Notes on
Cardinal invariants on
Boolean algebras

(Second revised edition)

(1) Problem 46 has a positive answer: $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$.

Proof. $\leq$ is clear. For $\geq$, assume that $X \subseteq A \oplus B$, $\prod X = 0$, and $\forall F \in [X]^{<\omega} \prod F \neq 0$, with $|X| = \mathfrak{p}(A \oplus B) < \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$. For each $x \in X$ write $x = \sum_{i \in m_x} a_{ix} \cdot b_{ix}$ with each $a_{ix} \in A$ and each $b_{ix} \in B$. For each $F \in [X]^{<\omega}$ let

$$C_F = \left\{ f \in \prod_{x \in X} m_x : \prod_{x \in F} (a_{f(x)x} \cdot b_{f(x)x}) \neq 0 \right\}.$$

We claim that each C_F is nonempty. For,

$$0 \neq \prod F = \prod_{x \in F} \sum_{i \in m_x} (a_{ix} \cdot b_{ix}) = \sum_{f \in \prod_{x \in F} m_x} \left(\prod_{x \in F} (a_{f(x)x} \cdot b_{f(x)x}) \right),$$

so there is an $f \in \prod_{x \in F} m_x$ such that $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x} \neq 0$. Thus any extension of f to a function in $\prod_{x \in X} m_x$ is in C_F.

Now each C_F is closed in $\prod_{x \in X} m_x$. For, suppose that $f \in \prod_{x \in X} m_x \backslash C_F$. Thus $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x}) = 0$. Then f is in the open set $\{g \in \prod_{x \in X} m_x : g \restriction F = f\}$, and this set is disjoint from C_F.

Now let $f \in \bigcap_{F \in [X]^{<\omega}} C_F$. Then $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x}) \neq 0$ for each $F \in [X]^{<\omega}$. Hence $\prod_{x \in F} a_{f(x)x} \neq 0$ for each $F \in [X]^{<\omega}$, so $\prod_{x \in X} a_{f(x)x} \neq 0$. Similarly, $\prod_{x \in X} b_{f(x)x} \neq 0$. Hence

$$0 \neq \prod_{x \in X} a_{f(x)x} \cdot \prod_{x \in X} b_{f(x)x} \leq \prod X = 0,$$

contradiction.

(2) Problem 49 has been solved by Malliaris and Shelah, who showed that $\mathfrak{p}(\mathscr{P}(\omega)/\mathrm{fin}) = \mathrm{tow}(\mathscr{P}(\omega)/\mathrm{fin})$ in ZFC. See

Malliaris, M.; Shelah, S. *Cofinality spectrum theorems in model theory, set theory, and general topology.* Publication 998 of Shelah.

(3) Problem 90 has been solved by Kunen, who showed assuming $2^{\aleph_1} = \aleph_2$ that there is an atomic BA A such that $\pi(A) = \aleph_1 < \mathrm{Irr}_{\mathrm{mm}}(A)$. See

Kunen, K. *Irredundant sets in atomic Boolean algebras.* http://arxiv.org/abs/1307.3533

(4) Problem 126 was answered, consistently: there is a model with $\mathrm{s}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) < \mathfrak{i}(\mathscr{P}(\omega)/\mathrm{fin})$. See

Cancino, J.; Guzmán, O.; Miller, A. *Irredundant generators.*

(5) Problem 149 is solved by the following result, which shows that $\mathrm{hd}_{\mathrm{m}}^{\mathrm{id}}$ is trivial:

Proposition. $\mathrm{hd}_{\mathrm{mm}}^{\mathrm{id}}(A) = \omega$ *for any infinite BA A.*

Proof. Let $\langle b_n : n \in \omega \rangle$ be a partition in $\overline{A}$. Define

$$I_n = \{a \in A : \forall m < n[a \cdot b_m = 0]\}.$$

Clearly $I_n \supseteq I_p$ if $n < p$. Suppose that $a \in \bigcap_{n\in\omega} I_n$ and $a \neq 0$. Choose $n \in \omega$ so that $a \cdot b_n \neq 0$, and choose $c \in A^+$ so that $c \leq a \cdot b_n$. Then $c \in I_{n+1}$, so $c \cdot b_n = 0$, contradiction. □

(6) Problems 156, 157 and 158 were answered by M. Hrusak. He showed that if $|X| < \min\{\pi(B \restriction a) : a \in B^+\}$ with $0, 1 \notin X$, X incomparable, then there is an element incomparable with each member of X. In particular, $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin} = 2^\omega$.

http://mathoverflow.net/questions/154708/families-of-pairwise-incomparable-subsets-of
-the-integers

We give the argument here. Let C be the subalgebra of B generated by X; then also $|C| < \min\{\pi(B \restriction a) : a \in B, a \neq 0\}$. Take any $a \in X$. Then $a \neq 0, 1$. Since $|C| < \pi(B \restriction -a)$, there is an $x \leq -a$ with $x \neq 0$ such that no nonzero element of C is below x. Similarly there is a $y \leq a$ with $y \neq 0$ such that no nonzero element of C is below y. Let $b = x + a \cdot -y$. Now $a \not\leq b$, for if $a \leq b$ then $a = b \cdot a = a \cdot -y \leq -y$, hence $y = a \cdot y = 0$, contradiction. Also, $b \not\leq a$, as otherwise $x \leq a$, hence $x = 0$, contradiction. Now suppose that $c \in X$ and $c \leq b$. Then $c \cdot -a \leq x$, $c \cdot -a \in C$, and $c \cdot -a \neq 0$, contradiction. Suppose that $c \in X$ and $b \leq c$. Then $a \cdot -y \leq c$, so $a \cdot -c \leq y$, again a contradiction. So b is incomparable with each element of X.

Also, Hrusak showed that it is consistent with ¬CH that there is a maximal tree in both $\mathscr{P}(\omega)$ and $\mathscr{P}(\omega)/\mathrm{fin}$ of size ω_1.

(7) Problem 160 has a negative solution, at least for atomless BAs. Namely, we claim that $\text{h-cof}_{\mathrm{mm}}(A) = \omega$ for any atomless BA A. For, let $\langle a_i : i \in \omega \rangle$ be a system of pairwise disjoint nonzero elements of A, and for each $i \in \omega$ let $\langle b^i_j : j \in \omega \rangle$ be a system of pairwise disjoint nonzero elements less than a_i. We consider the following sequence

$$\begin{array}{l} -a_0, -a_1, -a_2, \ldots \text{(rank 0)} \\ -a_1 + b^1_0, -a_2 + b^2_0, -a_3 + b^3_0 \ldots \text{(rank 1)} \\ -a_2 + b^2_0 + b^2_1, -a_3 + b^3_0 + b^3_1, -a_4 + b^4_0 + b^4_1 \ldots \text{(rank 2)} \\ \ldots\ldots \end{array}$$

This gives an h-cof sequence of length ω^2. Suppose that c is adjoined at the end. Then c has rank ω, so it includes cofinally many sets $-a_n$. Since $-a_n + -a_m = 1$ for $n \neq m$, it follows that $c = 1$, contradiction.