Leon Henkin and cylindric algebras
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Cylindric algebras are abstract algebras which stand in the same relationship to first-order logic as Boolean algebras do to sentential logic. There are two ways of passing from logic to cylindric algebras. For the first, we are given a first-order language $\mathscr{L}$ and a set Γ of sentences in $\mathscr{L}$. We assume that $\mathscr{L}$ has the sequence $v_0, v_1, \ldots$ of individual variables. We define an equivalence relation $\equiv$ on the set of formulas of $\mathscr{L}$ by defining $\varphi \equiv \psi$ iff $\Gamma \vdash \varphi \leftrightarrow \psi$. Then it is easy to see that there are the following operations on the set A of $\equiv$-classes:

$$[\varphi] + [\psi] = [\varphi \vee \psi]; \quad [\varphi] \cdot [\psi] = [\varphi \wedge \psi];$$
$$-[\varphi] = [\neg\varphi]; \quad c_i[\varphi] = [\exists v_i \varphi].$$

Then the following structure is a cylindric algebra:

$$\langle A, +, \cdot, -, [\neg(v_0 = v_0)], [v_0 = v_0], c_i, [v_i = v_j]\rangle_{i,j\in\omega}.$$

For the second method of obtaining a cylindric algebra, we suppose that a set A is given. We consider the following unary operations C_i of cylindrification acting upon subsets of ${}^{\omega}A$ (the set of infinite sequences of elements of A):

$$C_i X = \{a \in {}^{\omega}A : \exists b \in X[a_j = b_j \text{ for all } j \neq i]\}.$$

Then the following structure is a cylindric set algebra: $\langle B, \cup, \cap, -, \emptyset, {}^{\omega}A, C_i, D_{ij}\rangle_{i,j\in\omega}$, where B is a collection of subsets of ${}^{\omega}A$ closed under the operations $\cup$, $\cap$, $-$ (with $-X = {}^{\omega}A \backslash X$ for any $X \subseteq {}^{\omega}A$), and with $\emptyset$, ${}^{\omega}A$ and D_{ij} as members, where $D_{ij} = \{a \in {}^{\omega}A : a_i = a_j\}$.

Tarski and his students F. B. Thompson and L. H. Chin introduced an abstract notion of cylindric algebra which encompasses both of these cases. For any ordinal number α, a cylindric algebra of dimension α is an algebra of the form $\overline{A} = (A, +, \cdot, -, 0, 1, c_\xi, d_{\xi\eta})_{\xi,\eta<\alpha}$ such that the following conditions hold:

(1) $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra.
(2) $c_\xi 0 = 0$.
(3) $x + c_\xi x = c_\xi x$.
(4) $c_\xi(x \cdot c_\xi y) = c_\xi x \cdot c_\xi y$.
(5) $c_\xi c_\eta x = c_\eta c_\xi x$.
(6) $d_{\xi\xi} = 1$.
(7) If $\xi \neq \eta, \rho$, then $d_{\eta\rho} = c_\xi(d_{\eta\xi} \cdot d_{\xi\rho}$.
(8) If $\xi \neq h$, then $c_\xi(d_{\xi\eta} \cdot x) \cdot c_\xi(d_{\xi\eta} \cdot -x) = 0$.

Historical remarks on the development of cylindric algebras up to the time of Henkin's work on them can be found in [61]. The development went via the relation algebras of Tarski and the projective algebras of Everett and Ulam.

The work of Leon Henkin concerning cylindric algebra can be divided into these parts: on the algebraic theory of them, the theory of set algebras, representation theorems, construction of non-representable algebra, and applications to logic. Many of the publications of Henkin concerning cylindric algebras are devoted to exposition without proofs. Detailed proofs of most of his results are found in the two volumes [71] and [85], written jointly with Monk and Tarski.

Algebraic theory

The purely algebraic theory of cylindric algebras, exclusive of set algebras and representation theory, is fully developed in [71]. The parts of this theory due at mainly to Henkin are as follows.

If $\overline{A}$ is a CA_α and $\Gamma = \{\xi(0), \ldots, \xi(m-1)\}$ is a finite subset of α, then we define $c_{(\Gamma)}a = c_{\xi(0)} \cdots c_{\xi(m-1)}a$. This does not depend on the order of $\xi(0), \ldots, \xi(m-1)$, by axiom (5). An element a is *rectangular* iff $c_{(\Gamma)}a \cdot c_{(\Delta)}a = c_{(\Gamma\cap\Delta)}a$ for any finite subsets Γ, Δ of α. This notion was first introduced in [56]. Elementary properties of the notion are given in section 1.10 of [71]. Their use in representation theory is described below.

The dimension set Δx of an element x of a CA_α is the collection of all $\xi < \alpha$ such that $c_\xi x \neq x$. The CA_ω's obtained from first-order theories as above have the property that the dimension sets are always finite. A CA_α is *locally finite* iff Δx is always finite. This notion is due to Tarski. Henkin introduced the following generalization. A CA_α is *dimension complemented* iff $\Delta x \neq \alpha$ for all x. Algebraic properties of these two notions are worked on in section 1.11 of [71]. Both notions are important in representation theory. In [73] Henkin proved that every locally finite CA_α is isomorphic to one of the cylindric algebras described at the beginning of this article, an algebra of formulas modulo some theory in the language.

If $\overline{A}$ is a CA_α and $a \in A$, then the *relativization* of $\overline{A}$ to a is the structure

$$\overline{A} \restriction a = \langle A \restriction a, +', \cdot', -', 0', 1', c'_\xi, d'_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where $A \restriction a = \{x \in A : x \leq a\}$, $x +' y = x + y$ and $x \cdot' y = x \cdot y$ for any $x, y \in A \restriction a$, $-'x = a \cdot -x$ for any $x \in A \restriction a$, $0' = 0$, $1' = a$, $c'_\xi x = c_\xi x \cdot a$ for any $x \in A \restriction a$ and any $\xi < \alpha$, and $d'_{\xi\eta} = d_{\xi\eta} \cdot a$ for any $\xi, \eta < \alpha$. In general the relativization is not itself a CA_α. Algebraic properties of relativizations are developed in section 2.2 of [71]. This is a notion which Henkin worked on thoroughly. It is interesting in its own right, and is also useful in constructing non-representable cylindric algebras. In [75], written jointly by Henkin and his student Diane Resek, some simple equations are shown to characterize the class Cr_2 of two-dimensional relativized cylindric algebras. It is also shown there that the class Cr_3 is not closed under subalgebras. Additional results are stated without proof.

Given a CA_α $\overline{A}$ and an ordinal $\beta < \alpha$ we can associate the *β-reduct* of $\overline{A}$, which is the algebra $\langle A, +, \cdot, -, 0, 1, c_\xi, d_{\xi\eta}\rangle_{\xi,\eta<\beta}$. We say that $\overline{B}$ is *neatly embedded* in $\overline{A}$ provided that $\overline{B}$ is a subalgebra of the β-reduct of $\overline{A}$, and $c_\xi b = b$ for all $b \in B$ and $\xi \in [\beta, \alpha)$. Algebraic properties of reducts and neat embeddings are explored in section 2.6 of [71]. Neat embeddings play a prominent role in representation theory.

The duality theory of Boolean algebras can be adapted to cylindric algebras as follows. For a CA_α $\overline{A}$ we associate the following structure, called the *cannonical embedding algebra* $\mathrm{Em}A$, where M is the collection of all maximal ideals of the Boolean part of $\overline{A}$:

$$\langle \mathscr{P}(M), \cup, \cap, -, \emptyset, M, c_\xi, d_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where $-x = M\backslash x$ for any $x \subseteq M$, for any $x \subseteq M$ and $\xi < \alpha$ we define

$$c_\xi x = \{J \in M : c_\xi^{-1} x c[J] = \emptyset\} \cup \bigcup_{I\in x}\{J \in M : c_\xi^{-1}[J] \subseteq I\},$$

and for any $\xi, \eta < \alpha$ we define

$$d_{\xi\eta} = \{I \in M : d_{\xi\eta} \notin I\}.$$

The *canonical embedding function* $\mathrm{em}_{\overline{A}}$ is defined by $\mathrm{em}_{\overline{A}}(a) = \{I \in M : a \notin I\}$. The algebraic theory of canonical embedding algebras is developed in section 2.7 of [71]. In particular, $\mathrm{Em}\overline{A}$ is always a CA_α, and $rmem_{\overline{A}}$ is an isomorphic embedding of $\overline{A}$ into $\mathrm{Em}\overline{A}$. To show that $\mathrm{Em}\overline{A}$ is a CA_α one of course has to check the axioms for a CA_α. This procedure can be generalized to Boolean algebras with operators, and this has been carried out by Jónsson and Tarski. The question of extending equations valid in a Boolean algebra with operators to is canonical embedding algebra is difficult. Henkin in [70] contribiuted to answering this question.

The CA_α's $\mathrm{Em}\overline{A}$ are complete and atomic. This gives rise to another way of defining the class of cylindric algebras. A *cylindric atom structure* is a relational structure $\langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ such that the following conditions hold:

(1) T_ξ is an equivalence relation on B.
(2) $T\xi|T_\eta = T_\eta|T_\xi$.
(3) $E_{\xi\eta} = T_\mu[E_{\xi\mu} \cap E_{\mu\eta}]$ if $\mu \neq \xi, \eta$.
(4) If $\xi \neq \eta$ and $b, c \in E_{\xi\eta}$, then $bT_\xi c$ iff $b = c$.

Given a cylindric atom structure $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ we define its *complex algebra* $Cm(\overline{B}$ to be the algebra

$$\langle \mathscr{P}(B), \cup, \cap, -, \emptyset, B, c_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where $-x = B\backslash x$ for any $x \subseteq B$ and $c_\xi x = T_\xi[x]$ for any $\xi < \alpha$. Then $Cm(\overline{B}$ is a complete atomic CA_α, and every complete and atomic CA_α can be obtained in this way. Hence any CA_α is a subalgebra of $Cm(\overline{B}$ for some cylindric atom structure $\overline{B}$. The details of this correspondence are worked out in section 2.7 of [71].

Set algebras

The notion of a cylindric set algebra given in the introduction can be generalized as follows. An algebra $\overline{A}$ is a *cylindric-relativized set algebra of dimension* α iff there is a nonempty set U and a set $V \subseteq {}^\alpha U$ such that $\overline{A}$ has the form

$$\langle A, \cup, \cap, -, \emptyset, V, c_\xi, d_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where A is a collection of subsets of V closed under $\cup$, $\cap$, $-$ (with $-a = V\backslash a$), c_ξ, with

$$c_\xi a = \{y \in V : \exists x \in a[x_\eta = y_\eta \text{ for all } \eta \neq \xi]\}$$

and with $d_{\xi\eta} \in A$, where $d_{\xi\eta} = \{y \in V : y_\xi = y_\eta\}$.

In general such algebras do not satisfy the axioms for cylindric algebras. However, the following special cases do.

With $V = {}^\alpha U$, giving *cylindric set algebras.*

With $V = \bigcup_{i\in I} {}^\alpha Z_i$, where $\langle Z_i : i \in I\rangle$ is a system of nonempty pairwise disjoint sets, giving *generalized cylindric set algebras.* These set algebras were first introduced in [55a].

With $V = {}^\alpha W^{(p)}$, where $p \in {}^\alpha U$ and ${}^\alpha W^{(p)} = \{x \in {}^\alpha W : \{\xi < \alpha : x_\xi \neq p_\xi\}$ is finite, giving *weak cylindric set algebras.*

With $V = \bigcup_{i \in I} {}^\alpha W_i^{(p_i)}$ with ${}^\alpha W_i^{(p_i)} \cap {}^\alpha W_j^{(p_j)} = \emptyset$ for $i \neq j$, giving *generalized weak cylindric set algebras.*

It turns out that generalized cylindric set algebras and generalized weak cylindric set algebras are the natural algebras for representation theory; a CA_α is *representable* iff it is isomorphic to one of these. The theory of the various kinds of set algebras is described in section 3.1 of [85]; see also [81]. Some of the results were proved earlier in [74].

Representation theorems

That every infinite dimensional locally finite CA_α is representable is due to Tarski. In [85] this theorem is proved by an algebraic adaptation of Henkin's proof of the completeness theorem for first-order logic. In fact, call an element x of a CA_α ξ-thin iff there is an $\eta \neq \xi$ such that $x \cdot c_\xi(d_{\xi\eta} \cdot x) \leq d_{\xi\eta}$ and $c_\xi x = 1$. And call a CA_α $\overline{A}$ *rich* iff for every nonzero $y \in A$ such that $\Delta y \subseteq 1$ there is a 0-thin element x such that $x \cdot c_0 y \leq y$. The main technical lemma which implies the above representation theorem runs as follows:

If $2 \leq \alpha$ and $\overline{A}$ is a simple rich locally finite CA_α satisfying the equality

$$c_\xi(x \cdot y \cdot c_\eta(x \cdot -y)) \cdot -c_\eta(c_\xi x \cdot -d_{\xi\eta}) = 0$$

for all distinct $\xi, \eta < \alpha$ and all $x, y \in A$, then $\overline{A}$ is representable.

From this lemma one can derive using algebraic results and facts about set algebras the following additional representation theorems due to Henkin:

For $\alpha \geq 2$, a CA_α is representable iff it can be neatly embedded in a $\mathrm{CA}_{\alpha+\omega}$. This was first stated, for finite α, in [56]. For arbitrary α it was stated in [61].

Every dimension-complemented CA_α of infinite dimension is representable. This was first stated without proof in [55a].

By a direct proof given in [85] we have the following representation theorem of Henkin, first stated in [56]:

For $\alpha \geq 2$, a CA_α is representable iff it can be embedded in an atomic CA_α in which all atoms are rectangular.

A special representation theorem due to Henkin runs as follows.

Suppose that $\overline{A}$ is a CA_α, and the subalgebra of $\overline{A}$ generated by $\{d_{\xi,\eta} : \xi, \eta < \alpha\}$ is simple. Suppose that there is a positive integer m such that

$$c_0 \cdots c_{m-1} \left(\prod_{i,j<m} -d_{ij} \right) = 0.$$

Then $\overline{A}$ is representable.

It is easy to prove that every CA_0 and every CA_1 is representable. This is no longer true for CA_2's, but Henkin proved that one only needs to add two equations to obtain representablity:

A CA_2 $\overline{A}$ *is representable iff the following two equations hold in* $\overline{A}$*:*

$$c_1(x \cdot y \cdot c_0(x \cdot -y)) \cdot -c_0(c_1 x \cdot -d_{01}) = 0;$$
$$c_0(x \cdot y \cdot c_1(x \cdot -y)) \cdot -c_1(c_0 x \cdot -d_{10}) = 1;$$

Many of these representation theorems can be found in [86].

Non-representable cylindric algebras

It turns out that not every cylindric algebra is representable. Henkin invented three methods of constructing non-representable cylindric algebras, described in section 3.2 of [85].

Permutation models. Let U be a nonempty set, and consider the cylindric algebra $\overline{A}$ of all subsets of ${}^\alpha U$. Every permutation f of U extends in a natural way to an automorphism $\tilde{f}$ of $\overline{A}$. If H is a subgroup of the group of all permutations of U, then one can consider the set $\{a \in A : \tilde{f}(a) = a \text{ for all } f \in H\}$. This set forms a subalgebra $\mathrm{fix}_H(\overline{A})$ of $\overline{A}$. By choosing U and H suitably and taking a relativization of $\mathrm{fix}_H(\overline{A})$ one can obtain a non-representable cylindric algebra. This is carried out in [85] to show that the following inequality (which can be written as an equation) holds in every representable CA_α with $\alpha \geq 3$ but fails in a permutation model:

$$c_0 x \cdot c_1 y \cdot c_2 z \leq c_0 c_1 c_2 [c_2(c_1 x \cdot c_0 y) \cdot c_1(c_2 x \cdot c_0 z) \cdot c_0(c_2 y \cdot c_1 z)].$$

Dilation. While permutation models take a relativization of a subalgebra of some cylindric algebra, dilation does the opposite: starting with an algebra, one adds atoms. More precisely, let $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ be a cylindric atom structure. Suppose that $a \in {}^\alpha B$ satisfies the following conditions:

$[a_\xi]_{Y_\eta} \cap [a_\eta]_{T_\xi} \neq 0$ for all $\xi, \eta < \alpha$.
$a_\mu \notin E_{\xi\eta}$ for distinct ordinals ξ, η, μ.

Suppose that u is some object not in B. Then we form a new relational structure $\overline{B}' = \langle B', T'_\xi, E'_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ by setting

$B' = B \cup \{u\}$,

For any $\xi < \alpha$, T'_ξ is an equivalence relation on B' such that $T'_\xi \cap (B \times B) = T_\xi$, and for any $b \in B$, $bT'_\xi u$ iff $bT_\xi a_\xi$.

$E'_{\xi\eta} = E_{\xi\eta}$ for distinct $\xi, \eta < \alpha$, and $E_{\xi\xi} = B'$ for any $\xi < \alpha$.

By a suitable choice of a one obtains in this way a cylindric atom structure whose associated cylindric algebra is nonrepresentable. This is done in [85] to show that the following equation holds in every representable CA_α with $\alpha \geq 3$ but fails in some dilation model: $x;(y;z) = (x;y);z$, where in general $u;v = c_2(c_1(d_{12} \cdot c_2 u) \cdot c_0(d_{02} \cdot c_2 v))$.

Twisting. Roughly speaking this method consists of selecting two members x, y of a cylindric atom structure together with an index $\xi < \alpha$ and redefining the equivalence relation E_ξ using x and y. Formally we are given a cylindric atom structure $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$, two elements $x, y \in B$, an index $\xi < \alpha$ such that not$(xT_\xi y)$, and two partitions $[x]_{T_\xi} = X_0 \cup X_1$ and $[y]_{T_\xi} = Y_0 \cup Y_1$ such that the following conditions hold, where $M = [x]_{T_\xi} \cup [y]_{T_\xi}$:

(1) If $\eta \neq \xi$ and $(a, b) \in (M \times M) \cap T_\eta$ and $a \neq b$, then $(a, b) \in (X_0 \times Y_0) \cup (Y_0 \times X_0) \cup (X_1 \times Y_1) \cup (Y_1 \times X_1)$.

(2) If $\eta \neq \xi$ and $a \in M$, then there is a $b \in M\backslash\{a\}$ such that $aT_\eta b$.

(3) If $i \in \{0, 1\}$ and $\eta, \nu < \alpha$, then $X_i \cap E_{\xi\eta} \cap E_{\xi\nu} \neq 0$ iff $Y_i \cap E_{\xi\eta} \cap E_{\xi\nu} \neq 0$.

Then a new structure $\overline{B}' = \langle B, T'_\eta, E_{\eta\nu}\rangle_{\eta,\nu<\alpha}$ is defined as follows: $T'_\eta = T_\eta$ if $\eta \neq \xi$, while T_ξ is the equivalence relation on B with equivalence classes $[z]_{T_\xi}$ for $z \notin M$, along with $X_0 \cup Y_1$ and $X_1 \cup Y_0$.

It is shown in [85] that $\overline{B}'$ is a cylindric atom structure. This is used to show that the following equation holds in every representable CA_α but fails in some twisting model:

$$c_2(d_{20} \cdot c_0(d_{01} \cdot c_1(d_{12} \cdot x))) = c_2(d_{21} \cdot c_1(d_{01} \cdot c_0(d_{02} \cdot x))).$$

Applications to logic

In [67] Henkin considers first-order logic with only finitely many variables. In the case of just two variables x and y, he proves that the formula

$$\exists x(x = y \wedge \exists yGxy) \rightarrow \forall x(x = y \rightarrow \exists yGxy)$$

is universally valid but not derivable from the natural axioms (restricted to two variables). Here G is a binary relation symbol. The non-derivablity is proved using a modified cylindric set algebra. This example suggests adding all formulas of the following forms to the axioms for two-variable logic:

$$\text{(1)} \qquad \begin{aligned} &\exists x(x = y \wedge \varphi) \rightarrow \forall x(x = y \rightarrow \varphi)\\ &\exists y(x = y \wedge \varphi) \rightarrow \forall y(x = y \rightarrow \varphi) \end{aligned}$$

Henkin shows, again using a modified cylindric set algebra, that this axiom system is also incomplete; the following universally valid formula is not provable in the expanded axiom system:

$$\begin{aligned} &\exists xFx \wedge \forall x\forall y[Fx \wedge Fy \rightarrow x = y] \rightarrow\\ &\qquad [\exists x(Fx \wedge Gxy) \leftrightarrow \forall x(Fx \leftrightarrow Gxy)] \end{aligned}$$

An analysis of this situation leads to adding the following formulas to the axioms:

$$\text{(2)} \qquad \begin{aligned} &\exists x\forall y(\varphi \leftrightarrow y = x) \rightarrow [\exists y(\varphi \wedge \psi) \leftrightarrow \forall y(\varphi \rightarrow \psi)] \quad \text{with } x \text{ not free in } \varphi\\ &\exists y\forall x(\varphi \leftrightarrow x = y) \rightarrow [\exists x(\varphi \wedge \psi) \leftrightarrow \forall y(\varphi \rightarrow \psi)] \quad \text{with } y \text{ not free in } \varphi \end{aligned}$$

But again the resulting axiom system is not complete. By another modified cylindric set algebra Henkin shows that the following formula is universally valid but not derivable in this axiom system:

$$\exists xGxy \leftrightarrow \exists x(x = y \wedge \exists yGyx).$$

Finally, adding the following axioms results in a complete axiom system:

$$\exists x\varphi \leftrightarrow \exists x(x = y \wedge \exists y\varphi^r)$$
$$\exists y\varphi \leftrightarrow \exists y(y = x \wedge \exists y\varphi^r)$$

where φ^r is recursively defined by interchanging x and y if φ is atomic, with $(\neg\varphi)^r = \neg\varphi^r$, $(\varphi \vee \psi)^r = \varphi^r \vee \psi^r$, $(\exists x\varphi)^r = \exists y(x = y \wedge \exists x\varphi)$, and $(\exists y\varphi)^r = \exists x(x = y \wedge \exists y\varphi)$. The proof of completeness of the resulting axiom system is rather involved, but is completely carried out.

It is shown that the above axioms do not suffice for logic with three variables.

In [68], Henkin translates the notion of relativization of a cylindric algebra, described above, into a purely logical framework. Namely, given a first-order language $\mathscr{L}$ and a formula π of $\mathscr{L}$ (with no restriction on the number of free variables of π) one associates with each formula φ of $\mathscr{L}$ its relativization φ^π as follows: $\varphi^\pi = \varphi$ for φ atomic; $(\neg\varphi)^\pi = \neg\varphi^\pi$; $(\varphi \wedge \psi)^\pi = \varphi^\pi \wedge \psi^\pi$, and $(\exists x\varphi)^\pi = \exists x(\pi \wedge \varphi^\pi)$. The main theorem of [68] is as follows. Consider first-order logic with variables in the list $v_0, \ldots, v_n, \ldots$ with $n < \alpha$, where $\alpha \leq \omega$. A natural set of axioms for logic with these variables is explicitly described. Let π be a formula with free variables among $v_0, \ldots, v_{n-1}$. Then a set Δ of formulas having to do with relativization are described, namely all formulas of one of the following two forms:

$$\forall v_0 \ldots \forall v_m[(\exists v_i \exists v_j \varphi)^\pi \rightarrow (\exists v_j \exists v_i \varphi)^\pi],$$

where all free variables of φ are among $v_0, \ldots, v_m$;

$$\forall v_0 \ldots \forall v_m(\pi \rightarrow \pi'),$$

where there are $i, j < \alpha$ such that π' is obtained from π by replacing all free occurrences of v_i in π by free occurrences of v_j, and the free variables of $\pi \rightarrow \pi'$ are among $v_0, \ldots, v_m$.

The theorem is then that $\Gamma \vdash \varphi$ implies that $\Delta \cup \Gamma \vdash \varphi^\pi$.

This theorem is used, along with a suitable (ordinary) model to show that a certain sentence involving only three variables cannot be proved from the logical axioms. The sentence expresses that if a function has at most two elements in its domain, then it has at most two elements in its range.

Publications of Henkin concerning cylindric algebras

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