpha, 0) + \frac{1}{[a(\alpha, 1), a(\alpha, 2), \ldots, a(\alpha, n), r(\alpha, n+1)]} \\ &\qquad = a(\alpha, 0) + \frac{1}{[a(r(\alpha, 1), 0) a(r(\alpha, 1), 1), \ldots, a(r(\alpha, 1), n-1), r(r(\alpha, 1), n)]} \\ &\qquad = a(\alpha, 0) + \frac{1}{r(\alpha, 1)} \\ &\qquad = a(\alpha, 0) + s(\alpha, 1) \\ &\qquad = \alpha, \end{aligned}$$

completing the inductive proof.

(14) If $\alpha > 1$ is rational, then the above definition of $r(\alpha, i)$'s terminates after finitely many steps.

In fact, it suffices to show that if $r(\alpha, i) = \frac{b}{c}$ with b, c positive integers and g.c.d$(b, c) = 1$, and $r(\alpha, i+1)$ is defined, then $r(\alpha, i+1)$ has the form $\frac{d}{e}$, with d and e positive integers with $e < c$. To prove this, recall that $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$, with $s(\alpha, i+1)$ a nonnegative real < 1, and $r(\alpha, i+1) = \frac{1}{s(\alpha, i+1)}$. Thus

$$\frac{b}{c} = r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1) \quad \text{and hence}$$

$$b = ca(\alpha, i) + cs(\alpha, i+1); \tag{15}$$

Hence

$$\begin{aligned} r(\alpha, i+1) &= \frac{1}{s(\alpha, i+1)} \\ &= \frac{1}{r(\alpha, i) - a(\alpha, i)} \\ &= \frac{1}{\frac{b}{c} - a(\alpha, i)} \\ &= \frac{c}{b - ca(\alpha, i)} \\ &= \frac{c}{cs(\alpha, i+1)} \quad \text{by (15)}, \end{aligned}$$

and $cs(\alpha, i+1)$ is a positive integer $< c$, as desired.

(16) If α is rational, then there exist integers $a_0, a_1, \ldots, a_n$ with $a_i > 0$ for all $i > 0$ such that $\alpha = [a_0, a_1, \ldots, a_n]$.

In fact, let m be an integer such that $\alpha + m > 1$; if $\alpha > 1$, let $m = 0$. By (14), $n \stackrel{\text{def}}{=} l(\alpha + m)$ is finite. We then have $r(\alpha + m, n) = a(\alpha + m, n)$. Hence by (13) we have $\alpha + m = [a(\alpha + m, 0), \ldots, a(\alpha + m, n)]$, and the desired conclusion follows.

(17) If $\langle a_0, a_1, \ldots \rangle$ is a sequence of rational numbers each greater than 0, then also $[a_0, a_1, \ldots, a_n]$ is rational for each n.

This is clear from the basic definition, by induction.

(18) Let $\alpha > 1$ be irrational. Then by (17), the sequence

$$b \stackrel{\text{def}}{=} \langle a(\alpha, 0), a(\alpha, 1), \ldots \rangle$$

never terminates. We claim that for each positive integer n,

$$\alpha = \frac{p(b, n-1) r(\alpha, n) + p(b, n-2)}{q(b, n-1) r(\alpha, n) + q(b, n-2)}.$$

We prove by induction that for every positive integer n, this holds for all irrationals $\alpha > 1$. First, the case $n = 1$:

$$\begin{aligned} \frac{p(b,0) r(\alpha, 1) + p(b, -1)}{q(b,0) r(\alpha, 1) + q(b, -1)} &= \frac{a(\alpha, 0) r(\alpha, 1) + 1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + \frac{1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + s(\alpha, 1) \\ &= r(\alpha, 0) \\ &= \alpha, \end{aligned}$$

as desired. Now we assume our statement for n. In fact, we apply it to $r(\alpha, 1)$ rather than α. Note that $r(\alpha, 1) > 1$, and it is irrational by (17) and (13). Let

$$\begin{aligned} c &= \langle a(\alpha,1), a(\alpha,2), \ldots \rangle \\ &= \langle a(r(\alpha,1),0), a(r(\alpha,1),1), \ldots \rangle, \end{aligned}$$

by (12). Hence, starting with the inductive hypothesis,

$$\begin{aligned} r(\alpha,1) &= \frac{p(c,n-1)r(r(\alpha,1),n) + p(c,n-2)}{q(c,n-1)r(r(\alpha,1),n) + q(c,n-2)} \\ &= \frac{p(c,n-1)r(\alpha,n+1) + p(c,n-2)}{q(c,n-1)r(\alpha,n+1) + q(c,n-2)}. \end{aligned}$$

Hence, using the equations following (1),

$$\begin{aligned} \alpha &= r(\alpha,0) \\ &= a(\alpha,0) + s(\alpha,1) \\ &= a(\alpha,0) + \frac{1}{r(\alpha,1)} \\ &= a(\alpha,0) + \frac{q(c,n-1)r(\alpha,n+1) + q(c,n-2)}{p(c,n-1)r(\alpha,n+1) + p(c,n-2)} \\ &= \frac{a(\alpha,0)p(c,n-1)r(\alpha,n+1) + a(\alpha,0)p(c,n-2) + q(c,n-1)r(\alpha,n+1) + q(c,n-2)}{p(c,n-1)r(\alpha,n+1) + p(c,n-2)} \\ &= \frac{p(b,n)r(\alpha,n+1) + p(b,n-1)}{q(b,n)r(\alpha,n+1) + q(b,n-1)}, \end{aligned}$$

which finishes the inductive proof of (18).

We now omit the parameter b, as it is understood in what follows.

(19) Let $\alpha > 1$ be irrational. Then for every positive integer n,

$$\alpha - \frac{p_n}{q_n} = \frac{(p_{n-1}q_{n-2} - q_{n-1}p_{n-2})(r_n - a_n)}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})}.$$

To prove this, first note by (18) and (1) that

(20) $$\alpha - \frac{p_n}{q_n} = \frac{p_{n-1}r_n - p_{n-2}}{q_{n-1}r_n + q_{n-2}} - \frac{p_{n-1}a_n + p_{n-2}}{q_{n-1}a_n + q_{n-2}}.$$

Now we have

$$\begin{aligned} &(p_{n-1}r_n - p_{n-2})(q_{n-1}a_n + q_{n-2}) - (p_{n-1}a_n + p_{n-2})(q_{n-1}r_n + q_{n-2}) \\ &\quad = p_{n-1}q_{n-1}a_nr_n + p_{n-1}q_{n-2}r_n + p_{n-2}q_{n-1}a_n + p_{n-2}q_{n-2} \\ &\qquad - p_{n-1}q_{n-1}a_nr_n - p_{n-1}q_{n-2}a_n - p_{n-2}q_{n-1}r_n - p_{n-2}q_{n-2} \\ &\quad = (p_{n-1}q_{n-2} - q_{n-1}p_{n-2})(r_n - a_n). \end{aligned}$$

Hence from (20) we get (19).

(21) For irrational $\alpha > 1$ we have

$$\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots].$$

In fact, note from (4) that $p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = (-1)^{n-1}$, while by definition we have $r(\alpha, n) - a(\alpha, n) = s(\alpha, n+1) < 1$. Hence by (19),

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})} < \frac{1}{q_{n-2}^2},$$

and hence (21) follows from (11).

Now for any irrational $\alpha > 1$, define

$$f(\alpha) = \langle a(\alpha, 0), a(\alpha, 1), \ldots \rangle.$$

Then by the above results, f is a one-to-one function mapping the set $\mathcal{N}$ of irrationals $>$ 1 onto the set ${}^{\omega}(\omega\backslash 1)$. The latter set is clearly homeomorphic to ${}^{\omega}\omega$.

(22) The set of irrationals > 1 is homeomorphic to the entire set of irrationals.

To see this, define g by setting, for each irrational $x > 1$,

$$g(x) = \begin{cases} x + m & \text{if } 0 < m < x < m+1 \text{ with } m \in \omega, \\ x + 3m + 1 & \text{if } -m < x < -m+1 \text{ with } m \in \omega. \end{cases}$$

Then g maps $(m, m+1)_{\text{irr}}$ one-one onto $(2m, 2m+1)_{\text{irr}}$ for each positive integer m, and $(-m, -m+1)_{\text{irr}}$ one-one onto $(2m+1, 2m+2)_{\text{irr}}$ for each $m \in \omega$. Clearly g is the desired homeomorphism.

Thus to finish the proof of Theorem 1 it suffices to show that f, defined above, is a homeomorphism. To do this, we need the following fact.

(23) Suppose that $a_0, \ldots, a_n, b_0, \ldots, b_{n-1}$ are positive integers and r is a real number > 1. Assume that

$$\begin{array}{ll} [a_0, \ldots, a_{n-1}] < [b_0, \ldots, b_{n-1}, r] < [a_0, \ldots, a_n] & \text{if } n \text{ is odd} \\ [a_0, \ldots, a_{n-1}] > [b_0, \ldots, b_{n-1}, r] > [a_0, \ldots, a_n] & \text{if } n \text{ is even} \end{array}$$

Then $a_i = b_i$ for all $i < n$. Cf here (2), (8), (9), (10).

We prove (23) by induction on n. For $n = 1$ the assumption is that $a_0 < b_0 + \frac{1}{r} < a_0 + \frac{1}{a_1}$. So clearly $a_0 = b_0$. Now assume (23) for an odd n; we prove it for $n+1$ and $n+2$. So, first suppose that

$$[a_0, \ldots, a_n] > [b_0, \ldots, b_n, r] > [a_0, \ldots, a_{n+1}].$$

Thus

$$a_0 + \frac{1}{[a_1, \ldots, a_n]} > b_0 + \frac{1}{[b_1, \ldots, b_n, r]} > a_0 + \frac{1}{[a_1, \ldots, a_{n+1}]},$$

and it follows that $a_0 = b_0$ and

$$[a_1, \ldots, a_n] < [b_1, \ldots, b_n, r] < [a_1, \ldots, a_{n+1}];$$

then the inductive hypothesis yields $a_i = b_i$ for all $i = 1, \ldots, n$, which proves our statement for $n+1$.

The inductive step to $n+2$ is clearly similar. So (23) holds.

Now to show that f is continuous, suppose that $s \in {}^n(\omega\backslash 1)$; we want to show that $f^{-1}[O(s)]$ is open. We may assume that $n = 2m+1$ for some natural number m. Let $\alpha \in f^{-1}[O_s]$. Define $a_i = a(\alpha, i)$ for all i. Thus $a_0 = s_0, \ldots, a_{2m} = s_{2m}$. By (2) and (8)–(10) we have $[a_0, \ldots, a_{2m}] < \alpha < [a_0, \ldots, a_{2m+1}]$. Choose ε so that $[a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}]$. We claim:

(24) For every irrational $\beta > 1$, if $|\alpha - \beta| < \varepsilon$, then $\beta \in f^{-1}[O(s)]$.

This will prove continuity of f. To prove (24), assume its hypothesis, and let $b_i = b(\beta, i)$ for all i.

Case 1. $\beta < \alpha$. Thus $\alpha - \beta < \varepsilon$. Hence $[a_0, \ldots, a_{2m}] < [a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \beta + \varepsilon$, so $[a_0, \ldots, a_{2m}] < \beta$. If $[a_0, \ldots, a_{2m+1}] \leq \beta$, then by (8)–(10), $\alpha < \beta$, contradiction. So $\beta < [a_0, \ldots, a_{2m+1}]$. Now $\beta = [b_0, \ldots, b_{2m}, r_{2m+1}]$ by (13), so by (23), $a_i = b_i$ for all $i \leq 2m$, as desired.

Case 2. $\alpha < \beta$. Thus $\beta - \alpha < \varepsilon$, so $\beta < \alpha + \varepsilon$. Hence

$$[a_0, \ldots, a_{2m}] < \alpha < \beta < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}],$$

and the argument is finished as in Case 1.

So (24) holds, and f is continuous.

(25) f is an open mapping.

For, suppose that $\alpha > 1$ is irrational, and ε is a positive real number; we want to show that $f[S_\varepsilon(\alpha)]$ is open. Let $b \in f[S_\varepsilon(\alpha)]$; we want to find a finite sequence s such that $b \in O(s) \subseteq f[S_\varepsilon(\alpha)]$. Say $b = f(\beta)$ with $\beta \in S_\varepsilon(\alpha)$. So $|\alpha - \beta| < \varepsilon$. Choose m such that

$$\frac{1}{q(b, 2m)q(b, 2m+1)} < \varepsilon - |\alpha - \beta|.$$

This is possible by (11). Let $s = \langle b_0, \ldots, b_{2m+1}\rangle$. So $b \in O(s)$. Now suppose that $c \in O(s)$. Then

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < [c] < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Also,

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < \beta < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Now

$$\begin{aligned}[b_0, \ldots, b_{2m+1}] - [b_0, \ldots, b_{2m}] &= \frac{p(b, 2m+1)}{q(b, 2m+1)} - \frac{p(b, 2m)}{q(b, 2m)} \quad \text{by (2)} \\ &= \frac{1}{q(b, 2m)q(b, 2m+1)} \\ &< \varepsilon - |\alpha - \beta|.\end{aligned}$$

Hence

$$|[c] - \alpha| \leq |[c] - \beta| + |\beta - \alpha| < \varepsilon,$$

and so $c = f([c]) \in f[S_\varepsilon(\alpha)]$, as desired.

The basic ideals

Let I be an ideal on a set A.

$$\begin{aligned}
\text{add}(I) &= \min\left\{\kappa : \exists E \in [I]^\kappa \left[\bigcup E \notin I\right]\right\};\\
\text{cov}(I) &= \min\left\{\kappa : \exists E \in [I]^\kappa \left[A = \bigcup E\right]\right\};\\
\text{non}(I) &= \min\{\kappa : \exists X \in [A]^\kappa [X \notin I]\}\\
\text{cof}(I) &= \min\{\kappa : \exists X \in [I]^\kappa \forall A \in I \exists B \in X[A \subseteq B]\}.
\end{aligned}$$

Two notions of meager

Let Z be any topological space. A set $X \subseteq Z$ is *meager*$_1$ iff it is a countable union of nowhere dense sets; $X \subseteq Z$ is *meager*$_2$ iff it is a countable union of closed nowhere dense sets. Every closed nowhere dense set is nowhere dense, so meager$_2 \subseteq$ meager$_1$. If A is meager$_1$, say $A = \bigcup_{n\in\omega} B_n$ with each B_n nowhere dense, and let $A^* = \bigcup_{n\in\omega} \overline{B}_n$. So $A^* \in$ meager$_2$.

Lemma 2. add(meager$_2$) = add(meager$_1$).

Proof. First let $\kappa =$ add(meager$_1$), and let $X \subseteq$ meager$_1$ be such that $|X| = \kappa$ and $\bigcup X \notin$ meager$_1$. Then $\forall A \in X[A \subseteq A^* \in$ meager$_2]$. Suppose that $\bigcup_{A\in X} A^* \subseteq C \in$ meager$_2$. Then $\bigcup X \subseteq C \in$ meager$_1$, contradiction. It follows that add(meager$_2$) $\leq \kappa$.

Second, let $\kappa =$ add(meager$_2$), and let $X \subseteq$ meager$_2$ be such that $|X| = \kappa$ and $\bigcup X \notin$ meager$_2$. Then $X \subseteq$ meager$_1$. Suppose that $\bigcup X \in$ meager$_1$. Then $\bigcup X \subseteq (\bigcup X)^* \in$ meager$_2$, contradiction. Hence $\bigcup X \notin$ meager$_1$. It follows that add(meager$_1$) $\leq \kappa$. □

Lemma 3. cov(meager$_1$) = cov(meager$_2$).

Proof. First let $\kappa =$ cov(meager$_1$), and let $X \in [\text{meager}_1]^\kappa$ be such that $Z = \bigcup X$. Then $Z = \bigcup X \subseteq \bigcup_{A\in X} A^*$; so $Z = \bigcup_{A\in X} A^*$. We have $|\{A^* : A \in X\}| \leq \kappa$, so cov(meager$_2$) $\leq$ cov(meager$_1$).

Second let $\kappa =$ cov(meager$_2$), and let $X \in [\text{meager}_2]^\kappa$ be such that $Z = \bigcup X$. since $X \in [\text{meager}_1]^\kappa$, it follows that cov(meager$_1$) $\leq$ cov(meager$_2$).

Lemma 4. non(meager$_1$) = non(meager$_2$).

Proof. First let $\kappa =$ non(meager$_1$), and let $X \in [Z]^\kappa$ be such that $X \notin$ meager$_1$. Then $x \notin$ meager$_2$, so non(meager$_2$) $\leq$ non(meager$_1$).

Second let $\kappa =$ non(meager$_2$), and let $X \in [Z]^\kappa$ be such that $X \notin$ meager$_2$. Suppose that $X \in$ meager$_1$. Then $X^* \in$ meager$_2$ and $X \subseteq X^*$, contradiction. Hence non(meager$_1$) $\leq$ non(meager$_2$). □

Lemma 5. $\mathrm{cof}(\mathrm{meager}_1) = \mathrm{cof}(\mathrm{meager}_2)$.

Proof. First let $\kappa = \mathrm{cof}(\mathrm{meager}_1)$, and let $X \in [\mathrm{meag}_1]^\kappa$ be such that $\forall A \in \mathrm{meag}_1 \exists B \in X[A \subseteq B]$. Then for all $A \in \mathrm{meag}_2$ there is a $B \in X$ such that $A \subseteq B$, hence $A \subseteq B^*$. So $\{B^* : B \in X\}$ has size $\leq \kappa$, and for all $A \in \mathrm{meag}_2$ there is a $C \in \{B^* : B \in X\}$ such that $A \subseteq C$. So $\mathrm{cof}(\mathrm{meager}_2) \leq \kappa$.

Second let $\kappa = \mathrm{cof}(\mathrm{meager}_2)$, and let $X \in [\mathrm{meag}_2]^\kappa$ be such that $\forall A \in \mathrm{meag}_2 \exists B \in X[A \subseteq B]$. Then $X \in [\mathrm{meag}_1]^\kappa$ and for all $A \in \mathrm{meag}_1$ we have $A^* \in \mathrm{meag}_2$ and hence there is a $B \in X$ such that $A \subseteq A^* \subseteq B$. So $\mathrm{cof}(\mathrm{meager}_1) \leq \kappa$. $\square$

Meager for $\mathbb{R}$ and $(0,1)$.

Proposition 6. $\mathbb{R}$ *is homeomorphic to* $(0,1)$.

Proof. For each $x \in (0,1)$ let $f(x) = -\frac{1}{x} + \frac{1}{1-x}$. Then if $x < y$ we have

$$1 < \frac{y}{x}; \quad \frac{1}{y} < \frac{1}{x} \quad -\frac{1}{x} < -\frac{1}{y};$$

$$-y < -x; \quad 1-y < 1-x; \quad 1 < \frac{1-x}{1-y}; \quad \frac{1}{1-x} < \frac{1}{1-y};$$

$$f(x) < f(y).$$

In particular, f is one-one. Also, $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = \infty$. So the proposition follows. $\square$

Proposition 7. *(i)* $\mathrm{add}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{add}(\mathrm{meager}_{(0,1)})$*;*
(ii) $\mathrm{cov}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{cov}(\mathrm{meager}_{(0,1)})$*;*
(iii) $\mathrm{non}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{non}(\mathrm{meager}_{(0,1)})$*;*
(iv) $\mathrm{cof}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{cof}(\mathrm{meager}_{(0,1)})$. $\square$

Meager for $(0,1)$ and $[0,1]$

Proposition 8. *If* $A \subseteq (0,1)$ *is nowhere dense in* $(0,1)$*, then* A *is nowhere dense in* $[0,1]$.

Proof. Suppose that $A \subseteq (0,1)$ is nowhere dense in $(0,1)$. Take any $a < b$ with $(a,b) \cap [0,1] \neq \emptyset$; we want to show that $(a,b) \cap [0,1] \backslash A \neq \emptyset$. Clearly $(a,b) \cap (0,1) \neq \emptyset$, so $(a,b) \cap (0,1) \backslash A \neq \emptyset$. So $(a,b) \cap [0,1] \backslash A \neq \emptyset$. $\square$

Corollary 9. *If* $A \subseteq (0,1)$ *is meager in* $(0,1)$*, then* A *is meager in* $[0,1]$. $\square$

Proposition 10. *If* $A \subseteq [0,1]$ *is nowhere dense in* $[0,1]$*, then* $A \cap (0,1)$ *is nowhere dense in* $(0,1)$.

Proof. Suppose that $A \subseteq [0,1]$ is nowhere dense in $[0,1]$. Take any $a < b$ with $(a,b) \cap (0,1) \neq \emptyset$; we want to show that $(a,b) \cap (0,1) \backslash (A \cap (0,1)) \neq \emptyset$. Clearly $(a,b) \cap [0,1] \neq \emptyset$, so $(a,b) \cap [0,1] \backslash A \neq \emptyset$. So $(a,b) \cap (0,1) \backslash (A \cap (0,1)) \neq \emptyset$. $\square$

Corollary 11. *If* $A \subseteq [0,1]$ *is meager in* $[0,1]$*, then* $A \cap (0,1)$ *is meager in* $(0,1)$. $\square$

Proposition 12. *(i)* $\mathrm{add}(\mathrm{meager}_{[0,1]}) = \mathrm{add}(\mathrm{meager}_{(0,1)})$*;*

(ii) $\mathrm{cov}(\mathrm{meager}_{[0,1]}) = \mathrm{cov}(\mathrm{meager}_{(0,1)})$*;*

(iii) $\mathrm{non}(\mathrm{meager}_{[0,1]}) = \mathrm{non}(\mathrm{meager}_{(0,1)})$*;*

(iv) $\mathrm{cof}(\mathrm{meager}_{[0,1]}) = \mathrm{cof}(\mathrm{meager}_{(0,1)})$.

Proof. (i): First let $\kappa = \mathrm{add}(\mathrm{meager}_{[0,1]})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{[0,1]})]^\kappa$ with $\bigcup E \notin \mathrm{add}(\mathrm{meager}_{[0,1]})$. Then by Corollary 11, $E' = \{A \cap (0,1) : A \in E\} \subseteq \mathscr{P}(\mathrm{meager}_{(0,1)})$. If $\bigcup E' \in \mathrm{add}(\mathrm{meager}_{(0,1)})$, then clearly

$$\bigcup E \subseteq \bigcup E' \cup \{0,1\} \in \mathrm{add}(\mathrm{meager}_{[0,1]}),$$

contradiction.

Second let $\kappa = \mathrm{add}(\mathrm{meager}_{(0,1)})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{(0,1)})]^\kappa$ with $\bigcup E \notin \mathrm{add}(\mathrm{meager}_{(0,1)})$. Then by Corollary 9, $E \subseteq \mathscr{P}(\mathrm{meager}_{[0,1]})$. If $\bigcup E \in \mathrm{add}(\mathrm{meager}_{[0,1]})$, then by Corollary 11, $\bigcup E = (\bigcup E) \cap (0,1) \in \mathrm{add}(\mathrm{meager}_{(0,1)})$, contradiction.

(ii): First let $\kappa = \mathrm{cov}(\mathrm{meager}_{[0,1]})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{[0,1]})]^\kappa$ with $[0,1] = \bigcup E$. Then by Corollary 11, $E' = \{A \cap (0,1) : A \in E\} \subseteq \mathscr{P}(\mathrm{meager}_{(0,1)})$. Hence $(0,1) = \bigcup E'$.

Second let $\kappa = \mathrm{cov}(\mathrm{meager}_{(0,1)})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{(0,1)})]^\kappa$ with $(0,1) = \bigcup E$. Then by Corollary 9, $E \subseteq \mathscr{P}(\mathrm{meager}_{[0,1]})$. Now $[0,1] = \bigcup E \cup \{0,1\}$.

(iii): First let $\kappa = \mathrm{non}(\mathrm{meager}_{[0,1]})$ and $X \in [[0,1]]^\kappa$ with $X \notin \mathrm{non}(\mathrm{meager}_{[0,1]})$. Then $X\backslash\{0,1\} \in [(0,1)]^\kappa$ and $X\backslash\{0,1\} \notin \mathrm{non}(\mathrm{meager}_{(0,1)})$ by Corollary 9.

Second let $\kappa = \mathrm{non}(\mathrm{meager}_{(0,1)})$ and $X \in [(0,1)]^\kappa$ with $X \notin \mathrm{non}(\mathrm{meager}_{(0,1)})$. Then $X \in [(0,1)]^\kappa$ with $X \notin \mathrm{non}(\mathrm{meager}_{[0,1]})$ by Corollary 11.

(iv): First let $\kappa = \mathrm{cof}(\mathrm{meager}_{[0,1]})$ and $X \in [\mathrm{cof}(\mathrm{meager}_{[0,1]})]^\kappa$ such that $\forall A \in \mathrm{cof}(\mathrm{meager}_{[0,1]})\exists B \in X[A \subseteq B]$. Let $X' = \{B \cap (0,1) : B \in X\}$. Then

$$X' \in \mathscr{P}(\mathrm{cof}(\mathrm{meager}_{(0,1)}))$$

by Corollary 11. Suppose that $A \in \mathrm{cof}(\mathrm{meager}_{(0,1)})$. Then by Corollary 9,

$$A \in \mathrm{cof}(\mathrm{meager}_{[0,1]}).$$

So there is a $B \in X$ such that $A \subseteq B$. Hence $A \subseteq B \cap (0,1)$.

Second let $\kappa = \mathrm{non}(\mathrm{meager}_{(0,1)})$ and $X \in [\mathrm{cof}(\mathrm{meager}_{(0,1)})]^\kappa$ such that

$$\forall A \in \mathrm{cof}(\mathrm{meager}_{(0,1)})\exists B \in X[A \subseteq B].$$

Let $X' = \{B \cup \{0,1\} : B \in X\}$. Clearly $X' \in [\mathrm{cof}(\mathrm{meager}_{[0,1]})]^\kappa$. Suppose that $A \in \mathrm{cof}(\mathrm{meager}_{[0,1]})$. Then $A \cap (0,1) \in \mathrm{cof}(\mathrm{meager}_{(0,1)})$ by Corollary 11. Choose $B \in X$ such that $A \cap (0,1) \subseteq B$. Then $A \subseteq B \cup \{0,1\}$. □

Meager for irrat and $\mathbb{R}$

Lemma 13. *Suppose that $S \subseteq \mathbb{R}$. If $A \subseteq S$, then A is closed nowhere dense in S iff A is closed in S and $\forall a,b \in \mathbb{R}[a < b \to \exists c,d \in R[c < d$ and $(a,b)_S \cap (c,d)_S \neq \emptyset$ and $(c,d)_S \cap A = \emptyset$.*

Proof. $\Rightarrow$: Suppose that A is closed nowhere dense in S, $a, b \in \mathbb{R}$, and $a < b$. Then $(a,b)_S \backslash A \neq \emptyset$. Since $S \backslash A$ is open, there exist c, d as indicated.

$\Leftarrow$: Assume the indicated condition. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Choose c, d as indicated. Then $(a,b)_S \backslash A \neq \emptyset$. So A is closed nowhere dense in S.

Corollary 14. *If $S \subseteq \mathbb{R}$ and A is closed nowhere dense in S, then $\overline{A}$ is closed nowhere dense in $\mathbb{R}$.* □

Corollary 15. *If A is closed nowhere dense in $\mathbb{R}$, then $A \cap \mathrm{irrat}$ is closed nowhere dense in irrat.* □

Corollary 16. *If A is $\mathrm{meager}_{\mathbb{R}}$, then $A \cap \mathrm{irrat}$ is meager in irrat.* □

Corollary 17. *Let $S \subseteq \mathbb{R}$. Suppose that X is meager_S, say $X = \bigcup_{n\in\omega} B_n$, with each B_n closed nowhere dense in S. Let $X^* = \bigcup_{n\in\omega} \overline{B}_n$. Then X^* is $\mathrm{meager}_{\mathbb{R}}$.* □

Lemma 18. $\mathrm{add}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{add}(\mathrm{meager}_{\mathrm{irrat}})$.

Proof. First let $\kappa = \mathrm{add}(\mathrm{meager}_{\mathrm{irrat}})$, and let $E \subseteq \mathrm{meag}_{\mathrm{irrat}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \mathrm{meag}_{\mathrm{irrat}}$. For each $A \in E$ let $A = \bigcup_{n\in\omega} B_{nA}$ with each B_{nA} closed nowhere dense in irrat. Let $A^* = \bigcup_{n\in\omega} \overline{B}_{nA}$. So by Corollary 17, A^* is $\mathrm{meager}_{\mathbb{R}}$. Suppose that $\bigcup_{A\in E} A^* \subseteq C$ with C $\mathrm{meager}_{\mathbb{R}}$. Then $\bigcup E = \left(\bigcup_{A\in E} A^*\right) \cap \mathrm{irrat} \subseteq C \cap \mathrm{irrat}$, and by Corollary 16, $C \cap \mathrm{irrat}$ is meager in irrat, contradiction.

Seconnd, let $\kappa = \mathrm{add}(\mathrm{meager}_{\mathbb{R}})$, and let $E \subseteq \mathrm{meager}_{\mathbb{R}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \mathrm{meager}_{\mathbb{R}}$. Since $\mathrm{rat} \in \mathrm{meager}_{\mathbb{R}}$, we also have $((\bigcup E) \cap \mathrm{irrat}) \notin \mathrm{meager}_{\mathbb{R}}$. Now $\forall A \in E[A \cap \mathrm{irrat} \in \mathrm{meager}_{\mathrm{irrat}}]$ by Corollary 16. Suppose that $\bigcup_{A\in E}(A \cap \mathrm{irrat}) \subseteq C \in \mathrm{meager}_{\mathrm{irrat}}$. Say $C = \bigcup_{n\in\omega} B_n$ with each B_n nowhere dense in irrat. Let $C' = \bigcup_{n\in\omega} \overline{B}_n$. By Corollary 17, $C' \in \mathrm{meager}_{\mathbb{R}}$. Now

$$\left(\left(\bigcup E\right) \cap \mathrm{irrat}\right) = \bigcap_{A\in E} (A \cap \mathrm{irrat}) \subseteq C \subseteq C' \in \mathrm{meager}_{\mathbb{R}},$$

contradiction. □

Lemma 19. $\mathrm{cov}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{cov}(\mathrm{meager}_{\mathrm{irrat}})$.

Proof. First let $\kappa = \mathrm{cov}(\mathrm{meager}_{\mathbb{R}})$, and let $E \in [\mathrm{meager}_{\mathbb{R}}]^\kappa$ be such that $\mathbb{R} = \bigcup E$. For each $A \in E$ we have $A \cap \mathrm{irrat} \in \mathrm{meager}_{\mathrm{irrat}}$ by Corollary 16. Moreover, $\bigcup_{A\in E}(A \cap \mathrm{irrat}) = \mathrm{irrat}$. So $\mathrm{cov}(\mathrm{meager}_{\mathrm{irrat}}) \leq \kappa$.

Second let $\kappa = \mathrm{cov}(\mathrm{meager}_{\mathrm{irrat}})$, and let $E \in [\mathrm{meager}_{\mathrm{irrat}}]^\kappa$ be such that $\mathrm{irrat} = \bigcup E$. For each $A \in E$ write $A = \bigcup_{n\in\omega} B_{nA}$ with each B_{nA} closed nowhere dense in irrat. Let $A^* = \bigcup_{n\in\omega} \overline{B}_{nA}$. Then by Corollary 17, A^* is $\mathrm{meager}_{\mathbb{R}}$. We have $\mathrm{irrat} = \bigcup E \subseteq \bigcup_{A\in E} A^*$, so $\mathbb{R} = \bigcup_{A\in E} A^* \cup \mathrm{rat}$, so $\mathrm{cov}(\mathrm{meager}_{\mathbb{R}}) \leq \kappa$. □

Lemma 20. $\mathrm{non}(\mathrm{meager}_{\mathbb{R}}) = \mathrm{non}(\mathrm{meager}_{\mathrm{irrat}})$.

Proof. First let $\kappa = \text{cov}(\text{meager}_{\mathbb{R}})$, and let $X \in [\mathbb{R}]^\kappa$ such that $X \notin \text{meager}_{\mathbb{R}}$. Now rat $\in \text{meager}_{\mathbb{R}}$, so $X \cap \text{rat} \in \text{meager}_{\mathbb{R}}$. Hence $X \cap \text{irrat} \notin \text{meager}_{\mathbb{R}}$. Suppose that $X \cap \text{irrat} \in \text{meager}_{\text{irrat}}$. Say $X \cap \text{irrat} = \bigcup_{n\in\omega} B_n$ with each B_n closed and nowhere dense in irrat. Let $X^* = \bigcup_{n\in\omega} \overline{B}_n$. Then $X^* \in \text{meager}_{\mathbb{R}}$ by Corollary 17. Now $X \cap \text{irrat} \subseteq X^*$, so $X \cap \text{irrat} \in \text{meager}_{\mathbb{R}}$, contradiction.

Second let $\kappa = \text{cov}(\text{meager}_{\text{irrat}})$, and let $X \in [\text{irrat}]^\kappa$ such that $X \notin \text{meager}_{\text{irrat}}$. By Corollary 16, $X \notin \text{meager}_{\mathbb{R}}$. Hence $\text{non}(\text{meager}_{\mathbb{R}}) \leq \kappa$. □

Lemma 21. $\text{cof}(\text{meager}_{\mathbb{R}}) = \text{cof}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_{\mathbb{R}})$, and let $X \in [\text{meag}_{\mathbb{R}}]^\kappa$ be such that $\forall A \in \text{meag}_{\mathbb{R}} \exists B \in X[A \subseteq B]$. Given $A \in \text{meag}_{\text{irrat}}$, form A^* as in Corollary 17. So $A^* \in \text{meag}_{\mathbb{R}}$. Choose $B \in X$ such that $A^* \subseteq B$. By Corollary 8, $B \cap \text{irrat} \in \text{meager}_{\text{irrat}}$. We have $A \subseteq A^* \cap \text{irrat} \subseteq B \cap \text{irrat}$. So $X' \stackrel{\text{def}}{=} \{B \cap \text{irrat} : B \in X$ satisfies $\forall A \in \text{meag}_{\text{irrat}} \exists B' \in X'[A \subseteq B]$. Hence $\text{cof}(\text{meager}_{\text{irrat}}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{\text{irrat}})$, with $X \in [\text{meag}_{\text{irrat}}]^\kappa$ so that $\forall A \in \text{meag}_{\text{irrat}} \exists B \in X[A \subseteq B]$. For each $B \in X$ form B^* as in Corollary 17, and let $X' = \{B^* \cup \text{rat} : B \in X\}$. Note that for each $B \in X$, $B^* \cup \text{rat} \in \text{meager}_{\mathbb{R}}$. Suppose that $A \in \text{meager}_{\mathbb{R}}$. Then $A \cap \text{irrat} \in \text{meager}_{\text{irrat}}$ by Corollary 8, so $A \cap \text{irrat} \subseteq B \in X$ for some B. Hence $A \subseteq B^* \cup \text{rat}$. □

The Cantor set and ${}^\omega 2$.

Let

$$C = \left\{ x \in [0,1] : \exists t \in {}^{\omega\backslash 1}\{0,2\} \left[x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \right] \right\}.$$

For $a < b$ let

$$f([a,b]) = \left\{ \left[a, a + \frac{1}{3}(b-a) \right], \left[a + \frac{2}{3}(b-a), b \right] \right\}.$$

Define A with domain ω recursively by

$$A_0 = \{[0,1]\};$$
$$A_{n+1} = \bigcup_{X \in A_n} f(X).$$

Lemma 22. *For every positive integer n and every set Y, $Y \in A_n$ iff there is a $t : (n+1)\backslash 1 \to \{0,2\}$ such that*

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Proof. For $n = 1$ we have $A_1 = f([0,1]) = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$. With $t_1 = 0$ we have $[\sum_{i=1}^n \frac{t_i}{3^i}, \sum_{i=1}^n \frac{t_i}{3^i} + \frac{1}{3^n}] = [0, \frac{1}{3}]$, and with $t_1 = 2$ we have $[\sum_{i=1}^n \frac{t_i}{3^i}, \sum_{i=1}^n \frac{t_i}{3^i} + \frac{1}{3^n}] = [\frac{2}{3}, 1]$, as desired.

Now assume the equivalence for $n \geq 1$. First suppose that $Y \in A_{n+1}$. Then there is an $X \in A_n$ such that $Y \in f(X)$. By the inductive hypothesis choose $t : (n+1)\backslash 1 \to \{0,2\}$ such that

$$X = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Note that X has size $\frac{1}{3^n}$.

Case 1.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right].$$

Let $s \restriction ((n+1)\backslash 1) = t \restriction ((n+1)\backslash 1)$ and $s(n+1) = 0$. Then

$$(*) \qquad Y = \left[\sum_{i=1}^{n+1} \frac{s_i}{3^i}, \sum_{i=1}^{n+1} \frac{s_i}{3^i} + \frac{1}{3^{n+1}}\right].$$

Case 2.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Let $s \restriction ((n+1)\backslash 1) = t \restriction ((n+1)\backslash 1)$ and $s(n+1) = 2$. Then $(*)$ holds.

Second, suppose that $(*)$ holds. Let $t = s \restriction ((n+1)\backslash 1)$. If $s(n+1) = 0$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right];$$

If $s(n+1) = 1$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Hence in either case,

$$Y \in f\left(\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right]\right),$$

and

$$\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right] \in A_n$$

by the inductive hypothesis. Hence $Y \in A_{n+1}$. □

Theorem 23. $C = \bigcap_{m\in\omega} (\bigcup A_n)$.

Proof. Suppose that $x \in C$ and $n \in \omega$. Choose $s \in {}^{\omega\backslash 1}\{0,2\}$ such that $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$. Let $t = s \restriction ((n+1)\backslash 1)$. Then

$$x \in \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right]$$

since $\sum_{i=n+1}^{\infty}\frac{t_i}{3^i}\le\frac{1}{3^n}$. Thus by Lemma 1, $x\in\bigcup A_n$.

Now suppose that $x\notin C$. If $x\notin[0,1]$, clearly $x\notin\bigcap_{m\in\omega}(\bigcup A_n)$. So suppose that $x\in[0,1]$, and choose $t\in{}^{\omega\setminus 1}3$ such that $x=\sum_{i=1}^{\infty}\frac{t_i}{3^i}$. Choose n minimal such that $t_n=1$. Suppose $x\in\bigcup A_n$. By Lemma 14, choose $s\in{}^{(n+1)\setminus 1}\{0,1\}$ such that

$$\sum_{i=1}^{n}\frac{s_i}{3^i}\le x\le\sum_{i=1}^{n}\frac{s_i}{3^i}+\frac{1}{3^n}.$$

We claim that $t\restriction n=s\restriction n$. Otherwise there is a least $m<n$ such that $t_m\ne s_m$. If $t_m<s_m$, then $t_m=0$ since $t_m\in\{0,2\}$ because $m<n$. Hence

$$x=\sum_{i=1}^{\infty}\frac{t_i}{3^i}=\sum_{i=1}^{m}\frac{t_i}{3^i}+\sum_{i=m+1}^{\infty}\frac{t_i}{3^i}\le\sum_{i=1}^{m}\frac{t_i}{3^i}+\frac{1}{3^m}<\sum_{i=1}^{m}\frac{s_i}{3^i}\le\sum_{i=1}^{n}\frac{s_i}{3^i}\le x,$$

contradiction. If $s_m<t_m$, then $s_m=0$ and $t_m=2$, and

$$\sum_{i=1}^{n}\frac{s_i}{3^i}+\frac{1}{3^n}\le\sum_{i=1}^{m}\frac{s_i}{3^i}+\sum_{i=m+1}^{\infty}\frac{2}{3^i}=\sum_{i=1}^{m}\frac{s_i}{3^i}+\frac{1}{3^m}<\sum_{i=1}^{m}\frac{t_i}{3^i}\le x,$$

contradiction.

So $s\restriction n=t\restriction n$.

Case 1. $s_n=2$. Then

$$x=\sum_{i=1}^{\infty}\frac{t_i}{3^i}=\sum_{i=1}^{m}\frac{t_i}{3^i}+\sum_{i=m+1}^{\infty}\frac{t_i}{3^i}\le\sum_{i=1}^{m}\frac{t_i}{3^i}+\frac{1}{3^m}=\sum_{i=1}^{n}\frac{s_i}{3^i}\le x$$

It follows that $t_i=2$ for all $i\ge m+1$, and hence $x=(t\restriction n)^\frown\langle 2,0,0,0,\ldots\rangle\in C$, contradiction.

Case 2. $s_n=0$. Then

$$x=\sum_{i=1}^{n}\frac{s_i}{3^i}+\frac{1}{3^n}=\sum_{i=1}^{n}\frac{t_i}{3^i};$$

since $x=\sum_{i=1}^{\infty}\frac{t_i}{3^i}$, it follows that $t_i=0$ for all $i>n$; hence $x=(t\restriction n)^\frown\langle 0,2,2,2,\ldots\rangle\in C$, contradiction. □

Theorem 24. *C is homeomorphic to ${}^{\omega}2$.*

Proof. For each $t\in{}^{\omega}2$ let

$$f(t)=\sum_{i=1}^{\infty}\frac{2t_{i-1}}{3^i}.$$

Clearly f maps onto C. It is one-one; for suppose that $s,t\in{}^{\omega}2$ with $s\ne t$. Let n be minimum such that $s_n\ne t_n$. Say $s_n=0$ and $t_n=1$. Then

$$\begin{aligned}f(s)&=\sum_{i=1}^{\infty}\frac{2s_{i-1}}{3^i}=\sum_{i=1}^{n}\frac{2s_{i-1}}{3^i}+\sum_{i=n+1}^{\infty}\frac{2s_{i-1}}{3^i}\\&\le\sum_{i=1}^{n}\frac{2s_{i-1}}{3^i}+\sum_{i=n+1}^{\infty}\frac{2}{3^i}=\sum_{i=1}^{n}\frac{2s_{i-1}}{3^i}+\frac{1}{3^n}<\sum_{i=1}^{n}\frac{2s_{i-1}}{3^i}+\frac{2}{3^n}\le\sum_{i=1}^{\infty}\frac{2t_{i-1}}{3^i}=f(t).\end{aligned}$$

So f is one-one. Now f is continuous. For, suppose that U is open in $\mathbb{R}$ and $t \in f^{-1}[U]$. Say $(f(t)-\varepsilon, f(t)+\varepsilon) \subseteq U$. Choose n so that $3^{-n} < \varepsilon$. Let $V = \{s \in {}^{\omega}2 : s \restriction n = t \restriction n\}$. For any $s \in V$ we have

$$|f(s) - f(t)| = \left| \sum_{n+1 \le i}^{\infty} \frac{2(s(i-1) - t(i-1))}{3^i} \right| \le \sum_{n+1 \le i}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} < \varepsilon.$$

Thus $V \subseteq f^{-1}[U]$. So f is continuous. By Engelking Theorem 3.1.13, f is a homeomorphism. □

Ω and ${}^{\omega}2$.

If $z \in {}^{\omega}2$ let $z^{\circ} = \sum_{i=1}^{\infty}(z_{i-1}2^{-i})$. For $z \in [0,1)$ let $z' \in {}^{\omega}2$ be such that $z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i})$, with z' not eventually 1. Let $\Omega = \{x \in {}^{\omega}2 : x$ is not eventually 1 and $x \neq \langle 0,0,0,\ldots\rangle\}$. For each $m \in \omega$ and each $f \in {}^{m}2$ let $W_f = \{x \in {}^{\omega}2 : f \subseteq x\}$ and $W'_f = \{x \in \Omega : f \subseteq x\}$. Let $M = \{x \in {}^{\omega}2 : x$ is eventually 1 or $x = \langle 0,0,0,\ldots\rangle\}$.

Lemma 25. *If $x, y \in {}^{\omega}2$ and neither x nor y is eventually 1, and if $x \neq y$, then $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \neq \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$.*

Proof. Suppose that $x, y \in {}^{\omega}2$ and neither is eventually 1, and $x \neq y$. Let j be minimum such that $x_j \neq y_j$. Wlog $x_j = 0$ and $y_j = 1$. Choose $k > j$ such that $x_k = 0$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} x_{i-1}2^{-i} \le \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + 2^{-k-1} < \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+1}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{j}(x_{i-1}2^{-i}) + 2^{-k} \le \sum_{i=1}^{\infty}(y_i 2^{-i}), \end{aligned}$$

so $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \neq \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$. □

Lemma 26. *(i) If $x \in {}^{\omega}2$ is not eventually 1, then $x^{\circ\prime} = x$.*
(ii) If $x \in [0,1)$, then $x^{\prime\circ} = x$.

Proof. (i): Suppose that $x \in {}^{\omega}2$ is not eventually 1. Now $x^{\circ\prime} = z'$, where $x^{\circ} = \sum_{i=1}^{\infty}(z'_{i-1}2^{i})$ with z' not eventualy 1; but also $x^{\circ} = \sum_{i=1}^{\infty}(x_{i-1}2^{i})$. So by Lemma 20, $x = z'$.
(ii): obvious. □

Lemma 27. *Ω is dense in ${}^{\omega}2$.*

Proof. Given $m \in \omega$ and $f \in {}^{m}2$, let $x \in W'_f$ be such that x is not eventually 1. □

Lemma 28. *If $X \subseteq {}^\omega 2$ is closed nowhere dense in ${}^\omega 2$, then $X \cap \Omega$ is closed nowhere dense in Ω.*

Proof. Suppose that $X \subseteq {}^\omega 2$ is closed nowhere dense in ${}^\omega 2$. Clearly $X \cap \Omega$ is closed in Ω. Now suppose that W'_f is given. Now ${}^\omega 2 \backslash X$ is dense in ${}^\omega 2$, so $W_f \backslash X \neq \emptyset$. Since ${}^\omega 2 \backslash X$ is open, there is a g with $W_g \cap X = \emptyset$ and $W_f \cap W_g \neq \emptyset$. Choose $x \in W_f \cap W_g \cap \Omega$. Then $x \in \Omega \backslash (\Omega \cap X)$. This shows that $X \cap \Omega$ is closed nowhere dense in Ω. $\square$

Corollary 29. *If $X \subseteq {}^\omega 2$ is meager in ${}^\omega 2$, then $X \cap \Omega$ is meager in Ω.* $\square$

Lemma 30. *If $X \subseteq \Omega$ is nowhere dense in Ω, then X is nowhere dense in ${}^\omega 2$.*

Proof. Suppose that $X \subseteq \Omega$ is nowhere dense in Ω, We want to show that for any $f \in {}^{<\omega} 2$, $W_f \backslash \overline{X} \neq \emptyset$. Choose g so that $W'_f \cap W'_g \neq \emptyset$ and $W'_g \cap X = \emptyset$. This shows that X is nowhere dense in ${}^\omega 2$. $\square$

Corollary 31. *If $X \subseteq \Omega$ is meager in Ω, then X is meager in ${}^\omega 2$.* $\square$

Lemma 32. $\text{add}(\text{meager}_{^\omega 2}) = \text{add}(\text{meager}_\Omega)$.

Proof. First let $\kappa = \text{add}(\text{meager}_{^\omega 2})$ and suppose that $E \in [\text{meager}_{^\omega 2}]^\kappa$ with $\bigcup E \notin \text{meager}_{^\omega 2}$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 11, $E' \subseteq \text{meager}_\Omega$, and $|E'| \leq \kappa$. Suppose that $\bigcup E' \in \text{meager}_\Omega$. Then by Corollary 26, $\bigcup E' \in \text{meager}_{^\omega 2}$. Now M is countable, and $\bigcup E \subseteq \bigcup E' \cup M$, so $\bigcup E \in \text{meager}_{^\omega 2}$, contradiction. Thus $\text{add}(\text{meager}_\Omega) \leq \text{add}(\text{meager}_{^\omega 2})$.

Second let $\kappa = \text{add}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\bigcup E \notin \text{meager}_\Omega$. By Corollary 26, $E \in [\text{meager}_{^\omega 2}]^\kappa$. Suppose that $\bigcup E \in \text{meager}_{^\omega 2}$. By Corollary 24, $\bigcup E = (\bigcup E) \cap \Omega \in \text{meager}_\Omega$, contradiction. $\square$

Lemma 33. $\text{cov}(\text{meager}_{^\omega 2}) = \text{cov}(\text{meager}_\Omega)$.

Proof. First let $\kappa = \text{cov}(\text{meager}_{^\omega 2})$ and suppose that $E \in [\text{meager}_{^\omega 2}]^\kappa$ with $\bigcup E = {}^\omega 2$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 24, $E' \subseteq \mathscr{P}(\text{meager}_\Omega)$, and $|E'| \leq \kappa$. Clearly $\bigcup E' = \Omega$. So $\text{cov}(\text{meager}_\Omega) \leq \kappa$.

Second let $\kappa = \text{cov}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\bigcup E = \Omega$. By Corollary 26, $E \in [\text{meager}^\omega 2]^\kappa$. Then $M \cup \bigcup E = {}^\omega 2$. Hence $\text{cov}(\text{meager}_{^\omega 2}) = \text{cov}(\text{meager}_\Omega)$. $\square$

Lemma 34. $\text{non}(\text{meager}_{^\omega 2}) = \text{non}(\text{meager}_\Omega)$.

Proof. First let $\kappa = \text{non}(\text{meager}_{^\omega 2})$ and suppose that $X \in [{}^\omega 2]^\kappa$ with $X \notin \text{meager}_{^\omega 2}$. Then $|X \cap \Omega| \leq \kappa$. Suppose that $X \cap \Omega \in \text{non}(\text{meager}_\Omega)$. Then by Corollary 26, $X \cap \Omega \in \text{non}(\text{meager}_{^\omega 2})$ so $X \subseteq (X \cap \Omega) \cup M \in \text{meager}_{^\omega 2}$, contradiction.

Second let $\kappa = \text{non}(\text{meager}_\Omega)$ and suppose that $X \in [\Omega]^\kappa$ with $X \notin \text{meager}_\Omega$. Then $X \in [{}^\omega 2]^\kappa$ and by Corollary 24 $X \notin \text{meager}_{^\omega 2}$. $\square$

Lemma 35. $\text{cof}(\text{meager}_{^\omega 2}) = \text{cof}(\text{meager}_\Omega)$.

Proof. First let $\kappa = \mathrm{cof}(\mathrm{meager}_{^\omega 2})$ and suppose that $X \in [\mathrm{meager}_{^\omega 2}]^\kappa$ is such that $\forall A \in \mathrm{meager}_{^\omega 2} \exists B \in X[A \subseteq B]$. Let $X' = \{A \cap \Omega : A \in X\}$. So $|X| \leq \kappa$, and by Corollary 11, $X' \subseteq \mathscr{P}(\mathrm{meager}_\Omega)$. Suppose that $A \in \mathrm{meager}_\Omega$. Then by Corollary 26, $A \in \mathrm{meager}_{^\omega 2}$, so there is a $B \in X$ such that $A \subseteq B$. Then $A \subseteq B \cap \Omega \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_\Omega) \leq \kappa$.

Second let $\kappa = \mathrm{cof}(\mathrm{meager}_\Omega)$ and suppose that $X \in [\mathrm{meager}_\Omega]^\kappa$ is such that $\forall A \in \mathrm{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{A \cup M : A \in X\}$. So $|X'| \leq \kappa$. Suppose that $A \in \mathrm{meager}_{^\omega 2}$. Then by Corollary 24, $A \cap \Omega \in \mathrm{meager}_\Omega$, so there is a $B \in X$ such that $A \cap \Omega \subseteq B$. Then $A \subseteq B \cup M \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_{^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Omega)$. □

$(0,1)$ and Ω

Lemma 36. *If $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$, then*

$$\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) < \sum_{i=1}^{m}(h_{i-1}2^{-i}) + 2^{-m-1}.$$

Proof. Assume that $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$. Choose $n > m$ so that $h_n = 0$. Then

$$\begin{aligned}\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) &\leq \sum_{i=1}^{n+1}(h_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{n+1}(h_{i-1}2^{-i}) + 2^{-n-1}\\ &< \sum_{i=1}^{n+1}(h_{i-1}2^{-i}) + 2^{-n-1} + 2^{-n-2}\\ &\leq \sum_{i=1}^{m}(h_i 2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{m}(h_i 2^{-i}) + 2^{-m-1}.\end{aligned}$$

□

Lemma 37. *If $0 < a < b < 1$, then there is an f such that ${}^\circ W'_f \subseteq (a,b)$.*

Proof. Assume that $0 < a < b < 1$. Let m be minimum such that $a'_m \neq b'_m$. If $a'_m = 1$ and $b'_m = 0$, then

$$\begin{aligned}b = \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) &\leq \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + 2^{-m-1} = \sum_{i=1}^{m+1}(a'_{i-1}2^{-i}) \leq a,\end{aligned}$$

contradiction. Hence $a'_m = 0$ and $b'_m = 1$.

Choose $p > n > m$ such that $a'_n = a'_p = 0$. Let $f = \langle a'_i : i < n\rangle^\frown\langle 1\rangle^\frown\langle a'_i : n+1 \leq i \leq p\rangle$. We claim that ${}^\circ W'_f \subseteq (a,b)$. Take any $g \in W'_f$; we want to show that $a < g^\circ < b$, i.e.,

$$\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(g_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) \tag{1}$$

We have

$$\begin{aligned}
\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) &< \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n-1} \quad \text{by Lemma 31}\\
&= \sum_{i=1}^{n+1}(f_{i-1}2^{-i}) = \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) \leq \sum_{i=1}^{\infty}(g_{i-1}2^{-i})\\
&\leq \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + \sum_{i=p+2}^{\infty}2^{-i} = \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + 2^{-p-1}\\
&< \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i}\\
&= \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + 2^{-n-1} = \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n} + 2^{-n-1}\\
&\leq \sum_{i=1}^{m}b'_{i-1}2^{-i} + \sum_{i=m+1}^{\infty}2^{-i} = \sum_{i=1}^{m+1}b'_{i-1}2^{-i} \leq \sum_{i=1}^{\infty}b'_{i-1}2^{-i}.
\end{aligned}$$

□

Lemma 38. *For any $f \in {}^{<\omega}2$ there exist a, b with $0 < a < b < 1$ and $(a,b) \subseteq {}^\circ W'_f$.*

Proof. Say $f \in {}^m2$. Let $a = f^\frown\langle 1,0,0,0,\ldots\rangle$ and $b = f^\frown\langle 1,1,0,0,\ldots\rangle$. Clearly $0 < a^\circ < b^\circ < 1$. We claim that $(a^\circ, b^\circ) \subseteq {}^\circ W'_f$. Suppose that $a^\circ < z < b^\circ$. In particular, $z \in (0,1)$ and $z' \in \Omega$. If $a = z'$, then $a^\circ = z'^\circ = z$, contradiction. So $a \neq z'$. Similarly $b \neq z'$. Let n be minimum such that $a_n \neq z'_n$. Suppose that $n \leq m$.

Subcase 1. $a_n = 0$, $z'_n = 1$. Then

$$b^\circ = \sum_{i=1}^{\infty}(b_{i-1}2^{-i}) \leq \sum_{i=1}^{n}(b_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i} = \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + 2^{-n-1} \leq \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) = z,$$

contradiction.

Subcase 2. $a_n = 1$, $z'_n = 0$. Then

$$z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) \leq \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i} = \sum_{i=1}^{n}(a_{i-1}2^{-i}) + 2^{-n-1} \leq \sum_{i=1}^{\infty}(a_{i-1}2^{-i}) = a^\circ,$$

contradiction.

It follows that $m \le n$, hence $f \subseteq z'$ and so $z' \in W'_f$. Thus $z = z'^{\circ} \in {}^{\circ}W'_f$. □

Lemma 39. *If $A \subseteq \Omega$ is nowhere dense, then ${}^{\circ}[A]$ is nowhere dense in $(0,1)$.*

Proof. Suppose that $A \subseteq \Omega$ is nowhere dense, U is open, $U \cap (0,1) \neq \emptyset$. We want to show that $U \backslash \overline{{}^{\circ}[A]} \neq \emptyset$. Wlog $U = (a,b)$ with $0 < a < b < 1$. We want to find $0 < c < d < 1$ such that $(c,d) \cap {}^{\circ}[A] = \emptyset$ and $(a,b) \cap (c,d) \neq \emptyset$. By Lemma 32 choose f such that ${}^{\circ}W'_f \subseteq (a,b)$. Now $W'_f \backslash \overline{A} \neq \emptyset$, so there is a g such that $W'_f \cap W'_g \neq \emptyset$ and $W'_g \cap A = \emptyset$. By Lemma 33 choose $c < d$ so that $(c,d) \subseteq {}^{\circ}W'_{f\cup g}$. Then $(c,d) \subseteq {}^{\circ}W_{f\cup g} \subseteq {}^{\circ}W_f \subseteq (a,b)$, so $(a,b) \cap (c,d) \neq \emptyset$. Suppose that $x \in A$ and $x^{\circ} \in (c,d)$. Then $x^{\circ} \in {}^{\circ}W'_{f\cup g}$, so there is a $y \in W'_{f\cup g}$ such that $x^{\circ} = y^{\circ}$. Hence $x = x^{\circ\prime} = y^{\circ\prime} = y$. So $x \in A \cap W'_{f\cup g} \subseteq A \cap W_g$, contradiction. □

Corollary 41. *If $A \subseteq \Omega$ is meager, then ${}^{\circ}[A]$ is meager in $(0,1)$.* □

Lemma 41. *If $A \subseteq (0,1)$ is nowhere dense in $(0,1)$, then ${}'[A]$ is nowhere dense in Ω.*

Proof. Assume that $A \subseteq (0,1)$ is nowhere dense in $(0,1)$ Let W_f be given; we want to show that $W_f \cap \Omega \backslash {}'[A] \neq \emptyset$. By Lemma 33 choose $0 < a < b < 1$ so that $(a,b) \subseteq W'_f$. Choose $0 < c < d < 1$ so that $(a,b) \cap (c,d) \neq \emptyset$ and $(c,d) \cap A = \emptyset$. By Lemma 32 choose g so that ${}^{\circ}W'_g \subseteq (a,b) \cap (c,d)$. Suppose that $x \in W'_{f\cup g} \cap \Omega \cap {}'[A]$. Say $x = y'$ with $y \in A$. Then $x^{\circ} = y'^{\circ} = y$. Hence $y = x^{\circ} \in {}^{\circ}[W'_{f\cup g}] \subseteq {}^{\circ}[W'_g] \subseteq (a,b) \cap (c,d)$ and $y \in A$, contradiction. Thus $W_{f\cup g} \cap \Omega \backslash {}'[A] \neq \emptyset$. □

Corollary 42. *If $A \subseteq (0,1)$ is meager in $(0,1)$. then ${}'[A]$ is meager in Ω.*

Lemma 43. $\mathrm{add}(\mathrm{meager}_{\Omega}) = \mathrm{add}(\mathrm{meager}_{(0,1)})$.

Proof. First let $\kappa = \mathrm{add}(\mathrm{meager}_{\Omega})$ and suppose that $E \in [\mathrm{meager}_{\Omega}]^{\kappa}$ with $\bigcup E \notin \mathrm{meager}_{\Omega}$. Let $E' = \{{}^{\circ}[A] : A \in E\}$. So by Corollary 22, $E' \in \mathscr{P}(\mathrm{meager}_{(0,1)})$. Clearly $|E'| \le \kappa$. Suppose that $\bigcup E' \in \mathrm{meager}_{(0,1)}$. By Corollary 37, ${}'[\bigcup E'] \in \mathrm{meager}_{\Omega}$. Take any $A \in E$. Then ${}^{\circ}[A] \in E'$, so ${}^{\circ}[A] \subseteq \bigcup E'$. Hence ${}'{}^{\circ}[A] \subseteq {}^{\circ}[\bigcup E']$, so by Lemma 21, $A \subseteq {}^{\circ}[\bigcup E']$. Thus $\bigcup E \subseteq {}^{\circ}[\bigcup E']$, so $\bigcup E \in \mathrm{meager}_{\Omega}$, contradiction.

Second let $\kappa = \mathrm{add}(\mathrm{meager}_{(0,1)})$ and suppose that $E \in [\mathrm{meager}_{(0,1)}]^{\kappa}$ with $\bigcup E \notin \mathrm{meager}_{(0,1)}$. Let $E' = \{{}'[A] : A \in E\}$. So by Corollary 37, $E' \subseteq \mathscr{P}(\mathrm{meager}_{\Omega})$. Suppose that $\bigcup E' \in \mathrm{meager}_{\Omega}$. By Corollary 35, ${}^{\circ}[\bigcup E'] \in \mathrm{meager}_{(0,1)}$. If $A \in E$, then ${}'[A] \in E'$, so ${}'[A] \subseteq \bigcup E'$, hence $A = {}^{\circ\prime}[A] \subseteq {}^{\circ}[\bigcup E']$, so $\bigcup E \subseteq {}^{\circ}[\bigcup E']$, contradiction. □

Lemma 44. $\mathrm{cov}(\mathrm{meager}_{\Omega}) = \mathrm{cov}(\mathrm{meager}_{(0,1)})$.

Proof. First let $\kappa = \mathrm{cov}(\mathrm{meager}_{\Omega})$ and suppose that $E \in [\mathrm{meager}_{\Omega}]^{\kappa}$ with $\Omega = \bigcup E$. Let $E' = \{{}^{\circ}[A] : A \in E\}$. Then by Corollary 35, $E' \subseteq \mathscr{P}(\mathrm{meager}_{(0,1)})$. If $a \in (0,1)$, then $a' \in \Omega$, hence there is an $A \in E$ such that $a' \in A$. So $a = a'^{\circ} \in {}^{\circ}[A] \in E'$. Thus $(0,1) \subseteq \bigcup E'$.

Seecond let $\kappa = \mathrm{cov}(\mathrm{meager}_{(0,1)})$ and suppose that $E \in [\mathrm{meager}_{(0,1)}]^{\kappa}$ with $(0,1) = \bigcup E$. Let $E' = \{{}'[A] : A \in E\}$. Then $E' \subseteq \mathscr{P}(\mathrm{meager}_{\Omega})$ by Corollary 37. Suppose that

$x \in \Omega$. Then $x^\circ \in (0,1)$, so there is an $A \in E$ such that $x^\circ \in A$. Hence $x = x^{\circ\prime} \in {}'[A] \in E'$. This shows that $\bigcup E' = \Omega$. □

Lemma 45. $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{non}(\text{meager}_\Omega)$ and suppose that $X \in [\Omega]^\kappa$ such that $X \notin \text{meager}_\Omega$. Suppose that ${}^\circ[X]$ is meager in $(0,1)$. Then $X = {}'{}^\circ [X]$ is meager in Ω by Corollary 37, contradiction. It follows that $\text{non}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and suppose that $X \in [(0,1)]^\kappa$ such that $X \notin \text{meager}_{(0,1)}$. Suppose that ${}'[X] \in \text{meager}_\Omega$. Then $X = {}^{\circ\prime}[X] \in \text{meager}_{(0,1)}$ by Lemma 35, contradiction. Hence $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$. □

Lemma 46. $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_\Omega)$ and suppose that $X \in [\text{meager}_\Omega]^\kappa$ is such that $\forall A \in \text{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{{}^\circ[A] : A \in X\}$. Then by Corollary 35, $X' \subseteq \mathscr{P}(\text{meager}_{(0,1)})$, and clearly $|X| \leq \kappa$. Suppose that $A \in \text{meager}_{(0,1)}$. Then by Corollary 37, ${}'[A] \in \text{meager}_\Omega$. Hence there is a $B \in X$ such that ${}'[A] \subseteq B$. Then $A = {}^{\circ\prime}[A] \subseteq {}^\circ[B] \in X'$. It follows that $\text{cof}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{(0,1)})$ and suppose that $X \in [\text{meager}_{(0,1)}]^\kappa$ is such that $\forall A \in \text{meager}_{(0,1)} \exists B \in X[A \subseteq B]$. Let $X' = \{A] : A \in X\}$. Then by Corollary 37, $X' \subseteq \mathscr{P}(\text{meager}_\Omega)$, and clearly $|X'| \leq \kappa$. Suppose that $A \in \text{meager}_\Omega)$. Then ${}^\circ[A] \in \text{meager}_{(0,1)})$ by Corollary 35, so there is a $B \in X$ such that ${}^\circ[A] \subseteq B$. Then $A = {}'{}^\circ[A] \subseteq {}'[B] \in X'$. It follows that $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$. □

measures

If A is a σ-algebra of subsets of X, then a *measure* on A is a function $\mu : A \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i)$ if $a \in {}^\omega A$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$. Note that $a_i = \emptyset$ is possible for some $i \in \omega$.

We give some important properties of measures:

Proposition 47. *Suppose that μ is a measure on a σ-algebra A of subsets of X. Then:*
(i) If $Y, Z \in A$ and $Y \subseteq Z$, then $\mu(Y) \leq \mu(Z)$.
(ii) If $Y \in {}^\omega A$, then $\mu(\bigcup_{n\in\omega} Y_n) \leq \sum_{n\in\omega} \mu(Y_n)$.
(iii) If $Y \in {}^\omega A$ and $Y_n \subseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcup_{n\in\omega} Y_n) = \sup_{n\in\omega} \mu(Y_n)$.

Proof. (i): We have $\mu(Z) = \mu(Y) + \mu(Z \backslash Y) \geq \mu(Y)$.

(ii): Let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $\bigcup_{m\leq n} Z_m = \bigcup_{m\leq n} Y_m$, and hence $\bigcup_{m\in\omega} Z_m = \bigcup_{m\in\omega} Y_m$. Now

$$\mu\left(\bigcup_{m\in\omega} Y_m\right) = \mu\left(\bigcup_{m\in\omega} Z_m\right) = \sum_{m\in\omega} \mu(Z_m) \leq \sum_{m\in\omega} \mu(Y_m).$$

(iii): Again let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $Y_n = \bigcup_{m\leq n} Z_m$. Hence

$$\mu\left(\bigcup_{n\in\omega} Y_n\right) = \mu\left(\bigcup_{n\in\omega} Z_n\right)$$

$$
\begin{aligned}
&= \sum_{n\in\omega} \mu(Z_n) \\
&= \lim_{n\to\infty} \sum_{m\le n} \mu(Z_m) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m\le n} Z_m\right) \\
&= \lim_{n\to\infty} \mu(Y_n) \\
&= \sup_{n\in\omega} \mu(Y_n). \qquad \square
\end{aligned}
$$

measure spaces and outer measures

A *measure space* is a triple (X, Σ, μ) such that:

(1) X is a set

(2) Σ is a σ-algebra of subsets of X.

(3) μ is a measure on Σ.

Given a measure space as above, a subset A of X is a *μ-null* set iff there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu(E) = 0$.

Theorem 48. *If (X, Σ, μ) is a measure space, then the collection of μ-null sets is a σ-ideal of subsets of X.*

Proof. Let I be the collection of all μ-null sets. Clearly $\emptyset \in I$, and $B \subseteq A \in I$ implies that $B \in I$. Now suppose that $\langle A_i : i \in \omega\rangle$ is a system of members of I. For each $i \in \omega$ choose $E_i \in \Sigma$ such that $A_i \subseteq E_i$ and $\mu(E_i) = 0$. Then $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} E_i$, and

$$\mu\left(\bigcup_{i\in\omega} E_i\right) \le \sum_{i\in\omega} \mu(E_i) = 0. \qquad \square$$

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X) \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \le \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \le \sum_{n\in\omega} \mu(A_n)$.

If θ is an outer measure on a set X, then a subset E of X is *θ-measurable* iff for every $A \subseteq X$,

$$\theta(A) = \theta(A \cap E) + \theta(A \backslash E).$$

Note that every subset $E \subseteq X$ such that $\theta(E) = 0$ is automatically θ-measurable.

Theorem 49. *Let θ be an outer measure on a set X. Let Σ be the collection of all θ-measurable subsets of X. Then $(X, \Sigma, \theta \restriction \Sigma)$ is a measure space.*

Proof. Note that Σ is obviously closed under complementation. Obviously

(1) If $A, E \subseteq X$, then $\theta(A) \leq \theta(A \cap E) + \theta(A \backslash E)$.

Clearly $\emptyset \in \Sigma$ and Σ is closed under complements. Next we show that Σ is closed under $\cup$. Suppose that $E, F \in \Sigma$ and $A \subseteq X$. Then

$$\begin{aligned}
\theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) &= \theta((A \cap (E \cup F) \cap E)) + \theta(A \cap (E \cup F) \backslash E))) \\
&\quad + \theta(A \backslash (E \cup F)) \\
&= \theta(A \cap E) + \theta((A \backslash E) \cap F) + \theta((A \backslash E) \backslash F) \\
&= \theta(A \cap E) + \theta(A \backslash E) \\
&= \theta(A).
\end{aligned}$$

This proves that $E \cup F \in \Sigma$. Thus we have shown that Σ is a field of subsets of X.

Next we show that Σ is closed under countable unions. So, suppose that $E \in {}^{\omega}\Sigma$, and let $K = \bigcup_{n\in\omega} E_n$. For every $m \in \omega$ let

$$G_m = \bigcup_{n \leq m} E_n.$$

Then clearly each G_m is in Σ. Now we define $F_0 = G_0$, and for $m > 0$, $F_m = G_m \backslash G_{m-1}$. Then also each F_m is in Σ. By induction, $\bigcup_{n \leq m} F_n = G_m$. Hence $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} E_n$. Now temporarily fix a positive integer n and an $A \subseteq X$. Then

$$\theta(A \cap G_n) = \theta(A \cap G_n \cap G_{n-1}) + \theta(A \cap G_n \backslash G_{n-1}) = \theta(A \cap G_{n-1}) + \theta(A \cap F_n);$$

hence by induction $\theta(A \cap G_n) = \sum_{m \leq n} \theta(A \cap F_m)$.

Now we unfix n. Now $A \cap K = \bigcup_{n\in\omega}(A \cap F_n)$, so

$$\theta(A \cap K) \leq \sum_{n\in\omega} \theta(A \cap F_n) = \lim_{n\to\infty} \sum_{m \leq n} \theta(A \cap F_m) = \lim_{n\to\infty} \theta(A \cap G_m).$$

Also, note that if $m < n$ then $G_m \subseteq G_n$, hence $X \backslash G_n \subseteq X \backslash G_m$, and so

$$\theta(A \backslash K) = \theta\left(A \backslash \bigcup_{n\in\omega} G_n\right) = \theta\left(\bigcap_{n\in\omega}(A \backslash G_n)\right) \leq \inf_{n\in\omega} \theta(A \backslash G_n) = \lim_{n\to\infty} \theta(A \backslash G_n).$$

Hence

$$\begin{aligned}
\theta(A \cap K) + \theta(A \backslash K) &= \lim_{n\to\infty} \theta(A \cap G_n) + \lim_{n\to\infty} \theta(A \backslash G_n) \\
&= \lim_{n\to\infty} (\theta(A \cap G_n) + \theta(A \backslash G_n)) \\
&= \theta(A) \\
&\leq \theta(A \cap K) + \theta(A \backslash K).
\end{aligned}$$

This proves that $K\in\Sigma$, so that Σ is closed under countable unions.

Finally, suppose that $\langle E_n : n\in\omega\rangle$ is a system of pairwise disjoint members of Σ. Let $K=\bigcup_{n\in\omega}E_n$. By Lemma 2, $\theta(K)\le\sum_{n\in\omega}\theta(E_n)$. Conversely, for each $n\in\omega$ let $G_n=\bigcup_{m\le n}E_m$. Then

$$\theta(G_{n+1})=\theta(G_{n+1}\cap E_{n+1})+\theta(G_{n+1}\backslash E_{n+1})=\theta(E_{n+1})+\theta(G_n).$$

Hence by induction, $\theta(G_n)=\sum_{m\le n}\theta(E_m)$ for every n, and hence

$$\theta(K)\ge\theta(G_n)=\sum_{m\le n}\theta(E_m),$$

and so $\theta(K)\ge\sum_{n\in\omega}\theta(E_n)$. □

measure on ${}^{\kappa}2$

Let κ be an infinite cardinal. For each $f\in\mathrm{Fn}(\kappa,2,\omega)$ let $U_f=\{g\in{}^{\kappa}2 : f\subseteq g\}$. Hence $U_\emptyset={}^{\kappa}2$. Note that the function taking f to U_f is one-one. For each $f\in\mathrm{Fn}(\kappa,2,\omega)$ let $\theta_0(U_f)=1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset)=1$. Let $\mathcal{C}=\{U_f : f\in\mathrm{Fn}(\kappa,2,\omega)\}$. Note that ${}^{\kappa}2\in\mathcal{C}$. For any $A\subseteq{}^{\kappa}2$ let

$$\theta(A)=\inf\left\{\sum_{n\in\omega}\theta_0(C_n) : C\in{}^{\omega}\mathcal{C}\text{ and }A\subseteq\bigcup_{n\in\omega}C_n\right\}.$$

Proposition 60. *θ is an outer measure on ${}^{\kappa}2$.*

Proof. For (1), for any $m\in\omega$ let $f\in\mathrm{Fn}(\kappa,2,\omega)$ have domain of size m. Then $\emptyset\subseteq U_f$ and $\theta_0(U_f)=\frac{1}{m}$. Hence $\theta(\emptyset)=0$.

For (2), if $A\subseteq B\subseteq{}^{\kappa}2$, then

$$\left\{C\in{}^{\omega}\mathcal{C} : B\subseteq\bigcup_{n\in\omega}C_n\right\}\subseteq\left\{C\in{}^{\omega}\mathcal{C} : A\subseteq\bigcup_{n\in\omega}C_n\right\},$$

and hence $\mu(A)\le\mu(B)$.

For (3), assume that $A\in{}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega}\theta(A_n)<\infty$. Let $\varepsilon>0$; we show that $\theta(\bigcup_{n\in\omega}A_n)\le\sum_{n\in\omega}\theta(A_n)+\varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n\in\omega$ choose $C^n\in{}^{\omega}\mathcal{C}$ such that $A_n\subseteq\bigcup_{m\in\omega}C^n_m$ and $\sum_{m\in\omega}\theta_0(C^n_m)\le\theta(A_n)+\frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega}A_n\subseteq\bigcup_{n\in\omega}\bigcup_{m\in\omega}C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega}A_n\right)\le\sum_{n\in\omega}\sum_{n\in\omega}\theta_0(C^n_m)\le\sum_{n\in\omega}\theta(A_n)+\varepsilon,$$

as desired. □

Let Σ_0 be the set of all θ-measurable subsets of ${}^{\omega}2$.

Proposition 61. *If* $\varepsilon \in 2$ *and* $\alpha < \kappa$*, then* $\{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\} \in \Sigma_0$.

Proof. Let $E = \{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^{\kappa}2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\leq$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace C_n by U_g and U_h, where $g = f \cup \{(\alpha, 0)\}$ and $h = f \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^{\omega}\mathcal{C}$. Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Then there is a partition M, N of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X\backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X\backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X\backslash E)$. □

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2}f(0) + \frac{1}{2}f(1)$.

Proposition 62. *If* $f_n : 2 \to [0, \infty)$ *for each* $n \in \omega$ *and* $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$*, then* $\sum_{n\in\omega} \int f_n < \infty$*, and* $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2}\sum_{n\in\omega} f_n(0) + \frac{1}{2}\sum_{n\in\omega} f_n(1) = \sum_{n\in\omega} \left(\frac{1}{2}f_n(0) + \frac{1}{2}f_n(1)\right) = \sum_{n\in\omega} \int f_n.$$

Proposition 63. $\theta({}^{\kappa}2) = 1$.

Proof. It is obvious that ${}^{\kappa}2 \in \Sigma_0$, and that $\theta({}^{\kappa}2) \leq \theta_0({}^{\kappa}2) = 1$. Suppose that $\theta({}^{\kappa}2) < 1$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $2^{\kappa} = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega)\exists n \in \omega[f_n \subseteq g$ or $g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^{\kappa}2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \neq n[f_m \not\subseteq f_n]\}$. Then ${}^{\kappa}2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^{\kappa}2$ choose $n \in \omega$ such that $g \in C_n$. Thus $f_n \subseteq g$. Let $m \in \omega$ with $f_m \subseteq f_n$ and $|\mathrm{dmn}(f_m)|$ minimum. Then $f_m \subseteq g$ and $m \in M$, as desired.

(3) $|M| \geq 2$.

In fact, obviously $M \neq \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \neq \emptyset$. Then ${}^{\kappa}2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m + 1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. Because of (3), this contradicts (2).

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G\subseteq J}\{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \le 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \mathrm{dmn}(f_n) : \forall j < k[\alpha \neq i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\mathrm{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$f_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int f_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int f_{nk} &= \frac{1}{2} f_{nk}(0) + \frac{1}{2} f_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n\in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n\in M_k} \alpha_{nk} &= \sum_{n\in M_k} \int f_{nk} && \text{by (6)} \\ &= \int \sum_{n\in M_k} f_{nk} && \text{by Proposition 6.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n\in M_k} f_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k + 1[i_j \notin \mathrm{dmn}(f_n), \text{ or } i_j \in \mathrm{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $f_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n\in M_{k+1}} \alpha_{n,k+1} = \sum_{n\in M_{k+1}} f_{mk}(t_k) \le \left(\sum_{n\in M_k} f_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^{\kappa}2$ such that $g(i_j) = t_j$ for all $j \leq k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \mathrm{dmn}(f_n)$, or $i_j \in \mathrm{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^{\kappa}2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose $k \in \omega$ so that $\mathrm{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m \in M_k} \alpha_{mk} < 1$. □

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^{\kappa}2$ let $F(f) = \nu \circ f$.

Proposition 64.

(i) F is a permutation of ${}^{\kappa}2$.

(ii) For any $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu \circ f}$.

(iii) For any $X \subseteq {}^{\kappa}2$ we have $\theta(X) = \theta(F[X])$.

(iv) $\forall E \in \Sigma_0[F[E] \in \Sigma_0]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^{\kappa}2$. So (i) holds.

(ii): For any $g \in {}^{\kappa}2$,

$$\begin{aligned} g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\ &\text{iff} \quad \exists h \in {}^{\kappa}2[f \subseteq h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \exists h \in {}^{\kappa}2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \nu \circ f \subseteq g \\ &\text{iff} \quad g \in U_{\nu \circ f} \end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n \in \omega} C_n$ iff $F[A] \subseteq \bigcup_{n \in \omega} F[C_n]$. So (iii) holds.

(iv): Suppose that $E \in \Sigma_0$. Let $X \subseteq {}^{\kappa}2$. Then

$$\begin{aligned} \theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\ &= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\ &= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\ &= \theta(E) = \theta(F[E]). \end{aligned}$$

□

Proposition 65. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 8 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 7. □

Proposition 66. *For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $U_f \in \Sigma_0$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha \in \mathrm{dmn}(f)} U_{\{(\alpha, f(\alpha))\}}$. Note that if $\alpha \in \mathrm{dmn}(f)$, then $U_{\{(\alpha, f(\alpha))\}} = \{g \in {}^{\kappa}2 : g(\alpha) = f(\alpha)\}$; hence $U_{\{(\alpha, f(\alpha))\}} \in \Sigma_0$ by Proposition 5, and so $U_f \in \Sigma_0$. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 9. Now assume that it holds for $|\text{dmn}(f)| = m$. For any f with $|\text{dmn}(f)| = m$ and $\alpha \notin \text{dmn}(f)$ we have $2^{-|\text{dmn}(f)|} = \theta(U_f) = \theta(U_{f\cup\{(\alpha,0)\}}) + \theta(U_{f\cup\{(\alpha,1)\}})$. Since $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) \le \theta_0(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\text{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\text{dmn}(f)|-1}$ for each $\varepsilon \in 2$. □

Proposition 67. *If F is a finite subset of ${}^\kappa 2$, then $F \in \Sigma_0$ and $\theta(F) = 0$.*

Proof. This is obvious if $|F| \le 1$, and then the general case follows. □

measure on $\mathbb{R}$

For any $a, b \in \mathbb{R}$ let $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$. Note that if $a \ge b$, then $[a, b) = \emptyset$. Note that if $[a, b) = [c, d)$, $a < b$, and $c < d$, then $a = c$ and $b = d$. For any $a, b \in \mathbb{R}$ we define

$$\lambda([a,b)) = \begin{cases} 0 & \text{if } a \ge b, \\ b - a & \text{if } a < b. \end{cases}$$

A set of the form $[a, b)$ is called a *half-open interval*.

Lemma 68. *Suppose that I is a half-open interval, $\langle J_i : i \in \omega\rangle$ is a system of half-open intervals, and $I \subseteq \bigcup_{i\in\omega} J_i$. Then*

$$\lambda(I) \le \sum_{j\in\omega} \lambda(J_i).$$

Proof. If $I = \emptyset$ this is obvious. So suppose that $I \ne \emptyset$. Then there exist real numbers $a < b$ such that $I = [a, b)$. Let

$$A = \left\{ x \in [a,b] : x - a \le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) \right\}.$$

Obviously $a \in A$, and A is bounded above by b, so $c \stackrel{\text{def}}{=} \sup(A)$ exists. Now

$$\begin{aligned} c - a &= \sup_{x\in A}(x - a) \\ &\le \sup_{x\in A} \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) \\ &\le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)). \end{aligned}$$

Hence $c \in A$. Now suppose that $c < b$. Thus $c \in [a, b)$, so there is a $k \in \omega$ such that $c \in J_k$. Say $J_k = [u, v)$. Then $x \stackrel{\text{def}}{=} \min(v, b) > c$. Then $\lambda(J_j \cap (-\infty, c)) \le \lambda(J_j \cap (-\infty, x))$ for each j, and $\lambda(J_k \cap (-\infty, x)) = \lambda(J_k \cap (-\infty, c)) + x - c$. Hence

$$\begin{aligned} &\sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) \ge \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)) + x - c \\ &\ge c - a + x - c = x - a. \end{aligned}$$

Here we used the above inequality on $c - a$. Thus we have shown that $x \in A$. But $x > c = \sup(A)$, contradiction.

Hence $c = b$, so $b \in A$. $\square$

Now for any $A \subseteq \mathbb{R}$ let

$$\theta'(A) = \inf\left\{ \sum_{j\in\omega} \lambda(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals} \right.$$
$$\left. \text{such that } A \subseteq \bigcup_{j\in\omega} I_j \right\}.$$

Lemma 69. *(i) θ' is an outer measure on $\mathbb{R}$.*

(ii) $\theta'(I) = \lambda(I)$ for every half-open interval I.

Proof. (i): Clearly (1) and (2) hold. Now for (3), suppose that $\langle A_i : i \in \omega\rangle$ is a sequence of subsets of X. Let $B = \bigcup_{i\in\omega} A_i$. For each $i \in \omega$ let $\langle I_{ij} : j \in \omega\rangle$ be a sequence of half-open intervals such that $A_i \subseteq \bigcup_{j\in\omega} I_{ij}$ and

$$\sum_{j\in\omega} \lambda(I_{ij}) \le \theta'(A_i) + \frac{\varepsilon}{2^i}.$$

Note that this holds even if $\theta'(A_i) = \infty$. Let $p : \omega \to \omega \times \omega$ be a bijection.

$$B \subseteq \bigcup_{m\in\omega} I_{1^{st}(p(m)),2^{nd}(p(m))}. \tag{1}$$

In fact, if $b \in B$, choose $i \in I$ such that $b \in A_i$, and then choose $j \in \omega$ such that $b \in I_{ij}$. Let $m = p^{-1}(i, j)$. Then

$$b \in I_{1^{st}(p(m)),2^{nd}(p(m))},$$

as desired in (1).

$$\sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \le \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}). \tag{2}$$

In fact, let $m \in \omega$, and set

$$n = \max(\{1^{st}(p(i)) : i \le m\} \cup \{2^{nd}(p(i)) : i \le m\}).$$

Then

$$\sum_{i=0}^{m} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \le \sum_{i=0}^{n}\sum_{j=0}^{n} \lambda(I_{ij}) \le \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}),$$

and (2) follows.

Hence using (1) we have

$$\begin{aligned}
\theta'\left(\bigcup_{i\in\omega} A_i\right) &= \theta'(B)\\
&\le \sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))})\\
&\le \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij})\\
&\le \sum_{i\in\omega}\left(\theta'(A_i)+\frac{\varepsilon}{2^i}\right)\\
&= \sum_{i\in\omega}\theta'(A_i)+\sum_{i\in\omega}\frac{\varepsilon}{2^i}\\
&= \sum_{i\in\omega}\theta'(A_i)+2\varepsilon.
\end{aligned}$$

Hence (3) in the definition of outer measure holds.

Clearly $\theta'(I) \le \lambda(I)$. The other inequality follows from Lemma 12. □

Corollary 70. *For θ' the explicit outer measure defined above on $\mathbb{R}$, and with*

$$\begin{aligned}
\Sigma_1 = \{E \subseteq \mathbb{R} : &\text{ for every } A \subseteq X,\\
&\theta'(A) = \theta'(A\cap E) + \theta'(A\backslash E)\},
\end{aligned}$$

the system $(\mathbb{R}, \Sigma_1, \theta' \restriction \Sigma_1)$ is a measure space. □

Lemma 71. *$(-\infty, x)$ is measurable for every $x \in \mathbb{R}$.*

Proof. First we show

(1) $\lambda(I) = \lambda(I \cap (-\infty, x)) + \lambda(I\backslash(-\infty, x))$ for every half-open interval I.

This is obvious if $I \subseteq (-\infty, x)$ or $I \subseteq [x, \infty)$. So assume that neither of these cases hold. Then with $I = [a, b)$ we must have $a < x < b$. Then

$$\begin{aligned}
\lambda(I\cap(-\infty,x)) + \lambda(I\backslash(-\infty,x)) &= \lambda([a,x)) + \lambda([x,b))\\
&= \lambda([a,x)) + \lambda([x,b))\\
&= x - a + b - x\\
&= b - a\\
&= \lambda([a,b))\\
&= \lambda(I).
\end{aligned}$$

So (1) holds.

Now for the proof of the lemma, let $A \subseteq \mathbb{R}$ and let $\varepsilon > 0$. We show that $\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) \le \theta'(A) + \varepsilon$, which will prove the lemma. By the definition of θ', there is a sequence $\langle I_j : j \in \omega\rangle$ of half-open intervals such that $A \subseteq \bigcup_{j\in\omega} I_j$ and $\sum_{j\in\omega} \lambda(I_j) \le \theta'(A) + \varepsilon$. Now $\langle I_j \cap (-\infty, x) : j \in \omega\rangle$ and $\langle I_j \backslash (-\infty, x) : j \in \omega\rangle$ are sequences of half-open intervals, $A \cap (-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \cap (-\infty, x))$, and $A\backslash(-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \backslash (-\infty, x))$, so

$$\begin{aligned}\theta'(A \cap (-\infty, x)) + \theta'(A\backslash(-\infty, x)) &\le \sum_{j=0}^{\infty} \lambda(I_j \cap (-\infty, x)) + \sum_{j=0}^{\infty} \lambda(I_j \backslash (-\infty, x)) \\ &= \sum_{j=0}^{\infty} \lambda(I_j) \le \theta'(A) + \varepsilon. \qquad \square\end{aligned}$$

Theorem 72. *Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.*

Proof. It suffices to show that every open set is Lebesgue measurable. It then suffices to prove the following:

(1) If U is a nonempty open subset of $\mathbb{R}$, then there is a family $\mathscr{A}$ of half-open intervals with rational coefficients such that $U = \bigcup \mathscr{A}$.

To prove (1), let $\mathscr{A}$ be the set of all half-open intervals contained in U. Now take any $x \in U$. Since U is open, there are real numbers $y < z$ such that $x \in (y, z) \subseteq U$. Choose rational numbers r, s such that $y < r < x < s < z$. Then $x \in [r, s) \subseteq U$, as desired. □

Corollary 73. *Every Lebesgue null set is Lebesgue measurable. Every singleton is a null set, and every countable set is a null set.* □

Now we proceed to the case of $\mathbb{R}^n$, n any positive integer. A subset A of $\mathbb{R}^n$ is a *half-open interval* iff there is a system $\langle (a_i, b_i) : i < n\rangle$ of pairs of real numbers such that

$$A = \{x \in {}^n\mathbb{R} : a_i \le x_i < b_i \text{ for all } i < n\}.$$

Note that $A = \emptyset$ if $a_i \ge b_i$ for some $i < n$; so the empty set is counted as a half-open interval. If A is nonempty, then the indicated system is uniquely determined by A.

Now we define $\lambda(\emptyset) = 0$, while if A is a nonempty half-open interval, with notation as above, then

$$\lambda(A) = \prod_{i<n}(b_i - a_i).$$

Lemma 74. *If A is a half-open interval in $\mathbb{R}^n$ and $\langle I_i : i \in \omega\rangle$ is a sequence of half-open intervals such that $A \subseteq \bigcup_{i\in\omega} I_i$, then*

$$\lambda(A) \le \sum_{i\in\omega} \lambda(I_i).$$

Proof. We use induction on n, and for this reason we denote the above function by λ_n for this proof. The case $n = 1$ is given by Lemma 1.

Now suppose the result is true for n, and A is a half-open interval in $\mathbb{R}^{n+1}$. Say $\langle (a_i, b_i) : i < n+1 \rangle$ is a sequence of pairs of real numbers such that

$$A = \{x \in {}^{n+1}\mathbb{R} : a_i \le x_i < b_i \text{ for all } i < n+1\}.$$

Also assume for each $j \in \omega$ that $\langle (a_i^{(j)}, b_i^{(j)}) : i < n+1 \rangle$ is a sequence of pairs of real numbers such that

$$I_j = \{x \in {}^{n+1}\mathbb{R} : a_i^{(j)} \le x_i < b_i^{(j)} \text{ for all } i < n+1\}.$$

We may assume that $A \ne \emptyset$, and hence that $a_i < b_i$ for all $i < n+1$. Let $\zeta = \prod_{i<n}(b_i - a_i)$. Thus $\lambda_{n+1}(A) = \zeta(b_n - a_n)$. For each $\xi \in \mathbb{R}$ let $H_\xi = \{x \in {}^{n+1}\mathbb{R} : x_n < \xi\}$. Define

$$B = \left\{ \xi \in \mathbb{R} : a_n \le \xi \le b_n \text{ and } \zeta(\xi - a_n) \le (1+\varepsilon) \sum_{j\in\omega} \lambda_{n+1}(I_j \cap H_\xi) \right\}.$$

Obviously $a_n \in B$ and $B \subseteq [a_n, b_n]$. Let $\gamma = \sup(B)$. So $\gamma \in [a_n, b_n]$.

(1) $\gamma \in B$.

For,

$$\begin{aligned} \zeta(\gamma - a_n) &= \sup_{\xi\in B} \zeta(\xi - a_n) \\ &\le (1+\varepsilon) \sup_{\xi \in B} \sum_{j\in\omega} \lambda_{n+1}(I_j \cap H_\xi) \\ &\le (1+\varepsilon) \sum_{j\in\omega} \lambda_{n+1}(I_j \cap H_\gamma), \end{aligned}$$

so (1) holds.

We claim that $\gamma = b_n$. Suppose not. Let

$$\begin{aligned} C &= \{x \in {}^{n}\mathbb{R} : a_i \le x_i < b_i \text{ for all } i < n\}; \\ D_j &= \{x \in {}^{n}\mathbb{R} : x^\frown\langle\gamma\rangle \in I_j\} \quad \text{for each } j \in \omega. \end{aligned}$$

Then $C \subseteq \bigcup_{j\in\omega} D_j$. For, suppose that $x \in C$. Then $x^\frown\langle\gamma\rangle \in A$, and hence there is a $j \in \omega$ such that $x^\frown\langle\gamma\rangle \in I_j$. So $x \in D_j$, as desired. It follows by the inductive hypothesis that $\zeta \overset{\text{def}}{=} \lambda_n(C) \le \sum_{j\in\omega} \lambda_r(D_j)$. Clearly $\zeta > 0$, so there is an $m \in \omega$ such that $\zeta \le (1+\varepsilon)\sum_{j\le m} \lambda_n(D_j)$.

Now note that if $D_j \ne \emptyset$, then $a_n^{(j)} \le \gamma < b_n^{(j)}$. Let

$$\xi = \min(\{b_n\} \cup \{b_n^{(j)} : D_j \ne \emptyset\}).$$

Thus $\xi < \gamma$. If $D_j \neq \emptyset$, then $I_j \cap H_\xi \neq \emptyset$, and

$$\begin{aligned}\lambda_{n+1}(I_j \cap H_\xi) &= \lambda_n(D_j)(\xi - a_n)\\ &= \lambda_n(D_j)(\xi-\gamma) + \lambda_n(D_j)(\gamma - a_n)\\ &= \lambda_n(D_j)(\xi-\gamma) + \lambda_{n+1}(I_j \cap H_\gamma).\end{aligned}$$

So

$$\lambda_{n+1}(I_j \cap H_\xi) = \lambda_n(D_j)(\xi-\gamma) + \lambda_{n+1}(I_j \cap H_\gamma). \tag{2}$$

This also holds if $D_j = \emptyset$. Hence

$$\begin{aligned}\zeta(\xi - a_n) &= \zeta(\xi-\gamma) + \zeta(\gamma - a_n)\\ &\leq (1+\varepsilon)(\xi-\gamma)\sum_{j\leq m}\lambda_n(D_j) + (1+\varepsilon)\sum_{j\in\omega}\lambda_{n+1}(I_j\cap H_\gamma)\\ &= (1+\varepsilon)(\xi-\gamma)\sum_{j\leq m}\lambda_n(D_j) + (1+\varepsilon)\sum_{j\leq m}\lambda_{n+1}(I_j\cap H_\gamma)\\ &\quad + (1+\varepsilon)\sum_{m<j\in\omega}\lambda_{n+1}(I_j\cap H_\gamma)\\ &= (1+\varepsilon)\sum_{j\in\omega}\lambda_{n+1}(I_j\cap H_\xi) \quad \text{by (2)}.\end{aligned}$$

Hence $\xi \in B$, contradicting $\gamma < \xi$.

Thus $\gamma = b_n$, so $b_n \in B$ by (1). Hence

$$\lambda_{n+1}(A) = \zeta(b_n - a_n) \leq (1+\varepsilon)\sum_{j\in\omega}\lambda_{n+1}(I_j \cap H_{b_n}) \leq (1+\varepsilon)\sum_{j\in\omega}\lambda_{n+1}(I_j).$$

since ε is arbitrary, it follows that $\lambda_{n+1}(A) \leq \sum_{j\in\omega}\lambda_{n+1}(I_j)$, as desired. □

Now we define what we will prove is outer measure on $\mathbb{R}^n$. For any $A \subseteq {}^n\mathbb{R}$,

$$\theta'(A) = \inf\left\{\sum_{j\in\omega}\lambda(I_j) : \langle I_j : j\in\omega\rangle \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_{j\in\omega} I_j\right\}.$$

Lemma 75. *θ' is an outer measure on $\mathbb{R}^n$, and $\theta'(I) = \lambda(I)$ for every half-open interval I.*

Proof. The proof is essentially the same as that of Lemma 13. □

Thus we obtain a σ-field of subsets of $\mathbb{R}^n$, and Lebesgue measure on $\mathbb{R}^n$.

Lemma 76. *Suppose that $i < n$ and $\xi \in \mathbb{R}$. Let $H_{i\xi} = \{y \in {}^n\mathbb{R} : y_i < \xi\}$. Then $H_{i\xi}$ is Lebesgue measurable.*

Proof. Write H instead of $H_{i\xi}$. First we claim

(1) If I is a half-open interval in ${}^n\mathbb{R}$, then $\lambda(I) = \lambda(I \cap H) + \lambda(I \backslash H)$.

For, if $I \subseteq H$ or $I \cap H = \emptyset$, this is clear. So suppose this is not the case. In particular, $I \neq \emptyset$, so there are real numbers $\langle a_j : j < n\rangle$ and $\langle b_j : j < n\rangle$ such that $a_j < b_j$ for all $j < n$, $I = \bigcup_{j<n}(a_j, b_j)$, and $a_i < \xi < b_i$. Hence

$$\begin{aligned}\lambda(I \cap H) + \lambda(I \backslash H) &= (\xi - a_i)\prod_{j\neq i}(b_j - a_j) + (b_i - \xi)\prod_{j\neq i}(b_j - a_j)\\ &= \prod_{j<n}(b_j - a_j)\\ &= \lambda(I),\end{aligned}$$

as desired in (1).

Now the proof is finished as for Lemma 15. □

It follows as in the one-dimensional case that every Borel set is measurable, and all null sets are measurable.

We return to the one-dimensional case.

Lemma 77. *Suppose that μ is a measure and E, F, G are μ-measurable. Then*

$$\mu(E \triangle F) \leq \mu(E \triangle G) + \mu(G \triangle F).$$

Proof.

$$\begin{aligned}\mu(E\triangle F) &= \mu(E\backslash F) + \mu(F\backslash E)\\ &= \mu((E\backslash F)\cap G) + \mu((E\backslash F)\backslash G) + \mu(F\backslash E)\cap G) + \mu((F\backslash E)\backslash G)\\ &\leq \mu(G\backslash F) + \mu(E\backslash G) + \mu(G\backslash E) + \mu(F\backslash G)\\ &= \mu(E\triangle G) + \mu(G\triangle F).\end{aligned}$$

□

Lemma 78. *If E is Lebesgue measurable with finite measure, then for any $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \varepsilon$. Moreover, there is a system $\langle K_j : j < \omega\rangle$ of open intervals such that $U = \bigcup_{j<\omega} K_j$ and $\theta'(U) \leq \sum_{j<\omega}\theta'(K_j) \leq \theta'(E) + \varepsilon$.*

Proof. By the basic definition of Lebesgue measure,

$$0 = \theta'(E) = \inf\left\{\sum_{j\in\omega}\theta'(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals}\right.$$
$$\left.\text{such that } A \subseteq \bigcup_{j\in\omega} I_j\right\}.$$

Hence we can choose a sequence $\langle I_j : j \in \omega\rangle$ of half-open intervals such that $E \subseteq \bigcup_{j\in\omega} I_j$ and

$$\theta'\left(\bigcup_{j\in\omega} I_j\right) \le \sum_{j\in\omega} \theta'(I_j) \le \theta'(E) + \frac{\varepsilon}{2}.$$

Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define

$$\begin{aligned} K_j &= \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right); \quad \text{then} \\ E &\subseteq \bigcup_{j\in\omega} K_j \quad \text{and} \end{aligned}$$

$$\begin{aligned} \theta'\left(\bigcup_{j\in\omega} K_j\right) &\le \sum_{j\in\omega} \theta'(K_j) \\ &= \sum_{j\in\omega} \left(\frac{\varepsilon}{2^{j+2}} + \theta'(I_j)\right) \\ &= \sum_{j\in\omega} \frac{\varepsilon}{2^{j+2}} + \sum_{j\in\omega} \theta'(I_j) \\ &\le \frac{\varepsilon}{2} + \theta'(E) + \frac{\varepsilon}{2} = \theta'(E) + \varepsilon. \end{aligned}$$

□

Corollary 79. *(i) If A is Lebesgue measurable and $\theta'(A)$ is finite, then $\theta'(A) = \inf\{\theta'(U) : U$ open, $A \subseteq U\}$.*

(ii) If A is Lebesgue measurable with finite measure, then $\theta'(A) = \sup\{\theta'(C) : C$ closed, $C \subseteq A\}$.

(iii) If A is measurable and $\theta'(A) = \infty$, then $\sup_{\{} \theta'(C) : C$ closed, $C \subseteq A\} = \infty$.

Proof. Only (iii) needs a proof. Let $\varepsilon > 0$. For each $n \in \omega$ let

$$\begin{aligned} a_{2n} &= n; \\ b_{2n} &= n + 1; \\ a_{2n+1} &= -n - 1; \\ b_{2n+1} &= -n. \end{aligned}$$

For each $n \in \omega$ let C_n be a closed subset of $[a_n, b_n) \cap A$ such that

$$\theta'([a_n, b_n) \cap A \backslash C_n) < \frac{\varepsilon}{2^n}.$$

Then

$$\theta'(A) = \sum_{n\in\omega} \theta'([a_n, b_n) \cap A)$$

$$
\begin{aligned}
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A] \cup \ldots \cup [[a_n, b_n) \cap A]) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A \backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A \backslash C_n]) + \theta'(C_0 \cup \ldots \cup C_n) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A \backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A \backslash C_n]) + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n) \\
&= \varepsilon + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n),
\end{aligned}
$$

as desired.

The following is an elementary lemma concerning the topology of the reals.

Lemma 80. *Suppose that U is a bounded open set.*

(i) There is a collection $\mathscr{A}$ of pairwise disjoint open intervals such that $U = \bigcup \mathscr{A}$.

(ii) There exist a countable subset C of $\mathbb{R}$ and a collection $\mathscr{B}$ of pairwise disjoint open intervals with rational endpoints such that $U = C \cup \bigcup \mathscr{B}$ and $C \cap \bigcup \mathscr{B} = \emptyset$.

Proof. (i): For $x, y \in \mathbb{R}$, define $x \equiv y$ iff one of the following conditions holds: (1) $x = y$; (2) $x < y$ and $[x, y] \subseteq U$; (3) $y < x$ and $[y, x] \subseteq U$. Clearly $\equiv$ is an equivalence relation on $\mathbb{R}$. If $x < z < y$ and $x \equiv y$, then obviously $x \equiv z$. Thus each equivalence class is convex. If C is an equivalence class with more than one element, then it must be an open interval (a, b), since if for example the left endpoint a is in C then some real to the left of a must be in C, contradiction. It follows now that the collection $\mathscr{A}$ of all equivalence classes with more than one element is as desired in (i).

(ii): First note that the set $\mathscr{A}$ of (i) must be countable. Now take any $(a, b) \in \mathscr{A}$, $a < b$. Let $c_0 < c_1 < \cdots < c_m < \cdots$ be rational numbers in (a, b) which converge to b, and $c_0 = d_0 > d_1 > \cdots > d_m > \cdots$ rational numbers which converge to a. Then let $L^{ab}_{2i} = (c_i, c_{i+1})$ and $L^{ab}_{2i+1} = (d_{i+1}, d_i)$ for all $i \in \omega$. Let $D^{ab} = \{c_i : i < \omega\} \cup \{d_i : i < \omega\}$. Define $\mathscr{B} = \{L^{ab}_i : (a, b) \in \mathscr{A}, i < \omega\}$ and $C = \bigcup_{(a,b)\in\mathscr{A}} D^{ab}$. Clearly this works for (ii). □

Lemma 81. *If E is Lebesgue measurable and $\varepsilon > 0$, then there is an $m \in \omega$ and a sequence $\langle I_i : i < m\rangle$ of open intervals with rational endpoints such that $\theta'\left(E \triangle \bigcup_{i<m} I_i\right) \leq \varepsilon$.*

Proof. By Lemma 1 let $U \supseteq E$ be open such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \frac{\varepsilon}{2}$. Then choose C and $\mathscr{B}$ as above. Let $W = \bigcup \mathscr{B}$. So $\theta'(W) = \sum_{I\in\mathscr{B}} \theta'(I)$. Then choose $m \in \omega$ and $\langle I_i : i < m\rangle$ elements of $\mathscr{B}$ such that $\sum_{I\in\mathscr{B}} \theta'(I) - \sum_{i<m} \theta'(I_i) \leq \frac{\varepsilon}{2}$. Now $\theta'(W) = \sum_{I\in\mathscr{B}} \theta'(I)$ and $\theta'(\bigcup_{i<m} I_i) = \sum_{i<m} \theta'(I_i)$. Let $V = \bigcup_{i<m} I_i$. Thus $\theta'(W) - \theta'(V) \leq \frac{\varepsilon}{2}$. Hence $V \subseteq W \subseteq U$, and

$$
\begin{aligned}
\theta'(E \triangle V) &\leq \theta'(E \triangle U) + \theta'(U \triangle W) + \theta'(W \triangle V) \\
&= \theta'(U \backslash E) + \theta'(C) + \theta'(W \backslash V) \\
&= \theta'(U) - \theta'(E) + \theta'(W) - \theta'(V) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$
□

Connections between different measures

Here we follow Fremlin, **Measure theory**, vol. 2, 254K.

Lemma 82. *If (X, Σ, μ) is a measure space and $Y \subseteq X$, then*

$$(Y, \{A \cap Y : A \in \Sigma\}, \mu \restriction \{A \cap Y : A \in \Sigma\})$$

is a measure space. □

At this point we have two important measure spaces: $({}^{\omega}2, \Sigma_0, \theta)$ and $(\mathbb{R}, \Sigma_1, \theta')$. We now define $\Sigma_2 = \{A \cap \Omega : A \in \Sigma_0\}$ and $\theta_2 = \theta \restriction \{A \cap \Omega : A \in \Sigma_0\}$. By Lemma 27 we have

Corollary 83. *$(\Omega, \Sigma_2, \theta_2)$ is a measure space.* □

Let $\Sigma_3 = \{A \cap [0,1] : A \in \Sigma_1\}$ and $\theta_3 = \theta' \restriction \{A \cap [0,1] : A \in \Sigma_1\}$.

Corollary 84. *$([0,1], \Sigma_3, \theta_3)$ is a measure space.* □

If (X, Σ, μ) is a measure space, the $A \in \Sigma$ is an *atom* iff $\mu(A) > 0$. and for all $B \in \Sigma$ with $B \subseteq A$, either B or $A \backslash B$ has measure 0.

Let λ be the usual measure on ${}^{\omega}2$ and μ Lebesgue measure on $[0,1]$. Consider the measure spaces $({}^{\omega}2, \Sigma_0, \lambda)$ and $([0,1], \Sigma_1, \mu)$. For each $x \in {}^{\omega}2$ let $\varphi(x) = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$.

Theorem 85. *There is a bijection $\tilde{\varphi} : {}^{\omega}2 \to [0,1]$ which is equal to φ except at countably many points, and any such bijection is an isomorphism from $({}^{\omega}2, \Sigma_0, \lambda)$ to $([0,1], \Sigma_1, \mu)$. That is:*

(a) $\forall X \subseteq {}^{\omega}2[X \in \Sigma_0$ iff $\tilde{\varphi}[X] \in \Sigma_1]$;
(b) $\forall X \subseteq [0,1][X \in \Sigma_1$ iff $\tilde{\varphi}^{-1}[X] \in \Sigma_0]$;
(c) $\forall X \in \Sigma_0[\lambda(X) = \mu(\tilde{\varphi}[X])]$;
(d) $\forall X \in \Sigma_1[\mu(X) = \lambda(\tilde{\varphi}^{-1}[X])]$.

Proof. Let $H = \{x \in {}^{\omega}2 : \exists m \in \omega \forall i \geq m[x_i = x_m]\}$ and $H' = \{2^{-n}k : n \in \omega, k \leq 2^n\}$. Then H and H' are countable.

(1) $\varphi \restriction ({}^{\omega}2 \backslash H)$ is a bijection from ${}^{\omega}2 \backslash H$ onto $[0,1] \backslash H'$.

For, first we show that $\varphi \restriction ({}^{\omega}2 \backslash H)$ maps into $[0,1] \backslash H'$. Let $x \in ({}^{\omega}2 \backslash H)$. Thus

(2) $\forall m \in \omega \exists i > m[x_i \neq x_m]$.

It follows that $\varphi(x) \neq 1$, for by (2) there is a j such that $x_j = 0$, and then

$$\varphi(x) = \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} < 1.$$

Suppose that $\varphi(x) \in H'$. Thus there exist $n \in \omega$ and $k < 2^n$ such that

(3) $$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 2^{-n}k.$$

Since $\varphi(x) \neq 1$, we can write $2^{-n}k = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$ with each $y_i \in 2$. Thus by (3) we have

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = \sum_{i=0}^{n-1}(2^{-i-1}y_i). \tag{4}$$

Now we claim that $y \subseteq x$. For, suppose not, and let $j < n$ be minimum such that $x_j \neq y_j$. Hence by (4) we have

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) = \sum_{i=j}^{n-1}(2^{-i-1}y_i).$$

Case 1. $x_j = 0$ and $y_j = 1$. By (2) choose $k > j$ so that $x_k = 1$ and choose $l > k$ so that $x_l = 0$. Then

$$\begin{aligned}\sum_{i=j}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=j}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=j}^{l-1}(2^{-i-1}x_i) + 2^{-l-1} \\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} \leq \sum_{i=j}^{n-1}(2^{-i-1}y_i),\end{aligned}$$

contradiction.

Case 2. $x_j = 1$ and $y_j = 0$. Then

$$\sum_{i=j}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=j+1}^{n} 2^{-i-1} < \sum_{i=j+1}^{\infty} 2^{-i-1} = 2^{-j-1} \leq \sum_{i=j}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus $y \subseteq x$. Now by (2) there is a $j \geq n$ such that $x_j = 1$. Hence

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) > \sum_{i=j}^{n-1}(2^{-i-1}y_i),$$

contradiction.

Thus $\varphi(x) \notin H'$.

To show that $\varphi \restriction ({}^{\omega}2\backslash H)$ maps into $[0,1]\backslash H'$, let $t \in [0,1]\backslash H'$. Since $1 \in H'$, we can write $t = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$ with x not eventually 1. We claim that $x \notin H$. For, suppose that $x \in H$. Say $m \in \omega$ and $\forall i > m[x_i = x_m]$. Since x is not eventually 1, we have $x_m = 0$. Since $t \notin H'$, we have $t \neq 0$, so x is not the all 0 sequence. Choose n maximum such that $x_n \neq 0$. Thus $t = \sum_{i=0}^{n}(2^{-i-1}x_i)$. Hence

$$\begin{aligned}2^{n+1}t &= 2^{n+1}2^{-1}x_0 + 2^{n+1}2^{-2}x_1 + \cdots + x_0 \\ &= 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0.\end{aligned}$$

Hence with $k = 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0$ we have $k \leq 2^{n+1}$ and $t = 2^{-n-1}k \in H'$, contradiction. So $x \notin H$. Clearly $\varphi(x) = t$.

For $\varphi \restriction ({}^\omega 2\backslash H)$ one-one, suppose that $x, y \in ({}^\omega 2\backslash H)$ and $x \neq y$. So $\sum_{i=0}^{\infty}(2^{-i-1}x_i) = \sum_{i=0}^{\infty}(2^{-i-1}y_i)$. Let m be minimum such that $x_m \neq y_m$. By symmetry, say $x_m = 0$ and $y_m = 1$. Choose $n > m$ so that $x_n = 0$; this is possible since $x \notin H$. Then

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{n-1}(2^{-i-1}x_i) + 2^{-n} < \sum_{i=0}^{n-1}(2^{-i-1}y_i).$$

This finishes the proof of (1).

Now H and H' are countable and infinite. Hence there is an extension of $\varphi \restriction ({}^\omega 2\backslash H)$ to a bijection of ${}^\omega 2$ onto $[0,1]$. Let $\tilde{\varphi}$ be any bijection of ${}^\omega 2$ onto $[0,1]$ which is equal to φ except for countably many points. Let M be the countable set $\{x \in {}^\omega 2 : \varphi(x) \neq \tilde{\varphi}(x)\}$ and let N be the countable set $\varphi[M] \cup \tilde{\varphi}[M]$.

(5) $\forall A \subseteq {}^\omega 2[\varphi[A]\triangle\tilde{\varphi}[A] \subseteq N]$.

In fact, if $b \in \varphi[A]\backslash\tilde{\varphi}[A]$, then there is an $x \in A$ such that $b = \varphi(x)$. Since $b \notin \tilde{\varphi}[A]$, we have $\tilde{\varphi}(x) \neq b$. Hence $x \in M$, so $b \in \varphi[M] \subseteq N$. Now suppose that $b \in \tilde{\varphi}[A]\backslash\varphi[A]$. Say $b = \tilde{\varphi}(x)$ with $x \in A$. Since $b \notin \varphi[A]$, we have $\varphi(x) \neq b$. So $x \in M$ and $b \in \tilde{\varphi}[M] \subseteq N$. So (5) holds.

(6) If $t \in [0,1]$, then $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ and hence $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = \mu(\{t\})$.

We have $\tilde{\varphi}^{-1}[\{t\}] = \{\tilde{\varphi}^{-1}(t)\}$, so $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ by Proposition 11. $\mu(\{t\}) = 0$ by Corollary 17.

(7) If $n \in \omega$, $k < 2^n$, and $E = [2^{-n}k, 2^{-n}(k+1)]$, then $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E) = 2^{-n}$.

$\mu(E) = 2^{-n}$ by Lemma 13. Further,

$$\varphi^{-1}[E] = \left\{x \in {}^\omega 2 : 2^{-n}k \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq 2^{-n}(k+1)\right\}.$$

Let $k = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + y_{n-1}$ with each $y_i \in 2$. Then $2^{-n}k = 2^{-1}y_0 + 2^{-2}y_1 + \cdots + 2^{-n}y_{n-1} = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$.

Case 1. $y_{n-1} = 0$. Then $k + 1 = \sum_{i=0}^{n-2}(2^{n-i-1}y_i) + 1$ and so $2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n}$. Now suppose that $x \in \varphi^{-1}[E]$.

(8) If x is not eventually 1, then $\forall i < n[x_i = y_i]$.

For, suppose that $j < n$ is minimum such that $x_i \neq y_i$. Choose $l > k > j$ with $x_l = x_k = 0$.

Subcase 1.1. $x_j = 0$, $y_j = 1$. Then

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-l-1}$$
$$< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} \leq \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k.$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 1.2. $x_j = 1$, $y_j = 0$. Then

$$2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n} < \sum_{i=0}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus (8) holds.

(9) If $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$, then $x \in \varphi^{-1}[E]$.

In fact, assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned}2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \le \sum_{i=0}^{\infty}(2^{-i-1}x_i)\\ &\le \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{n-2}(2^{-i-1}x_i) + 2^{-n} = 2^{-n}(k+1).\end{aligned}$$

This proves (9).

Case 2. $y_{n-1} = 1$ and there is a $j < n-1$ such that $y_j = 0$. Take the greatest such j. Then $k+1 = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + 2^{n-j}y_{j-1} + 2^{n-j-1}$, and hence $2^{-n}(k+1) = \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1}$. Now suppose that $x \in \varphi^{-1}[E]$. Again we claim that (8) and (9) hold. For (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and $l < n$ is minimum such that $x_l \ne y_l$. Choose $t > s > l$ with $x_t = x_s = 0$.

Subcase 2.1. $x_l = 0$, $y_l = 1$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}(2^{-i-1}x_i) &\le \sum_{i=0}^{t-1}(2^{-i-1}x_i) + \sum_{i=t+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{t-1}(2^{-i-1}x_i) + 2^{-t-1}\\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} \le \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k.\end{aligned}$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 2.2. $x_l = 1$, $y_l = 0$. Then $l \le j$, and

$$\begin{aligned}2^{-n}(k+1) &= \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} = \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l}^{j-1}(2^{-i-1}y_i) + 2^{-j-1}\\ &< \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-i-1} \le \sum_{i=0}^{\infty}(2^{-i-1}x_i,\end{aligned}$$

contradiction.

Hence (8) holds.

Now for (9), assume that $x \in {}^\omega 2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned}2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \le \sum_{i=0}^{\infty}(2^{-i-1}x_i)\\ &\le \sum_{i=0}^{j-1}(2^{-i-1}x_i + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}y_i + 2^{-j-1} = 2^{-n}(k+1).\end{aligned}$$

So (9) holds. This finishes Case 2.

Case 3. $\forall i < n[y + i = 1$. Then $k+1 = 2^n$ and $2^{-n}k = 1$. To check (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and j is minimum such that $x_j \neq y_j$. Take $s > t > j$ with $x_s = x_t = 0$. Then $x_j = 0$, and

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \le \sum_{i=0}^{s-1}(2^{-i-1}x_i) + \sum_{i=s+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{s-1}(2^{-i-1}x_i) + 2^{-s-1} \le \sum_{i=0}^{n-1}(2^{-i-1}y_i),$$

contradiction. Thus (8) holds.

For (9), assume that $x \in {}^\omega 2$ and $\forall i < n[x_i = y_i]$. Clearly $x \in \varphi^{-1}[E]$.

So (8) and (9) hold in all cases.

Now let $S = \{x \in {}^\omega 2 : x \text{ is eventually } 1\}$. So S is countable. By (8) we have $\varphi^{-1}[E] \backslash S \subseteq \{x \in {}^\omega 2 : x \restriction n = y \restriction n\}$, and by (9) we have $\{x \in {}^\omega 2 : x \restriction n = y \restriction n\} \subseteq \varphi^{-1}[E]$. Let $T = \varphi^{-1}[E] \backslash \{x \in {}^\omega 2 : x \restriction n = y \restriction n\}$. Now $\{x \in {}^\omega 2 : x \restriction n = y \restriction n\} \in \Sigma_0$ and $\lambda(\{x \in {}^\omega 2 : x \restriction n = y \restriction n\}) = 2^{-n}$ by Proposition 10. Note that $T \subseteq S$, so T is countable. Since $\varphi^{-1}[E] = \{x \in {}^\omega 2 : x \restriction n = y \restriction n\} \cup T$, it follows that $\varphi^{-1}[E] \in \Sigma_0$ and $\lambda(\varphi^{-1}[E]) = 2^{-n}$. Since $\varphi^{-1}[E] = (\varphi^{-1}[E] \cap M) \cup (\varphi^{-1}[E] \backslash M)$ and M is countable, it follows that $(\varphi^{-1}[E] \backslash M) \in \Sigma_0$ and $\lambda(\varphi^{-1}[E] \backslash M) = 2^{-n}$. Clearly $\tilde{\varphi}^{-1}[E] \backslash M = \varphi^{-1}[E] \backslash M$, so $(\tilde{\varphi}^{-1}[E] \backslash M) \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E] \backslash M) = 2^{-n}$. Hence $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}$. This proves (7).

(10) If $n \in \omega$ and $k < l \le 2^n$, and $E = [2^{-n}k, 2^{-n}l]$, then $E \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}(l-k) = \mu(E)$.

This is true by (7) since $E = \bigcup_{k \le i < l}[2^{-n}i, 2^{-n}(i+1)) \cup \{2^{-n}(l)\}$.

(11) Suppose that $0 \le t < u \le 1$ and $E = [t, u)$. Then $\tilde{\varphi}^{-1}[E]) \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = u - t = \mu(E)$.

In fact, for each $n \in \omega$ let $k_n = \lfloor 2^n t \rfloor$ and $l_n = \lfloor 2^n u \rfloor$. Then $k_n \le 2^n t < k_n + 1$ and $l_n \le 2^n u < l_n + 1$. It follows that $\lim_{n\to\infty}(2^{-n}k_n) = t$ and $\lim_{n\to\infty}(2^{-n}l_n) = u$. So $\bigcup_{n\in\omega}[k_n, l_n] = (t, u)$ or $\bigcup_{n\in\omega}[k_n, l_n] = (t, u]$. Hence the indicated conclusion follows.

Now for each $X \subseteq {}^\omega 2$ define

$$\lambda^*(X) = \inf\{\lambda(E) : X \subseteq E \in \Sigma_0\}.$$

(12) For every $X \subseteq {}^\omega 2$ there is an $E \in \Sigma_0$ such that $X \subseteq E$ and $\lambda^*(X) = \lambda(E)$.

In fact, suppose that $X \subseteq {}^{\omega}2$. For each $n \in \omega$ choose $E_n \in \Sigma_0$ such that $X \subseteq E_n$ and $\lambda(E) \leq \lambda^*(X) + \frac{1}{2^n}$. Then $E \stackrel{\text{def}}{=} \bigcap_{n\in\omega} E_n \in \Sigma_0$, $X \subseteq E$, and

$$\lambda^*(X) \leq \lambda(E) \leq \inf_{n\in\omega} \lambda(E_n) \leq \lambda^*(X),$$

proving (12).

(13) If $E \in \Sigma_1$, then $\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \mu(E)$ and there is a $V \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[E] \subseteq V$ and $\lambda(V) \leq \mu(E)$.

To prove (13), assume that $E \in \Sigma_1$. By the basic definition of Lebesgue measure,

$$\mu(E\backslash\{1\}) = \inf\left\{\sum_{n\in\omega} \mu(I_n) : \langle I_n : n \in \omega\rangle \text{ is a sequence of half-open}\right.$$
$$\left.\text{subintervals of } [0,1] \text{ such that } E \subseteq \bigcup_{n\in\omega} I_n\right\}$$

Hence for every $\varepsilon > 0$ there is a system $\langle I_n : n \in \omega\rangle$ of half-open subintervals of $[0,1]$ such that $E \subseteq \bigcup_{n\in\omega} I_n$ and $\sum_{n\in\omega} \mu(I_n) \leq \mu(E\backslash\{1\}) + \varepsilon$. Hence

$$\tilde{\varphi}^{-1}[E] \subseteq \{\tilde{\varphi}^{-1}[\{1\}] \cup \bigcup_{n\in\omega} \tilde{\varphi}^{-1}[I_n],$$

and hence

$$\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \lambda\left(\bigcup_{n\in\omega} \tilde{\varphi}^{-1}[I_n]\right) \leq \sum_{n\in\omega} \lambda(\tilde{\varphi}^{-1}[I_n]) = \sum_{n\in\omega} \mu(I_n) \leq \mu(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \mu(E)$. By (12) there is a $V \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[E] \subseteq V$ and $\lambda^*(\tilde{\varphi}^{-1}[E]) = \lambda(V)$. So (13) holds.

(14) If $E \in \Sigma_1$, then $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E)$.

For, by symmetry with (13) there is a $V' \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[[0,1]\backslash E] \subseteq V'$ and $\lambda(V') \leq \mu([0,1]\backslash E)$. Then $V \cup V' = {}^{\omega}2$ and

$$\lambda(V) + \lambda(V') \leq \mu(E) + \mu([0,1]\backslash E) = 1 \leq \lambda(V \cup V') \leq \lambda(V) + \lambda(V').$$

It follows that $\lambda(V \cap V') = 0$. In particular, $V \cap V' \cap \tilde{\varphi}^{-1}[E] \in \Sigma_0$. Now $\tilde{\varphi}^{-1}[[0,1]\backslash E] = \tilde{\varphi}^{-1}[[0,1]]\backslash\tilde{\varphi}^{-1}[E] = ({}^{\omega}2)\backslash\tilde{\varphi}^{-1}[E] \subseteq V'$, so $({}^{\omega}2)\backslash V' \subseteq \tilde{\varphi}^{-1}[E]$. Hence $\tilde{\varphi}^{-1}[E] = ({}^{\omega}2)\backslash V') \cup (V' \cap \tilde{\varphi}^{-1}[E] = ({}^{\omega}2)\backslash V') \cup (V' \cap V \cap \tilde{\varphi}^{-1}[E]) \in \Sigma_0$.

Now

$$\lambda(\tilde{\varphi}^{-1}[E]) \leq \lambda(V) \leq \mu(E) \quad \text{and} \quad 1 - \lambda(\tilde{\varphi}^{-1}[E]) \leq \lambda(V') \leq 1 - \mu(E),$$

so $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E)$. Thus (14) holds.

(15) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^{\omega}2 : x \restriction (n+1) = \varepsilon\}$, then $\varphi[C] = [t, t + 2^{-n-1}]$.

For, first let $x \in C$. Then

$$t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) \le \sum_{i=0}^{\infty}(2^{-i-1}x_i) \le \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = t + 2^{-n-1}.$$

Thus $\varphi(x) \in [t, t + 2^{-n-1}]$.

Second, suppose that $u \in [t, t + 2^{-n-1}]$.

Case 1. $\varepsilon_n = 0$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1.

(16) $x \restriction (n+1) = \varepsilon$.

For, suppose that j is minimum such that $x_j \neq \varepsilon_j$. Choose $s > t > j$ such that $x_s = x_t = 0$.

Subcase 2.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) \le \sum_{i=0}^{s-1}(2^{-i-1}x_i) + \sum_{i=s+1}^{\infty} 2^{-i-1} < \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 2.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$t + 2^{n-1} < \sum_{i=0}^{\infty}(2^{-i-1}x_i) = u,$$

contradiction.

Thus (16) holds, as desired in Case 1.

Case 2. ε is the all 1 sequence, and $u = t + 2^{-n-1}$. Then $u = 1$. Let $x = \langle 1 : i \in \omega\rangle$. Then $\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 1$.

Case 3. ε is the all 1 sequence, and $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds again. Otherwise there is a $j \le n$ such that $x_j = 0$. Then $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n} 2^{-i-1} = t$, contradiction.

Case 4. $\varepsilon_n = 1$, ε not the all 1 sequence, $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds. Otherwise let j be minimum such that $x_j \neq \varepsilon_j$.

Subcase 4.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 4.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$t + 2^{-n-1} = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) + 2^{-n-1} \le \sum_{i=0}^{j}(2^{-i-1}\varepsilon_i) \le \sum_{i=0}^{j}(2^{-i-1}x_i) = u,$$

contradiction.

Case 5. $\varepsilon_n = 1$, ε not the all 1 sequence, $u = t + 2^{-n-1}$. Let $x = \varepsilon^\frown\langle 1 : i \in \omega\rangle$. Then with j maximum such that $\varepsilon_j = 0$ we have

$$u = t + 2^{-n-1} = \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + 2^{-j-1} = \sum_{i=0}^{\infty}(2^{-i-1}x_i).$$

This finishes the proof of (15).

(17) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^\omega 2 : x \upharpoonright (n+1) = \varepsilon\}$, then $\mu(\varphi[C]) = \lambda(C) = 2^{-n-1}$.

This is clear from (15).

(18) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^\omega 2 : x \upharpoonright (n+1) = \varepsilon\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C) = 2^{-n-1}$.

Recall that M is countable, and $N = \varphi[M] \cup \tilde{\varphi}[M]$ is countable. Clearly $\varphi[C]\backslash N = \tilde{\varphi}[C]\backslash N$. Hence

$$\begin{aligned}\lambda(C) = \mu(\varphi[C]) &= \mu(\varphi[C] \cap N) + \mu(\varphi[C]\backslash N)\\ &= \mu(\varphi[C]\backslash N) = \mu(\tilde{\varphi}[C]\backslash N)\\ &= \mu(\tilde{\varphi}[C]\backslash N) + \mu(\tilde{\varphi}[C] \cap N) = \mu(\tilde{\varphi}[C]).\end{aligned}$$

(19) If $F \in [\omega]^{<\omega}$, $h \in {}^F 2$, and $C = \{x \in {}^\omega 2 : h \subseteq x\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C)$.

In fact, choose $m \in \omega$ such that $F \subseteq m$. Then

$$C = \bigcup\{\{x \in {}^\omega 2 : k \subseteq x\} : k \in {}^m 2 \text{ and } h \subseteq k\}.$$

For each $k \in {}^m 2$ such that $h \subseteq k$ let $D_k = \{x \in {}^\omega 2 : k \subseteq x\}$. Note that $D_k \cap D_l = \emptyset$ when $k \neq l$. Let $I = \{k \in {}^m 2 : h \subseteq k\}$ Note that $|\{k \in {}^m 2 : h \subseteq k\}| = 2^{m-|F|}$. Now $\lambda(C) = 2^{-|F|}$ by Proposition 10, and by (18),

$$\mu(\tilde{\varphi}[C]) = \mu\left(\bigcup_{k \in I} \tilde{\varphi}[D_k]\right) = \sum_{k \in I} 2^{-m} = 2^{-m}2^{m-|F|} = 2^{-|F|}.$$

So (19) holds.

Now for each $X \subseteq [0,1]$ define

$$\mu^*(X) = \inf\{\mu(E) : X \subseteq E \in \Sigma_1\}.$$

(20) For every $X \subseteq [0,1]$ there is an $E \in \Sigma_1$ such that $X \subseteq E$ and $\mu^*(X) = \mu(E)$.

In fact, suppose that $X \subseteq [0,1]$. For each $n \in \omega$ choose $E_n \in \Sigma_1$ such that $X \subseteq E_n$ and $\mu(E) \leq \mu^*(X) + \frac{1}{2^n}$. Then $E \stackrel{\text{def}}{=} \bigcap_{n\in\omega} E_n \in \Sigma_1$, $X \subseteq E$, and

$$\mu^*(X) \leq \lambda(E) \leq \inf_{n\in\omega} \mu(E_n) \leq \mu^*(X),$$

proving (20).

(21) If $E \in \Sigma_0$, then $\mu^*(\tilde{\varphi}[E]) \leq \lambda(E)$ and there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu(V) \leq \lambda(E)$.

To prove (16), assume that $E \in \Sigma_0$. By the basic definition of measure on ${}^{\omega}2$,

$$\lambda(E) = \inf\left\{\sum_{n\in\omega} \theta_0(U_{f_n}) : E \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}.$$

Hence for every $\varepsilon > 0$ there is a system $\langle f_n : n \in \omega\rangle$ such that $E \subseteq \bigcup_{n\in\omega} U_{f_n}$ and $\sum_{n\in\omega} \lambda(U_{f_n}) \leq \lambda(E) + \varepsilon$. Hence

$$\tilde{\varphi}[E] \subseteq \bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}],$$

and hence

$$\mu^*(\tilde{\varphi}[E]) \leq \mu\left(\bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}]\right) \leq \sum_{n\in\omega} \mu(\tilde{\varphi}[U_{f_n}]) = \sum_{n\in\omega} \lambda(U_{f_n}) \leq \lambda(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\mu^*(\tilde{\varphi}[E]) \leq \lambda(E)$. By (15) there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu^*(\tilde{\varphi}[E]) = \mu(V)$. So (21) holds.

(22) If $E \in \Sigma_0$, then $\tilde{\varphi}[E] \in \Sigma_1$ and $\mu(\tilde{\varphi}[E]) = \lambda(E)$.

For, by symmetry with (16) there is a $V' \in \Sigma_1$ such that $\tilde{\varphi}[{}^{\omega}2\backslash E] \subseteq V'$ and $\mu(V') \leq \lambda({}^{\omega}2\backslash E)$. Then $V \cup V' = [0,1]$ and

$$\mu(V) + \mu(V') \leq \lambda(E) + \lambda({}^{\omega}2\backslash E) = 1 \leq \mu(V \cup V') \leq \mu(V) + \mu(V').$$

It follows that $\mu(V \cap V') = 0$. In particular, $V \cap V' \cap \tilde{\varphi}[E] \in \Sigma_1$. Now $\tilde{\varphi}[{}^{\omega}2\backslash E] = \tilde{\varphi}[{}^{\omega}2]\backslash\tilde{\varphi}[E] = [0,1]\backslash\tilde{\varphi}[E] \subseteq V'$, so $[0,1]\backslash V' \subseteq \tilde{\varphi}[E]$. Hence $\tilde{\varphi}[E] = ([0,1]\backslash V') \cup (V' \cap \tilde{\varphi}[E]) = [0,1]\backslash V') \cup (V' \cap V \cap \tilde{\varphi}[E]) \in \Sigma_1$.

Now

$$\mu(\tilde{\varphi}[E]) \leq \lambda(V) \leq \lambda(E) \quad \text{and} \quad 1 - \mu(\tilde{\varphi}[E]) \leq \mu(V') \leq 1 - \lambda(E),$$

so $\mu(\tilde{\varphi}[E]) = \lambda(E)$. Thus (17) holds. □

Lemma 86. *If $E \subseteq \mathscr{P}(\Sigma_0)$, then $\tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A\in E} \tilde{\varphi}[A]$.* □

Proposition 87. $\text{add}(\text{null}_{^\omega 2}) = \text{add}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{add}(\text{null}_{^\omega 2})$, and let $E \in [\text{null}_{^\omega 2}]^\kappa$ with $\bigcup E \notin \text{null}_{^\omega 2}$. For each $A \in E$ let $A' = \tilde{\varphi}[A]$, and let $E' = \{A' : A \in E\}$. Then by the theorem, $E' \subseteq \mathscr{P}(\text{null}_{[0,1]})$. Suppose that $\bigcup E' \in \text{null}_{[0,1]}$. By the lemma,

$$\tilde{\varphi}^{-1}\left[\bigcup E'\right] = \bigcup_{B \in E'} \tilde{\varphi}^{-1}[B] = \bigcup_{A \in E} \tilde{\varphi}^{-1}[\tilde{\varphi}[A]] = \bigcup E \in \Sigma_0,$$

contradiction.

Second let $\kappa = \text{add}(\text{null}_{[0,1]})$, and let $E \in [\text{null}_{[0,1]}]^\kappa$ with $\bigcup E \notin \text{null}_{[0,1]}$. For each $A \in E$ let $A' = \tilde{\varphi}^{-1}[A]$. Thus $A' \in \text{null}_{^\omega 2}$ by the theorem. Continue as in the first case. □

Proposition 88. $\text{cov}(\text{null}_{^\omega 2}) = \text{cov}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{cov}(\text{null}_{^\omega 2})$, and let $E \in [\text{null}_{^\omega 2}]^\kappa$ with ${}^\omega 2 = \bigcup E$.

$$[0,1] = \tilde{\varphi}[{}^\omega 2] = \tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A \in E} \tilde{\varphi}[A],$$

and each $\tilde{\varphi}[A] \in \text{null}_{[0,1]}$.

The other direction is similar. □

Proposition 89. $\text{non}(\text{null}_{^\omega 2}) = \text{non}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{non}(\text{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $X \notin \text{null}_{^\omega 2}$. If $\tilde{\varphi}[X] \in \text{null}_{[0,1]}$, this is a contradiction.

The other direction is similar. □

Proposition 90. $\text{cof}(\text{null}_{^\omega 2}) = \text{cof}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{non}(\text{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $\forall A \in \text{null}_{^\omega 2} \exists B \in X[A \subseteq B]$. Let $X' = \{\tilde{\varphi}[C] : C \in X\}$. Take any $A \in \text{null}_{[0,1]}$. Then $\tilde{\varphi}^{-1}[A] \in \text{null}_{^\omega 2}$, so there is a $B \in X$ such that $\tilde{\varphi}^{-1}[A] \subseteq B$. Then $\tilde{\varphi}[\tilde{\varphi}^{-1}[A]] = A \subseteq \tilde{\varphi}[B]$.

The other direction is similar. □