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Towers and maximal chains in Boolean algebras

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Abstract. A *tower* in a Boolean algebra A is a strictly increasing sequence, of regular length, of elements of A with supremum 1. We consider the following functions:

$$\mathfrak{t}_{\text{spect}}(A) = \{|T| : T \text{ is a tower in } A\};$$
$$\text{length}_{\text{spect}}(A) = \{|C| : C \text{ is a maximal chain in } A\}.$$

These are called the *spectrum* of towers of A and the *spectrum* of maximal chains of A, respectively. Our main results are (1) for any nonempty set K of regular cardinals there is an atomless Boolean algebra A such that $\mathfrak{t}_{\text{spect}}(A) = K$; (2) under GCH, an analogous result holds for maximal chains. Note that towers do not exist in some Boolean algebras, for example in the finite-cofinite algebra on an uncountable set.

1. Introduction

Our set-theoretical notation is standard, with some possible exceptions, as follows. If α and β are ordinals, then $[\alpha, \beta]_{\text{card}}$ is the collection of all cardinals κ such that $\alpha \leq \kappa \leq \beta$; similarly $[\alpha, \beta]_{\text{reg}}$ for the collection of all regular cardinals in this interval; and similarly for other intervals (half open, rays, etc.).

We follow Koppelberg [6] for Boolean algebraic notation. In several places we use the following construction. Let $\langle A_i : i \in I\rangle$ be a system of Boolean algebras, with I infinite. The *weak product* $\prod^{\text{w}}_{i\in I} A_i$ consists of all members x of the full product such that one of the two sets

$$\{i \in I : x_i \neq 0\} \quad \text{or} \quad \{i \in I : x_i \neq 1\}$$

is finite; this set, is called the *support* of x, is uniquely determined by x and is denoted by supp(x); x is called *of type* I or II respectively.

If L is a linear order, then Intalg(L) is the interval algebra over L (perhaps after adjoining a first element to L).

Unless otherwise indicated, all algebras are assumed to be atomless.

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Towers. Towers in Boolean algebras have been considered in McKenzie, Monk [10] and Monk [12], [13], [14]. For the important Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ the function $\mathfrak{t}_{\mathrm{mm}}$ has been extensively studied (most facts being consistency results); see for example Blass [1].

We first indicate what happens to towers under products of Boolean algebras.

Theorem 1.1. $\mathfrak{t}_{\mathrm{spect}}(A \times B) = \mathfrak{t}_{\mathrm{spect}}(A) \cup \mathfrak{t}_{\mathrm{spect}}(B)$.

Proof. $\supseteq$ is clear. For $\subseteq$, suppose that $\langle (a_\alpha, b_\alpha) : \alpha < \kappa \rangle$ is strictly increasing with sum $(1,1)$, where κ is regular. Then either $\langle a_\alpha : \alpha < \kappa \rangle$ or $\langle b_\alpha : \alpha < \kappa \rangle$ is not eventually constant. This gives rise to a strictly increasing subsequence with supremum 1 of one or the other of these sequences, and this subsequence must have length κ since κ is regular. $\square$

Theorem 1.2. *If $\langle A_i : i \in I \rangle$ is an infinite system of atomless Boolean algebras, then*

$$\mathfrak{t}_{\mathrm{spect}}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right) = \begin{cases} \{\omega\} \cup \bigcup_{i \in I} \mathfrak{t}_{\mathrm{spect}}(A_i) \textit{if } |I| = \omega, \\ \bigcup_{i\in I} \mathfrak{t}_{\mathrm{spect}}(A_i) \textit{if } |I| > \omega. \end{cases}$$

Proof. Clearly $\mathfrak{t}_{\mathrm{spect}}(A_j) \subseteq \mathfrak{t}_{\mathrm{spect}}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right)$ for each $j \in I$. Now suppose that $|I| = \omega$; without loss of generality, $I = \omega$. The following is a tower of length ω:

$$\langle 0, 0, \ldots \rangle, \langle 1, 0, 0, \ldots \rangle, \langle 1, 1, 0, 0, \ldots \rangle, \ldots$$

Thus $\omega \in \mathfrak{t}_{\mathrm{spect}}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right)$ when $|I| = \omega$. So far we have proved $\supseteq$.

Now for the other direction, suppose that $\langle x_\alpha : \alpha < \kappa \rangle$ is a tower in $\prod_{i\in I}^{\mathrm{w}} A_i$, and $\kappa > \omega$ if $|I| = \omega$.

Claim: There is an $\alpha < \kappa$ such that x_α has type II.

In fact, assume otherwise. Then $\mathrm{supp}(x_\alpha) \subseteq \mathrm{supp}(x_\beta)$ if $\alpha < \beta$. Now either $\kappa = \omega$ and $|I| > \omega$, or $\kappa > \omega$, and a clear contradiction is reached in either case. So the claim holds.

By the claim, we may assume that x_α is of type II for all $\alpha < \kappa$. For $\alpha < \beta$ we have $\mathrm{supp}(x_\alpha) \supseteq \mathrm{supp}(x_\beta)$. So there is a single finite set P such that $\mathrm{supp}(x_\alpha) = P$ for every sufficiently large α. Now the argument in the proof of Theorem 1.1 gives a tower in some A_i of length κ. $\square$

Theorem 1.3. *If $\langle A_i : i \in I \rangle$ is a system of atomless Boolean algebras, with I infinite, then*

$$\mathfrak{t}_{\mathrm{spect}}\left(\prod_{i\in I} A_i\right) = [\omega, |I|]_{\mathrm{reg}} \cup \bigcup_{i\in I} \mathfrak{t}_{\mathrm{spect}}(A_i).$$

Proof. Clearly $\bigcup_{i\in I} \mathfrak{t}_{\mathrm{spect}}(A_i) \subseteq \mathfrak{t}_{\mathrm{spect}}\left(\prod_{i\in I} A_i\right)$. Now suppose that $\lambda \in [\omega, |I|]_{\mathrm{reg}}$. Write $I = J \cup K$ with $J \cap K = \emptyset$ and $|J| = \lambda$. Say $J = \{j_\xi : \xi < \lambda\}$, with $j_\xi \neq j_\eta$

for $\xi \neq \eta$. Now we define x_α for $\alpha < \lambda$ by setting, for each $i \in I$,

$$x_\alpha(i) = \begin{cases} 0 & \text{if } i = j_\beta \text{ for some } \beta > \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $\langle x_\alpha : \alpha < \lambda \rangle$ is a tower in $\prod_{i \in I} A_i$. Hence $[\kappa, |I|]_{\text{reg}} \subseteq \mathfrak{t}_{\text{spect}}\big(\prod_{i \in I} A_i\big)$.

Thus $\supseteq$ in the proposition holds. Now suppose $\kappa \in \mathfrak{t}_{\text{spect}}\big(\prod_{i \in I} A_i\big)$, and $\kappa \notin [\kappa, |I|]_{\text{card}}$. Let $\langle x_\alpha : \alpha < \kappa \rangle$ be a tower in $\prod_{i \in I} A_i$. Then $\langle x_\alpha(i) : \alpha < \kappa \rangle$ is an increasing sequence in A_i, for each $i \in I$. Suppose that for every $i \in I$ there is an $\alpha_i < \kappa$ such that $x_\beta(i) = x_{\alpha_i}(i)$ for all $\beta > \alpha_i$. Since κ is regular and $|I| < \kappa$, we have $\sup_{i \in I} \alpha_i < \kappa$. For any $\beta, \gamma > \sup_{i \in I} \alpha_i < \kappa$ we have $x_\beta = x_\gamma$, a contradiction. Thus there is an $i \in I$ such that the sequence $\langle x_\alpha(i) : \alpha < \kappa \rangle$ has a strictly increasing subsequence of type κ, so that $\kappa \in \mathfrak{t}_{\text{spect}}(A_i)$. □

The above results indicate some ways of building up a bigger spectrum from smaller ones. The following result indicates a direct determination of a small spectrum.

Theorem 1.4. *For each regular cardinal κ there is an atomless Boolean algebra A such that $\mathfrak{t}_{\text{spect}}(A) = \{\kappa\}$.*

Proof. (For an alternate proof, see Lemma 1.9 below.) By Hausdorff [4], let L be a dense linear order such that:

(1) L has cofinality and coinitiality κ.

(2) Each element of L has character (κ, κ).

(3) Each gap in L has character (λ, μ) with $\lambda \neq \mu$.

Then by Theorem 41 of Monk [13] the theorem follows. □

Corollary 1.5. *If M is a nonempty set of regular cardinals, then there is an atomless Boolean algebra A such that $\mathfrak{t}_{\text{spect}}(A) = M$.*

Proof. According to Theorem 1.3, it suffices to take a system $\langle B_i : i \in I \rangle$ of Boolean algebras with I uncountable and $M = \bigcup_{i \in I} \mathfrak{t}_{\text{spect}}(B_i)$ and let $A = \prod^{\text{w}}_{i \in I} B_i$. The existence of such a system follows from Theorem 1.4. □

We continue with additional remarks on $\mathfrak{t}_{\text{spect}}$.

Proposition 1.6. $\mathfrak{t}_{\text{spect}}(\mathcal{P}(X)) = [\omega, |X|]_{\text{reg}}$.

Proof. $\subseteq$ is clear. Now let κ be any regular cardinal such that $\omega \leq \kappa \leq |X|$. We can write $X = \bigcup_{\alpha < \kappa} Y_\alpha$ with $\langle Y_\alpha : \alpha < \kappa \rangle$ a strictly increasing sequence of subsets of X. This shows that $\kappa \in \mathfrak{t}_{\text{spect}}(\mathcal{P}(X))$. □

We now consider the question: what is the relationship between the spectra $\mathfrak{t}_{\text{spect}}(A)$ and $\mathfrak{t}_{\text{spect}}(B)$ when $A \leq B$?

Lemma 1.7. *Let A be any infinite Boolean algebra, and let $\mathcal{S}$ be the Stone isomorphism of A onto the Boolean algebra of all clopen subsets of* $\mathrm{Ult}(A)$. *Then no tower in $\mathcal{S}[A]$ remains a tower in $\mathcal{P}(\mathrm{Ult}(A))$.*

Proof. Suppose that $\langle a_\alpha : \alpha < \kappa \rangle$ is a tower in A, with κ a regular cardinal. Let $X = \bigcup_{\alpha<\kappa} \mathcal{S}(a_\alpha)$. By compactness, $X \neq \mathrm{Ult}(A)$. □

Lemma 1.8. *If A is an atomless Boolean algebra, then there is an atomless Boolean algebra B such that A is a subalgebra of B and no tower in A is a tower in B.*

Proof. Applying 1.7, just embed $\mathcal{P}(\mathrm{Ult}(A))$ into an atomless Boolean algebra. □

Lemma 1.9. *Let A be an atomless Boolean algebra, and let κ be a regular cardinal. Then $A \leq B$ for some atomless Boolean algebra B such that $\mathfrak{t}_{\mathrm{spect}}(B) = \{\kappa\}$.*

Proof. We define a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ of Boolean algebras as follows. Let $C_0 = A$. If C_α has been defined, let $C_\alpha \subseteq C_{\alpha+1}$, where $C_{\alpha+1}$ is obtained from C_α by applying Lemma 1.8. For $\alpha < \kappa$ limit, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Finally, let $B = \bigcup_{\alpha<\kappa} C_\alpha$. We claim that B is as desired.

Since it is easily seen that every atomless Boolean algebra has a tower, it suffices to show that B has no tower of length different from κ. Suppose that $\langle b_\alpha : \alpha < \lambda \rangle$ is a tower in B, where $\lambda \neq \kappa$ is a regular cardinal. For each $\alpha < \lambda$ there is a $\xi_\alpha < \kappa$ such that $b_\alpha \in C_{\xi_\alpha}$.

First suppose that $\lambda < \kappa$. Choose $\beta < \kappa$ such that $\xi_\alpha < \beta$ for all $\alpha < \lambda$. Then $\langle b_\alpha : \alpha < \lambda \rangle$ is a tower in C_β. By construction and Lemma 1.8 it is not a tower in $C_{\beta+1}$ and hence also not in B, contradiction.

Second suppose that $\kappa < \lambda$. There is an $\beta < \kappa$ such that $\xi_\alpha = \beta$ for λ many $\alpha < \lambda$. Then $\langle b_\alpha : \alpha < \lambda \rangle$ has a cofinal subsequence in C_β; this gives a tower in C_β, and a contradiction follows as above. □

Theorem 1.10. *Let K and L be nonempty sets of regular cardinals. Then there are atomless Boolean algebras A, B such that $A \leq B$, $\mathfrak{t}_{\mathrm{spect}}(A) = K$, and $\mathfrak{t}_{\mathrm{spect}}(B) = L$.*

Proof. Choose A such that $\mathfrak{t}_{\mathrm{spect}}(A) = K$, by Corollary 1.5. Let κ be the least member of L. Let C be such that $A \leq C$ and $\mathfrak{t}_{\mathrm{spect}}(C) = \{\kappa\}$, by Lemma 1.9. Choose D such that $\mathfrak{t}_{\mathrm{spect}}(D) = L$, by Corollary 1.5. Now $B \stackrel{\mathrm{def}}{=} C \times D$ is as desired, by Theorem 1.1. □

Next we consider free products. An element of $A \oplus B$ can be written in the form

$$\sum_{i<m} a_i \cdot b_i$$

where $0 \neq a_i \in A$, $0 \neq b_i \in B$, $b_i \cdot b_j = 0$ for $i \neq j$, and m minimum subject to these conditions. Note that then $a_i \neq a_j$ for $i \neq j$. We call this the *standard form*, and m is called the *length* of the element.

Lemma 1.11. *Suppose that κ is uncountable and regular, and neither A nor B has a tower of length κ. Suppose that $m \in \omega$. Also suppose that $0 \neq u \in A$ and $0 \neq v \in B$. Then there is no tower in $(A \oplus B) \restriction (u \cdot v)$ of length κ such that each element has length m.*

Proof. The proof uses some ideas in the proof of 1.3.1 of McKenzie, Monk [9].

Recall from Theorem 1.1 that also neither $A \restriction u$ nor $B \restriction v$ has a tower of length κ. We prove the lemma by induction on m. First consider the case $m = 1$. Suppose that $\langle c_\alpha : \alpha < \kappa \rangle$ is a tower in $(A \oplus B) \restriction (u \cdot v)$ with each c_α of length 1, say $c_\alpha = a_\alpha \cdot b_\alpha$ with $0 \neq a_\alpha \in A$ and $0 \neq b_\alpha \in B$. Then $a_\alpha \leq a_\beta \leq u$, and $b_\alpha \leq b_\beta \leq v$ if $\alpha < \beta$. Since $c_\alpha < u \cdot v$, it follows that $a_\alpha < u$ or $b_\alpha < v$ for each $\alpha < \kappa$. Hence either $a_\alpha < u$ for all $\alpha < \kappa$ or $b_\alpha < v$ for all $\alpha < \kappa$. But then we get a subsequence which is a tower in $A \restriction u$ or in $B \restriction v$ of length κ, contradiction.

Now assume that $m > 1$ and the lemma holds for $m' < m$, for any u, v as indicated. Note that this implies that there is no tower of length κ with each element of length less than m, even if the lengths vary from term to term of the tower. (Consider subsequences.) Assume that there is a tower $\langle c_\alpha : \alpha < \kappa \rangle$ in $(A \oplus B) \restriction (u \cdot v)$ with each element of length m; say a standard form for c_α is

$$c_\alpha = \sum_{i<m} a_{\alpha,i} \cdot b_{\alpha,i},$$

with assumptions as in the definition of standard form. Clearly if $\alpha < \beta$ then $\sum_{i<m} b_{\alpha,i} \leq \sum_{i<m} b_{\beta,i}$. It follows that from some point on each such sum is equal to v, and so we may assume that $\sum_{i<m} b_{\alpha,i} = v$ for all $\alpha < \kappa$.

(1) If $\alpha < \beta$ and $i, j < m$ and $b_{\alpha,i} \cdot b_{\beta,j} \neq 0$, then $a_{\alpha,i} \leq a_{\beta,j}$.

In fact, $a_{\alpha,i} \cdot b_{\alpha,i} \cdot \prod_{k<m}(-a_{\beta,k} + -b_{\beta,k}) = 0$; multiplying by $b_{\beta,j}$ gives the conclusion of (1). Now we claim

(2) There is a $g \in {}^\kappa m$ such that $a_{\alpha,g(\alpha)} \leq a_{\beta,g(\beta)}$ whenever $\alpha < \beta$.

This follows by an easy compactness argument from (1). For completeness we sketch this argument. If $\alpha < \beta$, then the set $\{g \in {}^\kappa m : a_{\alpha,g(\alpha)} \leq a_{\beta,g(\beta)}\}$ is closed in ${}^\kappa m$, where m has the discrete topology and then ${}^\kappa m$ has the product topology. If $\alpha_0 < \cdots < \alpha_m < \kappa$, then the set

$$\{g \in {}^\kappa m : a_{\alpha_0,g(\alpha_0)} \leq \cdots \leq a_{\alpha_m,g(\alpha_m)}\} = \bigcap_{i<j\leq m} \{g \in {}^\kappa m : a_{\alpha_i,g(\alpha_i)} \leq a_{\alpha_j,g(\alpha_j)}\}$$

is nonempty; this follows from (1) using an easy inductive argument. Now (2) follows by compactness.

From (2), there is a $\gamma < \kappa$ such that $a_{\alpha,g(\alpha)} = a_{\beta,g(\beta)}$ whenever $\gamma \leq \alpha < \beta$, so we may assume that $a_{\alpha,g(\alpha)} = a_{\beta,g(\beta)}$ for all $\alpha, \beta < \kappa$; relabeling, we may assume that $a_{\alpha,0} = a_0$ does not depend on α.

(3) We may assume that $a_{\alpha,i} \leq a_0$ for all $\alpha < \kappa$ and $i < m$.
For, this is obvious if $a_0 = u$. Suppose that $a_0 \neq u$. Now $c_\alpha \cdot -a_0$ has length less than m, and $\langle c_\alpha \cdot -a_0 : \alpha < \kappa \rangle$ is a tower in $(A \oplus B) \restriction (u \cdot -a_0 \cdot v)$, so by the inductive hypothesis there is a $\gamma < \kappa$ such that $c_\alpha \cdot -a_0 = c_\beta \cdot -a_0$ whenever $\gamma \leq \alpha \leq \beta$. We may assume that $c_\alpha \cdot -a_0 = c_\beta \cdot -a_0$ for all $\alpha, \beta < \kappa$. Then $\langle c_\alpha \cdot a_0 : \alpha < \kappa \rangle$ is a tower in $(A \oplus B) \restriction (a_0 \cdot v)$. So again (3) holds.

(4) If $\alpha < \beta$, $i < m$, and $b_{\alpha,0} \cdot b_{\beta,i} \neq 0$, then $i = 0$.
In fact, if $i > 0$ then by (1) we get $a_0 \leq a_{\beta,i} \leq a_0$, so $a_0 = a_{\beta,i}$, contradicting a remark in the definition of standard form.

From (4) it follows that if $\alpha < \beta$, then $b_{\alpha,0} \leq b_{\beta,0}$. Thus we may assume that $b_{\alpha,0} = b_0$ does not depend on α. Since $m > 1$, we must have $b_0 \neq v$. But now $\langle c_\alpha \cdot -b_0 : \alpha < \kappa \rangle$ is a tower in $(A \oplus B) \restriction (u \cdot v \cdot -b_0)$ with each element of length less than m, contradicting the inductive hypothesis. □

Lemma 1.12. *Suppose that κ is uncountable and regular, and neither A nor B has a tower of length κ. Then also $A \oplus B$ does not have a tower of length κ.*

Proof. Suppose to the contrary that $\langle c_\alpha : \alpha < \kappa \rangle$ is a tower in $A \oplus B$. For each $\alpha < \kappa$ let m_α be the length of c_α. Then there is an n such that $m_\alpha = n$ for κ many α's. This contradicts Lemma 1.11. □

Lemma 1.13. *Suppose that neither A nor B has a tower of length ω. Then also $A \oplus B$ does not have a tower of length ω.*

Proof. It is convenient to consider the dual of towers for this proof. So, suppose that $\langle c_i : i < \omega \rangle$ is a strictly decreasing sequence of elements of $A \oplus B$ with meet 0; we want to get a contradiction. For each $i < \omega$ write

$$c_i = \sum_{k < m_i} a_{i,k} \cdot b_{i,k}$$

with $0 \neq a_{i,k} \in A$, $0 \neq b_{i,k} \in B$, and $b_{i,k} \cdot b_{i,l} = 0$ if $k \neq l$; we do not assume that m_i is minimal. Taking intersections, we may assume in addition that:

(1) If $i < j$ and $k < m_j$, then there is an $l < m_i$ such that $b_{j,k} \leq b_{i,l}$.
Next we claim:

(2) $\exists k < m_0 \forall i > 0 \exists l < m_i (b_{i,l} \leq b_{0,k})$.
In fact, otherwise for every $k < m_0$ there is an $i(k) > 0$ such that for every $l < m_{i(k)}$ we have $b_{i(k),l} \not\leq b_{0,k}$, and hence by (1) $b_{i(k),l} \cdot b_{0,k} = 0$. So $c_{i(k)} \cdot b_{0,k} = 0$. If $j > i(k)$ for all $k < m_0$, we then get $c_j \cdot c_0 = 0$, contradiction. So (2) holds. We fix such a k.

(3) For every $i > 0$ there is an $l \in \prod_{0 < j \leq i} m_j$ such that $b_{0,k} \geq b_{1,l(1)} \geq \cdots \geq b_{i,l(i)}$.
In fact, let $l(0) = k$. By (2), choose $l(i) < m_i$ such that $b_{i,l(i)} \leq b_{0,k}$. Then for $0 < j < i$, by (1) choose $l(j) < m_j$ such that $b_{i,l(i)} \leq b_{j,l(j)}$. Now suppose that

$0 \leq j < n \leq i$. Then $b_{i,l(i)} \leq b_{j,l(j)} \cdot b_{n,l(n)}$, so by (1) we get $b_{n,l(n)} \leq b_{j,l(j)}$. So (3) holds.

By (3) and König's tree lemma we get $l \in \prod_{0<j<\omega} m_j$ such that $b_{0,k} \geq b_{1,l(1)} \geq \cdots \geq b_{i,l(i)} \geq \cdots$. Since B has no tower of length ω, it follows that the meet of all $b_{i,l(i)}$'s is not zero; say that $d \neq 0$ and $d \leq b_{i,l(i)}$ for every positive integer i. Now if $0 < i < j$, then $0 \neq c_j \cdot d \leq c_i \cdot d$, and for any $i > 0$ we have $c_i \cdot d = a_{l(i)} \cdot d$. Hence $0 = \prod_{0<i<\omega}(c_i \cdot d) = \prod_{0<i<\omega}(a_{l(i)} \cdot d)$, and it follows that $\langle a_{l(i)} : 0 < i < \omega \rangle$ is a decreasing sequence with meet 0, and hence some subsequence is a (dual) tower in A, a contradiction. □

Theorem 1.14. $\mathfrak{t}_{\mathrm{spect}}(A \oplus B) = \mathfrak{t}_{\mathrm{spect}}(A) \cup \mathfrak{t}_{\mathrm{spect}}(B)$.

Proof. Clearly any tower in A remains a tower in $A \oplus B$, and similarly for B. Thus $\supseteq$ holds. $\subseteq$ holds by Lemmas 1.12 and 1.13. □

Now we consider infinite free products.

Theorem 1.15. *Suppose that $\langle A_i : i \in I \rangle$ is a system of atomless Boolean algebras, with I also infinite. Then*

$$\mathfrak{t}_{\mathrm{spect}}\Big(\bigoplus_{i\in I} A_i\Big) = \{\omega\} \cup \bigcup_{i\in I} \mathfrak{t}_{\mathrm{spect}}(A_i).$$

Proof. Clearly $\mathfrak{t}_{\mathrm{spect}}(A_j) \subseteq \mathfrak{t}_{\mathrm{spect}}\big(\oplus_{i\in I} A_i\big)$ for each $j \in I$. To show that $\omega \in \mathfrak{t}_{\mathrm{spect}}\big(\oplus_{i\in I} A_i\big)$, first choose $k \in {}^{\omega}I$ one-one, and then choose $a_i \in A_{k(i)}$ with $0 < a_i < 1$ for each $i < \omega$. We set $b_i = \prod_{j\leq i} a_j$ for each $i < \omega$. Clearly $b_i > b_{i+1}$ for all $i < \omega$. We claim that the meet of all b_i's is 0; this will prove that $\omega \in \mathfrak{t}_{\mathrm{spect}}\big(\oplus_{i\in I} A_i\big)$. Suppose to the contrary that $c \neq 0$ is below all b_i's. Now c is in $\oplus_{j\in K} A_j$ for some finite $K \subseteq I$. Choose $j < \omega$ such that $k(j) \notin K$. Let f be a homomorphism from $A_{k(j)}$ into $\oplus_{i\in I} A_i$ such that $f(a_j) = 0$. Let h be an endomorphism of $\oplus_{i\in I} A_i$ such that h is the identity on each A_l with $l \neq k(i)$, while h agrees with f on $A_{k(i)}$. Then $h(c) = c$ but $h(b_i) = 0$, a contradiction.

Thus we have shown $\supseteq$. For the other direction, let $\langle c_\alpha : \alpha < \kappa \rangle$ be a tower in $\oplus_{i\in I} A_i$ with κ regular and uncountable, but suppose that $\kappa \notin \bigcup_{i\in I} \mathfrak{t}_{\mathrm{spect}}(A_i)$. For each $\alpha < \kappa$ there is a finite subset K_α of I such that $c_\alpha \in \oplus_{i\in K_\alpha} A_i$. We may assume that $\langle K_\alpha : \alpha < \kappa \rangle$ forms a Δ-system, say with root L. Then we can write each c_α in the following form:

$$c_\alpha = \sum_{s<m_\alpha} a_{\alpha,s} \cdot b_{\alpha,s},$$

where $0 \neq a_{\alpha,s} \in \oplus_{i\in L} A_i$, $0 \neq b_{\alpha,s} \in \oplus_{i\in K_\alpha \backslash L} A_i$, and $b_{\alpha,s} \cdot b_{\alpha,t} = 0$ for $s \neq t$. We may assume that $m_\alpha = m$ does not depend on α. Now if $\alpha < \beta$, then $b_{\alpha,0} \cdot \prod_{s<m} -b_{\beta,s} = 0$. Since $(K_\alpha \backslash L) \cap (K_\beta \backslash L) = \emptyset$, it then follows that $\sum_{s<m} b_{\beta,s} = 1$.

Now

$$(1) \qquad c_\alpha \cdot -c_\beta = \sum_{s<m} a_{\alpha,s} \cdot b_{\alpha,s} \cdot \prod_{t<m} (-a_{\beta,t} + -b_{\beta,t}).$$

Hence

(2) If $\alpha < \beta$, $s < m$, and $t < m$, then $a_{\alpha,s} \leq a_{\beta,t}$.

This follows from (1) by multiplying by $b_{\beta,t}$. In particular, $a_{\alpha,s} \leq a_{\beta,s}$ whenever $\alpha < \beta < \kappa$ and $s < m$. Hence we may assume that $a_{\alpha,s} = a_{\beta,s}$ whenever $\alpha < \beta < \kappa$ and $s < m$. By (2) we then see that $a_{\alpha,s}$ is always the same, say a. But then each c_α is equal to a too, a contradiction. □

2. Maximal chains

Maximal chains in Boolean algebras have been discussed before in Day [3], Jakubík [5], and Koppelberg [7]. It turns out that an important role is played here by saturated Boolean algebras (in the model-theoretic sense). For background on this, see Comfort, Negrepontis [2] and Monk [11].

Lemma 2.1. *Suppose that X is a dense maximal chain in A, and (Y,Z) is a gap in X, i.e., $X = Y \cup Z$, $Y < Z$, Y has no greatest element, and Z has no least element. Then for any Boolean algebra B, the set $\{(y,0) : y \in Y\} \cup \{(z,1) : z \in Z\}$ is a maximal chain in $A \times B$.*

Proof. Clearly $\{(y,0) : y \in Y\} \cup \{(z,1) : z \in Z\}$ is a chain. Suppose that it is not maximal; say that (a,b) is an element of $A \times B$ not in this set, but comparable with each member of it. Then a is comparable with each member of X, and hence $a \in X$. If $a \in Y$, then $b \neq 0$; choose $y \in Y$ with $a < y$. Then (a,b) is not comparable with $(y,0)$, a contradiction. A similar contradiction is reached if $a \in Z$. □

It is easy to see that a Boolean algebra A is incomplete $\iff$ some maximal chain has a gap. This will be used below. Note that if κ is an uncountable regular cardinal and A is κ-saturated, then A is incomplete. For example, any strictly increasing sequence of elements of A fails to have a supremum.

Lemma 2.2. *Suppose that K is a nonempty set of regular cardinals, A_λ is λ-saturated for each $\lambda \in K$, X is a chain in $\prod^{\mathrm{w}}_{\lambda\in K} A_\lambda$, $\kappa \in K$, W is a convex subset of X such that $|W| < \kappa$, and there are elements $p, x \in W$ with $p < x$, and for all $z \in W$, if $z < x$ then $z(\kappa) < x(\kappa)$. Then X is not maximal.*

Proof. Choose $s \in A_\kappa$ such that $z(\kappa) < s < x(\kappa)$ for all $z \in W$ such that $z < x$. Define $t(\mu) = x(\mu)$ for all $\mu \in K \backslash \{\kappa\}$, and $t(\kappa) = s$. We claim that $t \notin X$, and $X \cup \{t\}$ is a chain (as desired).

For, suppose that $z \in X$. If $z < p$, then $z < x$, and $z(\kappa) \leq p(\kappa) < s = t(\kappa)$; so $z < t$. If $z \in W$ and $z < x$, then $z(\kappa) < s = t(\kappa)$, so $z < t$. If $x \leq z$, then clearly $t < z$. □

Theorem 2.3. *If K is a nonempty finite set of cardinals, and if A_κ is κ-saturated for each $\kappa \in K$, then*

$$\mathrm{length}_{\mathrm{spect}}\left(\prod_{\kappa\in K} A_\kappa\right) \cap \mathrm{Reg} \subseteq \bigcup_{\kappa\in K}[\kappa, |A_\kappa|]$$

and

$$[\kappa, |A_\kappa|] \cap \mathrm{length}_{\mathrm{spect}}\left(\prod_{\kappa\in K} A_\kappa\right) \cap \mathrm{Reg} \neq \emptyset \textit{ for all } \kappa \in K.$$

Proof. For brevity let $B = \prod_{\kappa\in K} A_\kappa$. First suppose that $\lambda \in \mathrm{length}_{\mathrm{spect}}(B) \cap \mathrm{Reg}\setminus \bigcup_{\kappa\in K}[\kappa, |A_\kappa|]$. Let $L = \{\kappa \in K : \kappa < \lambda\}$, $M = \{\kappa \in K : \lambda < \kappa\}$, and $C = \prod_{\kappa\in L} A_\kappa$. So $|C| < \lambda$. Let X be a maximal chain in B such that $|X| = \lambda$. Then

$$X = \bigcup_{y\in C}\{x \in X : x \restriction L = y\},$$

so by the regularity of λ, there is a $y \in C$ such that $X' \stackrel{\mathrm{def}}{=} \{x \in X : x \restriction L = y\}$ has size λ. Note that X' is a convex subset of X. Fix $p, x \in X'$ with $p < x$.

(1) There is a $\kappa \in M$ such that for all $z \in X'$, if $z < x$ then $z(\kappa) < x(\kappa)$.

In fact, otherwise for each $\kappa \in M$ choose $z^{(\kappa)} \in X'$ such that $z^{(\kappa)} < x$ and $z^{(\kappa)}(\kappa) = x(\kappa)$. Let $w = \sup_{\kappa\in M} z^{(\kappa)}$. Then $w < x$. But clearly $w = x$, a contradiction. So (1) holds.

Now by (1) and Lemma 2.2, X is not maximal, a contradiction. For the second assertion of the theorem, take a maximal chain X in A_κ which has a gap. Thus $\kappa \leq |X| \leq |A_\kappa|$, and by Lemma 2.1 the assertion follows. □

Corollary 2.4 (GCH)**.** *If K is a nonempty finite set of regular cardinals, and if A_κ is the κ-saturated Boolean algebra of size κ for each $\kappa \in K$, then*

$$\mathrm{length}_{\mathrm{spect}}(\prod_{\kappa\in K} A_\kappa) \cap \mathrm{Reg} = K.$$

Theorem 2.5. *If K is a nonempty infinite set of cardinals, and if A_κ is κ-saturated for each $\kappa \in K$, then*

$$\mathrm{length}_{\mathrm{spect}}\left(\prod_{\kappa\in K}^{\mathrm{w}} A_\kappa\right) \cap \mathrm{Reg} \subseteq \bigcup_{\kappa\in K}[\kappa, |A_\kappa|]$$

and

$$[\kappa, |A_\kappa|] \cap \mathrm{length}_{\mathrm{spect}}\left(\prod_{\kappa\in K}^{\mathrm{w}} A_\kappa\right) \cap \mathrm{Reg} \neq \emptyset \textit{ for all } \kappa \in K.$$

Proof. For brevity let $B = \prod^{\mathrm{w}}_{\kappa\in K} A_\kappa$. First suppose that $\lambda \in \mathrm{length}_{\mathrm{spect}}(B) \cap \mathrm{Reg}\backslash \bigcup_{\kappa\in K}[\kappa, |A_\kappa|]$. Let X be a maximal chain in B of size λ. First we show that $\lambda > \omega$. For, suppose that $\lambda = \omega$.

Case 1. There exist distinct $x, p \in X$ of type I. Say $p < x$. Let

$$X' = \{z \in X : \{\mu \in K : z(\mu) \neq 0\} \subseteq \mathrm{supp}(x)\}.$$

So X' is a convex subset of X, and $p, x \in X'$.

(1) There is a $\kappa \in \mathrm{supp}(X)$ such that for all $z \in X'$, if $z < x$ then $z(\kappa) < x(\kappa)$. In fact, otherwise for each $\kappa \in \mathrm{supp}(x)$ choose $z^{(\kappa)} \in X'$ such that $z^{(\kappa)} < x$ and $z^{(\kappa)}(\kappa) = x(\kappa)$. Then $\sup_{\kappa\in\mathrm{supp}(x)} z^{(\kappa)} < x$. But clearly this supremum is x itself, a contradiction. So (1) holds, and then Lemma 2.2 gives a contradiction.

Case 2. Otherwise, since λ is infinite, there are distinct $x, p \in X'$ of type II; say $p < x$. Let $X' = \{z \in X : \{\mu \in K : z(\mu) \neq 1\} \subseteq \mathrm{supp}(p)\}$. Then X' is a convex subset of X, and $p, x \in X'$.

(2) There is a $\kappa \in \mathrm{supp}(p)$ such that for all $z \in X'$, if $z < x$ then $z(\kappa) < x(\kappa)$. In fact, otherwise for each $\kappa \in \mathrm{supp}(p)$ choose $z^{(\kappa)} \in X'$ such that $z^{(\kappa)} < x$ and $z^{(\kappa)}(\kappa) = x(\kappa)$. Let $q = \max(p, \sup_{\kappa\in\mathrm{supp}(p)} z^{(\kappa)})$. So $q < x$. But $q(\kappa) \geq p(\kappa) = 1$ for all $\kappa \notin \mathrm{supp}(p)$, hence $q(\kappa) = 1 = x(\kappa)$, and for $\kappa \in \mathrm{supp}(p)$ we have $x(\kappa) \geq q(\kappa) \geq z^{(\kappa)}(\kappa) = x(\kappa)$, so also $x(\kappa) = q(\kappa)$. Hence $q = x$, a contradiction. So (2) holds, and Lemma 2.2 gives a contradiction.

Thus we have shown that $\lambda > \omega$. We now consider two cases.

Case 1. $X' \overset{\mathrm{def}}{=} \{x \in X : x \text{ is of type I}\}$ has size λ. Then $x, y \in X'$ and $x < y$ imply that $\mathrm{supp}(x) \subseteq \mathrm{supp}(y)$. Define $x \equiv y \iff x, y \in X'$ and $\mathrm{supp}(x) = \mathrm{supp}(y)$. Clearly there are countably many equivalence classes. So, because $\lambda > \omega$, some equivalence class X'' has λ elements. Let N be the finite subset of K such that $\mathrm{supp}(x) = N$ for each $x \in X''$. Let $P = N \cap \lambda$. Let $C = \prod_{\kappa\in P} A_\kappa$. So $|C| < \lambda$. Now we can write

$$X'' = \bigcup_{y\in C} \{x \in X'' : x \restriction P = y\},$$

so there is a $y \in C$ such that $Y \overset{\mathrm{def}}{=} \{x \in X'' : x \restriction P = y\}$ has size λ. Clearly Y is a convex subset of X. Choose $p, x \in Y$ such that $p < x$. Let $Q = N\backslash\lambda$.

(3) $\exists \kappa \in Q \forall z \in Y [z < x \rightarrow z(\kappa) < x(\kappa)]$.

In fact, otherwise for all $\kappa \in Q$ choose $z^{(\kappa)} \in Y$ such that $z^{(\kappa)} < x$ and $z^{(\kappa)}(\kappa) < x(\kappa)$. Let $q = \sup_{\kappa\in Q} z^{(\kappa)}$. So $q < x$. If $\kappa \in Q$, then $q(\kappa) = x(\kappa)$. If $\kappa \in P$, then $q(\kappa) = y(\kappa) = x(\kappa)$. If $\kappa \notin N$, then $q(\kappa) = 0 = x(\kappa)$. So $q = x$, a contradiction. So (3) holds, and this gives a contradiction.

Case 2. $X' \overset{\mathrm{def}}{=} \{x \in X : x \text{ is of type II}\}$ has size λ. This can be treated similarly to Case 1.

The second assertion of the theorem is proved as before. □

Corollary 2.6 (GCH)**.** *If K is a nonempty infinite set of regular cardinals, and if A_κ is the κ-saturated Boolean algebra of size κ for each $\kappa \in K$, then*

$$\mathrm{length}_{\mathrm{spect}}\Big(\prod_{\kappa\in K}^{\mathrm{w}} A_\kappa\Big) \cap \mathrm{Reg} = K.$$

The results in this paper leave some questions open. Concerning towers, it would be natural to delete the assumption that towers have regular length and try to characterize the resulting spectrum. Clearly no longer do arbitrary nonempty sets work.

For maximal chains, one can make the following conjecture.

Conjecture. For each nonempty set K of infinite cardinals there is a Boolean algebra A such that $\mathrm{length}_{\mathrm{spect}}(A) = K$.

Note that in this conjecture, K is allowed to have singular members, and GCH is not assumed. A first step towards this conjecture might be to construct a Boolean algebra A, using any set-theoretical assumptions, in which $\mathrm{length}_{\mathrm{spect}}(A) = \{\aleph_\omega\}$; or to construct a Boolean algebra A in ZFC such that one has $\mathrm{length}_{\mathrm{spect}}(A) = \{\aleph_1\}$.

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