Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Annals of Pure and Applied Logic 126 (2004) 335–348

ANNALS OF
PURE AND
APPLIED LOGIC

www.elsevier.com/locate/apal

The spectrum of maximal independent subsets of a Boolean algebra

J. Donald Monk

University of Colorado at Boulder, Campus Box 395, Boulder CO 80309, USA

Abstract

Recall that a subset X of a Boolean algebra (BA) A is *independent* if for any two finite disjoint subsets F, G of X we have

$$\prod_{x\in F} x \prod_{y\in G} -y \neq 0.$$

The *independence* of a BA A, denoted by $\mathrm{Ind}(A)$, is the supremum of cardinalities of its independent subsets. We can also consider the maximal independent subsets. The smallest size of an infinite maximal independent subset is the cardinal invariant $\mathfrak{i}(A)$, well known in the case $A=\mathscr{P}(\omega)/\mathrm{fin}$. In this article we consider the collection of all cardinalities of infinite maximal independent subsets of a BA A; we call this set the *spectrum of infinite maximal independent subsets*, denoted by $\mathrm{Spind}(A)$. Note that infinite maximal independent subsets exist in any BA which is not superatomic.

The main result is that any set of infinite cardinals can occur as $\mathrm{Spind}(A)$ for some infinite BA A. Beyond this we give results concerning the way that $\mathrm{Spind}(A)$ changes under various algebraic operations. However, the basic components of most algebras that we deal with are free algebras.

MSC: primary 06E05; secondary 03E35

Keywords: Maximal independence

For notation and facts about Boolean algebras, see [2]. Information on $\mathrm{Ind}(A)$ can be found in Monk [5]. The invariant $\mathfrak{i}(A)$ for Boolean algebras in general is treated in [6].

For any element a of a BA we let $a1=a$ and $a0=-a$. The free BA on κ many generators is denoted by $\mathrm{Fr}(\kappa)$. If A is freely generated by X and $a\in A$, then there is

E-mail address: don.monk@colorado.edu (J.D. Monk).

0168-0072/$ - see front matter
doi:10.1016/j.apal.2003.11.028

a unique smallest finite subset F of X such that a is in the subalgebra of A generated by F. We denote this set F by $\mathrm{supp}(a)$. $\mathrm{supp}(a)$ is called the *support* of a. Note that $\mathrm{supp}(0)=\mathrm{supp}(1)=\emptyset$.

1. Elementary results

The following obvious proposition indicates the relationship of the set $\mathrm{Spind}(A)$ to $\mathrm{Ind}(A)$ and $\mathfrak{i}(A)$. If $\kappa\leqslant\lambda$ are cardinals, $[\kappa,\lambda]_{\mathrm{card}}$ denotes the set of all cardinals μ such that $\kappa\leqslant\mu\leqslant\lambda$.

Proposition 1.1. *Assume that A is atomless.*

(i) $\mathrm{Spind}(A)\subseteq[\omega,\mathrm{Ind}(A)]_{\mathrm{card}}$.
(ii) $\sup(\mathrm{Spind}(A))=\mathrm{Ind}(A)$, *and* $\mathrm{Ind}(A)$ *is attained iff* $\mathrm{Ind}(A)\in\mathrm{Spind}(A)$.
(iii) $\mathfrak{i}(A)=\min(\mathrm{Spind}(A))$.

The following fact is used in the proof of the main result.

Lemma 1.2. $\mathrm{Spind}(A)\subseteq\mathrm{Spind}(A\times B)$.

Proof. Let X be a maximal independent subset of A. Define

$$Y=\{(a,1):a\in X\}.$$

Clearly Y is an independent subset of $A\times B$. Now suppose that $(c,d)\notin Y$.

Case 1: $c\in X$. Thus $(c,d)\neq(c,1)$, so

$$(c,d)\cdot-(c,1)=(0,0)$$

shows that $Y\cup\{(c,d)\}$ is dependent.

Case 2: $c\notin X$. Therefore there exist a finite $F\subseteq X$, an $\varepsilon\in{}^F2$, and a $\delta\in 2$, such that $\prod_{a\in F}a^{\varepsilon(a)}c^{\delta}=0$. Choose $b\in X\backslash F$. Then

$$\prod_{a\in F}(a,1)^{\varepsilon(a)}\cdot(c,d)^{\delta}\cdot-(b,1)=(0,0)$$

shows again that $Y\cup\{(c,d)\}$ is dependent. $\square$

Ralph McKenzie has shown that actually equality holds in Lemma 1.2; see [4].

2. The main theorem

Note first that if A is superatomic, then A has no infinite independent subsets, and hence $\mathrm{Spind}(A)=\emptyset$. The following lemma treats a special case of the main result.

Lemma 2.1. Spind $\left(\prod_{\lambda\in\Gamma}\mathrm{Fr}(\lambda)\right)=\Gamma$ *if* Γ *is a finite nonempty set of infinite cardinals.*

Proof. $\supseteq$ holds by Lemma 1.2. Now suppose that κ is a member of the set Spind $\left(\prod_{\lambda\in\Gamma}\mathrm{Fr}(\lambda)\right)\setminus\Gamma$; we want to get a contradiction. Say that X is maximal independent with $|X|=\kappa$. Let $\Delta=\{\lambda\in\Gamma:\kappa<\lambda\}$. For each $\lambda\in\Delta$ let b_λ be a free generator of $\mathrm{Fr}(\lambda)$ not in the support of any element x_λ for $x\in X$. Let $b_\lambda=0$ if $\lambda\in\Gamma\cap\kappa$. Then $(b_\lambda)_{\lambda\in\Gamma}\notin X$, and so there exist a finite subset F of X, an $\varepsilon\in{}^X2$, and a $\delta\in 2$ such that $\prod_{x\in F}x^{\varepsilon(x)}\cdot b^\delta=0$. In particular we must have $\prod_{x\in F}(x\restriction\Delta)^{\varepsilon(x)}\cdot(b\restriction\Delta)^\delta=0$, and hence $\prod_{x\in F}(x\restriction\Delta)^{\varepsilon(x)}=0$. Now $|\prod_{\lambda\in\Gamma\setminus\Delta}\mathrm{Fr}(\lambda)|<\kappa$, so we can choose distinct $x,y\in X\setminus F$ such that $x\restriction(\Gamma\setminus\Delta)=y\restriction(\Gamma\setminus\Delta)$. Then

$$x\cdot -y\cdot\prod_{z\in F}z^{\varepsilon(z)}=0,$$

contradiction. □

A construction which will be used several times below is the *weak product* of a system $\langle A_i : i\in I\rangle$ of BAs; by definition it is the set of all $f\in\prod_{i\in I}A_i$ such that $\{i\in I: f(i)\neq 0\}$ is finite or $\{i\in I: f(i)\neq 1\}$ is finite, and it is denoted by $\prod^{\mathrm{w}}_{i\in I}A_i$.

Theorem 2.2. *If* I *is any set of infinite cardinals, then there is a BA* A *such that* $\mathrm{Spind}(A)=I$.

Proof. By Lemma 2.1 we may assume that I is infinite. Let κ be the smallest member of I. Define

$$A=\left(\prod^{\mathrm{w}}_{\lambda\in I}\mathrm{Fr}(\lambda)\right)\oplus\mathrm{Fr}(\kappa).$$

Here $\oplus$ is the free product operation. First fix any $\lambda\in I$; we show that $\lambda\in\mathrm{Spind}(A)$. Let $\langle x_\alpha:\alpha<\lambda\rangle$ enumerate free generators of $\mathrm{Fr}(\lambda)$. For each $\alpha<\lambda$, define $y_\alpha\in\prod^{\mathrm{w}}_{\mu\in I}\mathrm{Fr}(\mu)$ by defining, for any $\mu\in I$,

$$y_\alpha(\mu)=\begin{cases}x_\alpha & \text{if } \mu=\lambda,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\{y_\alpha:\alpha<\lambda\}$ is an independent system of elements of A; extend it to a maximal independent set X.

(1) $|X|=\lambda$. In fact, suppose not. Thus $|X|>\lambda$. For each $z\in X$ write

$$z=\sum_{i<m_z}u_i^z\cdot v_i^z,$$

where $u_i^z\in\prod^{\mathrm{w}}_{\mu\in I}\mathrm{Fr}(\mu)$ and $v_i^z\in\mathrm{Fr}(\kappa)$. Clearly each $m_z\neq 0$. Let X' be a subset of X of size λ^+ such that for some n, $m_z=n$ for all $z\in X'$, and the two sequences

$$\langle u_i^z(\lambda):i<n\rangle \quad\text{and}\quad \langle v_i^z:i<n\rangle$$

do not depend on the particular $z \in X'$. Take any distinct $w, z \in X'$, and choose $\alpha < \lambda$ so that $y_\alpha \neq w, z$. Then $w \cdot -z \cdot y_\alpha = 0$, contradiction. In fact,

$$w \cdot -z \cdot y_\alpha = \left(\sum_{i<n} u_i^w \cdot v_i^w\right) \cdot \prod_{i<n}(-u_i^z + -v_i^z) \cdot y_\alpha$$
$$= \left(\sum_{\substack{i<n \\ J \subseteq n}} \left(u_i^w \cdot v_i^w \cdot \prod_{j \in J} -u_j^z \cdot \prod_{j \in n \setminus J} -v_j^z\right)\right) \cdot y_\alpha.$$

Now take any $i < n$ and $J \subseteq n$. If $i \in J$, then

$$u_i^w \cdot \prod_{j \in J} -u_j^z \cdot y_\alpha = 0$$

as desired. If $i \notin J$, then

$$v_i^w \cdot \prod_{j \in n \setminus J} -v_j^z = 0$$

as desired.

Thus (1) holds.

Now suppose that $\mu \notin I$ but X is a maximal independent subset of A of size μ; we want to get a contradiction. For each $x \in X$ write

$$x = \sum_{i<m_x} u_i^x \cdot v_i^x$$

with $u_i^x \in \prod^{\mathrm{w}}_{\lambda \in I} \mathrm{Fr}(\lambda)$, $v_i^x \in \mathrm{Fr}(\kappa)$, and $v_i^x \cdot v_j^x = 0$ for distinct i, j. Let $\langle x_\alpha : \alpha < \kappa \rangle$ be a system of free generators of $\mathrm{Fr}(\kappa)$.

(2) $\kappa < |X|$. For, suppose that $|X| < \kappa$. Choose α so that x_α is not in the support of any v_i^x. We claim that $X \cup \{x_\alpha\}$ is independent (contradiction). For, suppose that F is a finite subset of X, and $\varepsilon \in {}^F2$. Then we can write

$$\prod_{x \in F} x^{\varepsilon(x)} = \sum_{i<n} s_i \cdot t_i,$$

where each s_i is in $\prod^{\mathrm{w}}_{\lambda \in I} \mathrm{Fr}(\lambda)$ and each t_i is in the subalgebra of $\mathrm{Fr}(\kappa)$ generated by $\bigcup_{x \in F, i < m_x} \mathrm{supp}(v_i^x)$. Clearly then $\prod_{x \in F} x^{\varepsilon(x)} \cdot x_\alpha^\delta \neq 0$ for $\delta = 0, 1$, giving the indicated contradiction. So (2) holds.

(3) Suppose that x, y are distinct members of X, $m_x = m_y$, and $v_i^x = v_i^y$ for all $i < m_x$. Then

$$x \cdot -y = \sum_{i<m_x} u_i^x \cdot -u_i^y \cdot v_i^x.$$

For,

$$x\cdot -y=\left(\sum_{i<m_x} u_i^x\cdot v_i^x\right)\cdot\prod_{i<m_y}(-u_i^y+-v_i^y)$$
$$=\sum_{i<m_x}\left(u_i^x\cdot\prod_{j<m_y} v_i^x\cdot(-u_j^y+-v_j^y)\right)$$
$$=\sum_{i<m_x} u_i^x\cdot -u_i^y\cdot v_i^x.$$

So (3) holds.

Now for each $x\in X$ and $i<m_x$ let

$$\delta_{ix}=\begin{cases}0 & \text{if } \{\lambda\in I: u_i^x(\lambda)\neq 0\} \text{ is finite,}\\ 1 & \text{if } \{\lambda\in I: u_i^x(\lambda)\neq 1\} \text{ is finite}\end{cases}$$

and let $F_{ix}=\{\lambda\in I: u_i^x(\lambda)\neq\delta_{ix}\}$. Now let Y be an uncountable subset of X such that the following conditions hold:

(4) There is an n such that $m_x=n$ for all $x\in Y$.

(5) $\delta_{ix}=\delta_i$ for all $x\in Y$ and all $i<n$.

(6) $v_i^x=v_i$ for all $x\in Y$ and all $i<n$.

(7) For each $i<n$, $\langle F_{ix}: x\in Y\rangle$ is a Δ-system, say with kernel G_i.

Let $H=\bigcup_{i<n} G_i$. Then

(8) If x,s,t,w are distinct members of Y, $i<n$, and $\lambda\in I\backslash H$, then $(u_i^x\cdot u_i^s\cdot -u_i^t\cdot -u_i^w)(\lambda)=0$.

For, if $\delta_i=0$, then since $F_{ix}\cap F_{is}=G_i$ we get $(u_i^x\cdot u_i^s)(\lambda)=0$, and if $\delta_i=1$ similarly $(-u_i^t\cdot -u_i^w)(\lambda)=0$, so (8) holds.

Now for each $x\in X$ let

$$x^H=\sum_{i<m_x}(u_i^x\restriction H)\cdot v_i^x,$$

considered as a member of $(\prod_{\lambda\in H}\mathrm{Fr}(\lambda))\oplus\mathrm{Fr}(\kappa)$. Now we claim

(9) $\langle x^H: x\in X\rangle$ is independent in $(\prod_{\lambda\in H}\mathrm{Fr}(\lambda))\oplus\mathrm{Fr}(\kappa)$.

To prove this, suppose that K and L are disjoint finite subsets of X and

$$(10)\qquad \prod_{y\in K} y^H\cdot\prod_{y\in L} -y^H=0;$$

we want to get a contradiction. Let $P=\prod_{y\in K} m_y$ and $N=\{(y,i): y\in L \text{ and } i<m_y\}$. Then

$$(11)\qquad \prod_{y\in K} y^H\cdot\prod_{y\in L} -y^H$$
$$=\prod_{y\in K}\left(\sum_{i<m_y}(u_i^y\restriction H)\cdot v_i^y\right)\cdot\prod_{y\in L}\prod_{i<m_y}[(-u_i^y\restriction H)+-v_i^y]$$

$$= \sum_{\substack{f\in P\\ M\subseteq N}} \left(\prod_{y\in K} \left((u^y_{f(y)} \restriction H) \cdot v^y_{f(y)} \right) \cdot \prod_{(y,i)\in M} (-u^y_i \restriction H) \cdot \prod_{(y,i)\in N\setminus M} -v^y_i \right).$$

Now choose distinct $x,s,t,w \in Y\setminus(K\cup L)$. Then, using (3),

$$(12)\quad x\cdot s\cdot -t\cdot -w\cdot \prod_{y\in K} y\cdot \prod_{y\in L} -y$$

$$= \sum_{\substack{i<n, f\in P\\ M\subseteq N}} \Bigg(u^x_i \cdot u^s_i \cdot -u^t_i \cdot -u^w_i \cdot v_i$$

$$\cdot \prod_{y\in K} (u^y_{f(y)} \cdot v^y_{f(y)}) \cdot \prod_{(y,j)\in M} -u^y_j \cdot \prod_{(y,j)\in N\setminus M} -v^y_j \Bigg).$$

Next,

(13) If $f\in P$, $M\subseteq N$, $\lambda\in H$, and

$$\prod_{y\in K} v^y_{f(y)} \cdot \prod_{(y,j)\in N\setminus M} -v^y_j \neq 0,$$

then

$$\left(\prod_{y\in K} u^y_{f(y)} \cdot \prod_{(y,j)\in M} -u^y_j \right)(\lambda) = 0.$$

In fact, under the hypothesis of (13), using (10) and (11) we get

$$\prod_{y\in K} (u^y_{f(y)} \restriction H) \cdot \prod_{(y,j)\in M} (-u^y_j \restriction H) = 0,$$

and so the conclusion of (13) follows.

By (8), (12), and (13) we get

$$x\cdot s\cdot -t\cdot -w\cdot \prod_{y\in K} y\cdot \prod_{y\in L} -y = 0,$$

contradiction. This proves (9).

Now let $L=\{\lambda\in I : \lambda<\mu\}$ and $K=\{\lambda\in I : \mu<\lambda\}$. For each $\lambda\in H\cap K$ let $w(\lambda)$ be a free generator of $\mathrm{Fr}(\lambda)$ not in the support of $u^x_i(\lambda)$ for any $x\in X$ and $i<m_x$, and let $w(\lambda)=0$ if $\lambda\in I\setminus(H\cap K)$. Clearly $w\notin X$, so there exist a finite $M\subseteq X$, an $\varepsilon\in {}^M2$, and a $\delta\in 2$ such that $w^\delta\cdot\prod_{y\in M} y^{\varepsilon(y)} = 0$. Choose distinct $x,z,s,t\in Y\setminus M$. Now $\prod_{\lambda\in H\cap L}\lambda<\mu$, and $\kappa<\mu$, so there exist distinct $\alpha,\beta\in X\setminus(M\cup H\cup\{x,z,s,t\})$ such that $m_\alpha=m_\beta$, $v^\alpha_i=v^\beta_i$ for all $i<m_\alpha$, and for all $i<m_\alpha$ and all $\lambda\in L\cap H$ we have

$u_i^\alpha(\lambda) = u_i^\beta(\lambda)$. Now

(14) $(x \cdot z \cdot -s \cdot -t \cdot \alpha \cdot -\beta) \restriction (I \backslash (H \cap K)) = 0$.

In fact, by (8) we just need to take any $\lambda \in H \cap L$ and show that $(\alpha \cdot -\beta)(\lambda) = 0$. Now by (3) we have

$$\alpha \cdot -\beta = \sum_{i<m_\alpha} u_i^\alpha \cdot -u_i^\beta \cdot v_i^\alpha.$$

The choice of α and β now gives $(\alpha \cdot -\beta)(\lambda) = 0$. So (14) holds.

Now

$$w^\delta \cdot \prod_{y \in M} y^{\varepsilon(y)} \cdot x \cdot z \cdot -s \cdot -t \cdot \alpha \cdot -\beta = 0,$$

so by (14) and the choice of w we get

$$\prod_{y \in M} y^{\varepsilon(y)} \cdot x \cdot z \cdot -s \cdot -t \cdot \alpha \cdot -\beta = 0,$$

contradiction. □

3. Additional results on direct products of free algebras

The following characterizes Spind(A) for A a weak product of free algebras.

Proposition 3.1. *Suppose that I is an infinite set, and $\langle \lambda_i : i \in I \rangle$ is a system of infinite cardinals. Then*

$$\mathrm{Spind}\left(\prod_{i \in I}^{\mathrm{w}} \mathrm{Fr}(\lambda_i) \right) = \{\lambda_i : i \in I\} \cup \{\omega\}.$$

Proof. $\supseteq$ holds using Proposition 8 of Monk [6]. For $\subseteq$, suppose to the contrary that $\kappa \in \mathrm{Spind}(\prod_{i \in I} \mathrm{Fr}(\lambda_i))$ and $\kappa \notin \{\lambda_i : i \in I\} \cup \{\omega\}$. Let $J = \{i \in I : \lambda_i < \kappa\}$ and $L = \{i \in I : \kappa < \lambda_i\}$. By Corollary 10.4 of Monk [5], $L \neq 0$.

Let X be a maximal independent subset of $\prod_{i \in I} \mathrm{Fr}(\lambda_i)$ of size κ. Wlog for all $x \in X$ the set $F_x = \{i \in I : x(i) \neq 0\}$ is finite. Let Y be an uncountable subset of X such that $\langle F_x : x \in Y \rangle$ forms a Δ-system, say with kernel G. Obviously $G \neq 0$.

$(*)$ $\langle x \restriction G : x \in Y \rangle$ is independent in $\prod_{i \in G} \mathrm{Fr}(\lambda_i)$.

In fact, suppose that $K \in [x]^{<\omega}$ and $\varepsilon \in {}^K 2$. Choose distinct $x, z \in Y \backslash K$. Then $xz \prod_{y \in K} y^{\varepsilon(y)} \neq 0$, so $\prod_{y \in K} (y \restriction G)^{\varepsilon(y)} \neq 0$, as desired in $(*)$.

Now for each $i \in G \cap L$ let $w(i)$ be a free generator of $\mathrm{Fr}(\lambda_i)$ not in the support of any $x(i)$ with $x \in X$, and let $w(i) = 0$ if $i \in I \backslash (G \cap L)$. Clearly $w \notin X$, so there exist a finite $K \subseteq X$, an $\varepsilon \in {}^K 2$, and a $\delta \in 2$ such that $w^\delta \cdot \prod_{y \in K} y^{\varepsilon(y)} = 0$. Choose distinct $x, z \in Y \backslash K$. Now $\prod_{i \in J \cap G} \lambda_i < \kappa$, so there exist distinct $u, v \in Y \backslash \{x, z\}$ such that

$u \restriction (G \cap J) = v \restriction (G \cap J)$. Hence

$$w^\delta \cdot x \cdot z \cdot u \cdot -v \cdot \prod_{y \in K} y^{\varepsilon(y)} = 0$$

and

$$\left(x \cdot z \cdot u \cdot -v \cdot \prod_{y \in K} y^{\varepsilon(y)} \right) \restriction (I \backslash G) = 0,$$

so by the choice of w, using $(*)$, we get

$$x \cdot y \cdot u \cdot -v \cdot \prod_{y \in K} y^{\varepsilon(y)} = 0,$$

contradiction. □

Concerning arbitrary infinite products of free algebras we have the following results.

Proposition 3.2. *If* $\langle \lambda_i : i \in I \rangle$ *is a system of infinite cardinals with* $I \neq 0$, *then* $\prod_{i \in I} \mathrm{Fr}(\lambda_i)$ *has a maximal independent subset of size* $\prod_{i \in I} \lambda_i$.

Proof. This is true by Lemma 2.1 if I is finite. For I infinite, for each $i \in I$ let X_i be a set of free generators of $\mathrm{Fr}(\lambda_i)$ of size λ_i. Let Y be a finitely distinguished subset of $\prod_{i < \mathrm{cf}\kappa} X_i$ of size $|A| = \prod_{i \in I} \lambda_i$. (See [2, p. 197]) Clearly Y is independent, and $|Y| = \prod_{i \in I} \lambda_i$. That is the size of the whole product, so the Proposition follows. □

Corollary 3.3. *If* $\langle \lambda_i : i \in I \rangle$ *is a system of infinite cardinals with* I *infinite*, *then for each infinite nonempty* $J \subseteq I$ *we have* $\prod_{j \in J} \lambda_j \in \mathrm{Spind}(\prod_{i \in I} \mathrm{Fr}(\lambda_i))$.

The methods of proof for the above results give the following.

Proposition 3.4. *Suppose that* Γ *is a nonempty set of infinite cardinals*, *and* κ *is an infinite cardinal not in* Γ *such that*

$$\prod_{\substack{\lambda \in \Gamma, \\ \lambda < \kappa}} \lambda < \kappa.$$

Then $\kappa \notin \mathrm{Spind}\left(\prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda)\right)$.

Proposition 3.5. *Suppose that* Γ *is an infinite set of infinite cardinals*, κ *is a cardinal not in* Γ,

$$\kappa \leqslant \prod_{\substack{\lambda \in \Gamma, \\ \lambda < \kappa}} \lambda$$

and

$$\forall \mu < \kappa \left[\prod_{\substack{\lambda \in \Gamma, \\ \lambda < \mu}} \lambda < \kappa \right].$$

Then κ is a limit cardinal, and $\kappa \notin \mathrm{Spind}(\prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda))$.

Proof. For each $\lambda \in \Gamma$ let X_λ be a set of free generators of $\mathrm{Fr}(\lambda)$.

The two conditions clearly imply that $\sup\{\lambda \in \Gamma : \lambda < \kappa\} = \kappa$, and hence κ is a limit cardinal. Now suppose that $Y \subseteq \prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda)$ is maximal independent, with $|Y| = \kappa$. Write $Y = \{y^\beta : \beta < \kappa\}$. Now the order type of $\{\lambda \in \Gamma : \lambda < \kappa\}$ is $\leqslant \kappa$. Let $\langle \mu_\xi : \xi < \alpha \rangle$ enumerate this set in strictly increasing order. So, α is a limit ordinal $\leqslant \kappa$. We now define a member x of $\prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda)$, as follows. For any $\xi < \alpha$,

$$x_{\mu_\xi} = \begin{cases} 0 & \text{if } \xi \text{ is a limit ordinal,} \\ \text{a member of } X_{\mu_{\eta+1}} \setminus \bigcup_{\beta \leqslant \mu_\eta} \mathrm{Supp}(y^\beta_{\mu_{\eta+1}}) & \text{if } \xi = \eta + 1; \end{cases}$$

$x_\lambda \in X_\lambda \setminus \bigcup_{\beta < \kappa} \mathrm{Supp}(y^\beta_\lambda)$ for $\kappa < \lambda$. Clearly $x \notin Y$. Hence there exist a finite subset F of κ, an $\varepsilon \in {}^F 2$, and a $\delta \in 2$ such that $\prod_{\beta \in F} (y^\beta)^{\varepsilon(\beta)} x^\delta = 0$. Choose $\xi < \alpha$ such that $\beta \leqslant \mu_\xi$ for all $\beta \in F$. Now

$$\kappa \setminus F = \bigcup_{w \in \prod_{\eta \leqslant \xi} \mathrm{Fr}(\mu_\eta)} \{\beta \in \kappa \setminus F : y^\beta \upharpoonright (\Gamma \cap \mu_{\xi+1}) = w\}$$

and $\prod_{\eta \leqslant \xi} \mu_\eta < \kappa$, so there are distinct $\gamma, \delta \in \kappa \setminus F$ such that

$$y^\gamma \upharpoonright (\Gamma \cap \mu_{\xi+1}) = y^\delta \upharpoonright (\Gamma \cap \mu_{\xi+1}).$$

Hence $(y^\gamma \cdot -y^\delta) \upharpoonright (\Gamma \cap \mu_{\xi+1}) = 0$. It follows that there is a $\lambda \in \Gamma$ with $\mu_{\xi+1} \leqslant \lambda$ such that $(y^\gamma \cdot -y^\delta \cdot \prod_{\beta \in F} (y^\beta)^{\varepsilon(\beta)})_\lambda \neq 0$. So $(\prod_{\beta \in F} (y^\beta)^{\varepsilon(\beta)})_\lambda \neq 0$, but $(\prod_{\beta \in F} (y^\beta)^{\varepsilon(\beta)})_\lambda \cdot x^\delta_\lambda = 0$, contradicting the definition of x. □

Corollary 3.6. *If κ is a strong limit cardinal and $\kappa \notin \Gamma$, then κ is not in the set* $\mathrm{Spind}(\prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda))$.

Corollary 3.7. *If the order type of $\Gamma \cap \kappa$ is ω, $\sup(\Gamma \cap \kappa) = \kappa$, and $\kappa \notin \Gamma$, then $\kappa \notin$* $\mathrm{Spind}(\prod_{\lambda \in \Gamma} \mathrm{Fr}(\lambda))$.

Corollary 3.8. (GCH) $\mathrm{Spind}\left(\prod_{i < \omega} \mathrm{Fr}(\aleph_i)\right) = \{\aleph_i : i < \omega\} \cup \{\aleph_{\omega+1}\}$.

The following consistency result clarifies these results.

Proposition 3.9. *It is consistent that if $A = \prod_{\alpha < \omega_1} \mathrm{Fr}(\aleph_\alpha)$, then A has a maximal independent subset of size $\aleph_{\omega_1}$.*

Proof. Take a model in which $2^\omega = \aleph_{\omega_1}$. By Holz et al. [1, 1.6.15 (a) and exercise, 9 p. 78] we have

$$\prod_{\alpha<\omega} \aleph_\alpha = \aleph_\omega^\omega = \aleph_{\omega_1}.$$

Now apply Corollary 3.2. □

Problem 1. *Is it true that for any infinite set Γ of infinite cardinals one has*

$$\mathrm{Spind}\left(\prod_{\lambda\in\Gamma} \mathrm{Fr}(\lambda)\right) = \Gamma \cup \left\{\prod_{\lambda\in\Delta} \lambda : \Delta \subseteq \Gamma\right\}?$$

4. On free products

Theorem 4.1. *If $\langle A_i : i\in I\rangle$ is a system of BAs each with at least 4 elements, and if I is infinite, then* $\mathrm{Spind}(\oplus_{i\in I} A_i) \subseteq [|I|, \infty)_{\mathrm{card}}$. □

Corollary 4.2. *If $|A| \leqslant \kappa$, then* $\mathrm{Spind}(A \oplus \mathrm{Fr}(\kappa)) = \{\kappa\}$.

Theorem 4.3. *If $\lambda \leqslant \kappa$ for all $\lambda \in \mathrm{Spind}(A)$, then* $\mathrm{Spind}(A \oplus \mathrm{Fr}(\kappa)) = \{\kappa\}$.

Proof. Suppose that X is maximal independent, $\kappa < |X|$. For each $x \in X$ write

$$x = \sum_{i<m_x} a_{ix} \cdot b_{ix}$$

with $a_{ix} \in A$, $b_{ix} \in \mathrm{Fr}(\kappa)$, and $b_{ix} b_{jx} = 0$ for $i \neq j$. Let Y be a subset of X of size κ^+ such that $m_x = m$ is constant for $x \in Y$, and so is $\langle b_{ix} : i < m\rangle$. Now for each $i < m$, the system $\langle a_{ix} : x \in Y\rangle$ is dependent in A. So by induction we can define pairwise disjoint finite $F_i \subseteq Y$ for $i < m$ along with $\varepsilon_i \in {}^{F_i}2$ such that

$$\prod_{x\in F_i} a_{ix}^{\varepsilon_i(x)} = 0$$

for each $i<m$. Let $G = \bigcup_{i<m} F_i$. Then choose $y \in Y \backslash G$. Let $\delta = \bigcup_{i<m} \varepsilon_i$. Then for each $i<m$,

$$y \cdot b_i \cdot \prod_{x\in G} x^{\delta(x)} \leqslant y \cdot b_i \cdot \prod_{x\in F_i} a_{ix}^{\varepsilon_i(x)} = 0,$$

so

$$y \cdot \prod_{x\in G} x^{\delta(x)} = 0,$$

contradiction. □

5. Mixed products

Proposition 5.1. *Suppose that I and J are sets of infinite cardinals, with J infinite, and κ is an infinite cardinal. Assume that $\mu<\kappa<\lambda$ for all $\mu\in I$ and $\lambda\in J$. Furthermore, assume that $|J|<\kappa$.*

Then

$$\kappa \notin \text{Spind}\left(\prod_{\mu\in I}^{\text{w}} \text{Fr}(\mu) \times \prod_{\lambda\in J}^{\text{w}} \text{Fr}(\lambda)\right).$$

Proof. Suppose the contrary, and let X be a maximal independent set of size κ. Wlog for all $x\in X$ the set $F_x \stackrel{\text{def}}{=} \{\lambda\in J : x_1(\lambda)\neq 0\}$ is finite. Here $x=(x_0,x_1)$ for each $x\in \prod^{\text{w}}_{\mu\in I}\text{Fr}(\mu) \times \prod^{\text{w}}_{\lambda\in J}\text{Fr}(\lambda)$. Now

$$X = \bigcup_{G\in[J]^{<\omega}} \{x\in X : F_x = G\},$$

so we can choose $G\in[J]^{<\omega}$ such that $\{x\in X : F_x=G\}$ is infinite. Obviously $G\neq 0$.

Now for each $\lambda\in G$ let $w(\lambda)$ be a free generator of $\text{Fr}(\lambda)$ not in the support of any $x_1(\lambda)$ with $x\in X$ and $0<x_1(\lambda)<1$, and let $w(\lambda)=0$ for all $\lambda\in J\backslash G$. Clearly then $(1,w)\notin X$, so we can choose a finite $K\subseteq X$, an $\varepsilon\in {}^K2$, and a $\theta\in 2$ such that $(1,w)^\theta\prod_{y\in K} y^{\varepsilon(y)}=(0,0)$. By the choice of w we then get

(1) If $\lambda\in G$, then $\left(\prod_{y\in K} y_1^{\varepsilon(y)}\right)(\lambda)=0$.

Now fix $x\in X\backslash K$, and choose $\delta\in 2$ so that $H\stackrel{\text{def}}{=}\{\mu\in I : x_0^\delta(\mu)\neq 0\}$ is finite. Choose $v,z\in X\backslash(K\cup\{x\})$ such that $v_0\restriction H=z_0\restriction H$. It follows that

(2) $x_0^\delta v_0 - z_0 = 0$.

Next, choose $y\in X\backslash(K\cup\{x,v,z\})$ such that $F_y=G$. Then by (1) and (2) we obtain

$$y\cdot x^\delta\cdot v\cdot -z\cdot \prod_{y\in K} y_1^{\varepsilon(y)} = 0,$$

contradiction. □

Proposition 5.2. *If $\langle\lambda_\alpha : \alpha<\kappa\rangle$ and $\langle\mu_\alpha : \alpha<\nu\rangle$ are systems of infinite cardinals, with both κ and ν infinite, then*

$$\omega\in\text{Spind}\left(\left(\prod_{\alpha<\kappa}^{\text{w}}\text{Fr}(\lambda_\alpha)\right)\oplus\left(\prod_{\alpha<\nu}^{\text{w}}\text{Fr}(\mu_\alpha)\right)\right).$$

Proof. An element of a weak product is of *type* 1 it is 0 except for finitely many places; otherwise it is of *type* 2.

For each $\alpha<\kappa$ let $\langle x_{\alpha,i} : i<\omega\rangle$ be an independent system of elements of $\text{Fr}(\lambda_i)$, and for each $\alpha<\nu$ let $\langle y_{\alpha,i} : i<\omega\rangle$ be an independent system of elements of $\text{Fr}(\mu_i)$.

Now for $n\in\omega$, $\alpha<\kappa$, and $\beta<v$ we define

$$w_n(\alpha)=\begin{cases} x_{\alpha,n-\alpha-1} & \text{if } \alpha<n,\\ 1 & \text{if } \alpha=n,\\ 0 & \text{if } n<\alpha \end{cases}$$

and

$$z_n(\beta)=\begin{cases} y_{\beta,n-\beta-1} & \text{if } \beta<n,\\ 1 & \text{if } \beta=n,\\ 0 & \text{if } n<\beta. \end{cases}$$

By the proof of Proposition 8 of Monk [6], the set

$$X\stackrel{\text{def}}{=}\{w_n : n\in\omega\}\cup\{z_n : n\in\omega\}$$

is independent in the free product. We claim that it is maximal independent. For, take any element w of the free product. Then we can write

$$w=\sum_{i<m} u_i\cdot v_i, \tag{1}$$

$$-w=\sum_{i<n} u_i'\cdot v_i', \tag{2}$$

where $u_i,u_i'\in\prod^{\text{w}}_{\alpha<\kappa}\text{Fr}(\lambda_\alpha)$ and $v_i,v_i'\in\prod^{\text{w}}_{\alpha<v}\text{Fr}(\mu_\alpha)$.

Case 1: For every $i<m$, u_i is of type 1 or v_i is of type 1. Choose $k\in\omega$ such that for all $i\in\omega$, if u_i is of type 1 then $u_i(n)=0$ for each natural number $n\geqslant k$, and if v_i is of type 1 then $v_i(n)=0$ for each natural number $n\geqslant k$. Then

$$y_{k+1}\cdot\prod_{p\leqslant k}-y_p\cdot z_{k+1}\cdot\prod_{p\leqslant k}-z_p\cdot w=0,$$

as desired.

Case 2: There is an $i<\omega$ such that both u_i and v_i are of type 2. Then Case 1 applies to $-w$. □

Proposition 5.3. *If I and J are nonempty finite sets of infinite cardinals, then*

$$\mathfrak{i}\left(\left(\prod_{\alpha\in I}\text{Fr}(\lambda)\right)\oplus\left(\prod_{\lambda\in J}\text{Fr}(\lambda)\right)\right)=\max(\min(I),\min(J)).$$

Proof. Wlog $\min(|I|)\leqslant\min(|J|)$. Let $\mu=\min(I)$ and $v=\min(J)$. For all $\lambda\in I\cup J$ let $\langle x^\lambda_\alpha : \alpha<\lambda\rangle$ be a system of free generators of $\text{Fr}(\lambda)$. If $|X|<v$, clearly X is not maximal independent. Now define, for $\alpha<\mu$ and $\lambda\in I$

$$y_\alpha(\lambda)=\begin{cases} x^\mu_\alpha & \text{if } \lambda=\mu,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly define, for $\alpha < \nu$ and $\lambda \in J$

$$z_\alpha(\lambda) = \begin{cases} x_\alpha^\nu & \text{if } \lambda = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{y_\alpha : \alpha < \mu\} \cup \{z_\alpha : \alpha < \nu\}$ is independent; extend it to a maximal independent subset X. So $|X| \geqslant \nu$; suppose that $|X| > \nu$. For each $w \in X$ write

$$w = \sum_{i < m_w} u_i^w \cdot v_i^w$$

with each $u_i^w \in \prod_{\lambda \in I} \mathrm{Fr}(\lambda)$ and each $v_i^w \in \prod_{\lambda \in J} \mathrm{Fr}(\lambda)$. Let X' be a subset of X of size ν^+ such that for $w \in X'$ we have m_w constant, $\langle u_i^w : i < m_w \rangle$ constant, $\langle v_i^w : i < m_w \rangle$ constant. A contradiction is reached as in the proof of 2.1. □

Familiar arguments also given.

Proposition 5.4. *If I and J are sets of infinite cardinals, with J finite then*

$$\mathfrak{i}\left(\left(\prod_{\lambda \in I}^{w} \mathrm{Fr}(\lambda)\right) \oplus \left(\prod_{\lambda \in J} \mathrm{Fr}(\lambda)\right)\right) = \max(\min(I), \min(J)).$$

6. Particular algebras

The following elementary result leads to a natural problem.

Proposition 6.1. *Let κ be an infinite cardinal and $A = \overline{\mathrm{Fr}(\kappa)}$, the completion of* $\mathrm{Fr}(\kappa)$. *Then* $\kappa, \kappa^\omega \in \mathrm{Spind}(A) \subseteq [\kappa, \kappa^\omega]_{\mathrm{card}}$.

Problem 2. *Let $A = \overline{\mathrm{Fr}(\omega)}$. Consistently, what are the possibilities for the set* $\mathrm{Spind}(A)$? *In particular, is there a model with 2^ω arbitrarily large in which* $\mathrm{Spind}(A) = \{\omega, 2^\omega\}$? *Or in which* $\mathrm{Spind}(A) = [\omega, 2^\omega]_{\mathrm{card}}$?

Several consistency results are known concerning $\mathrm{Spind}(A)$ where $A = \mathscr{P}\omega/\mathrm{fin}$. Kunen [3, Theorem 2.6, p 258], shows by Cohen forcing that it is consistent to have 2^ω large and $\mathrm{Spind}(A) = \{2^\omega\}$. In exercise (A13), page 289, he shows that it is consistent to have 2^ω large and $\omega_1 \in \mathrm{Spind}(A)$. In the model of Shelah [7] we have $2^\omega = \omega_2$ and $\mathrm{Spind}(\mathscr{P}\omega/\mathrm{fin}) = \{\omega_1, \omega_2\}$. On the other hand, in Shelah [8] a model is constructed in which $\mathfrak{i}(\mathscr{P}\omega/\mathrm{fin})$, itself large, is much smaller than the continuum, which can be arbitrarily large.

These results appear to leave the following problem open.

Problem 3. *Let $A = \mathscr{P}\omega/\mathrm{fin}$. Is there a model in which 2^ω is arbitrarily large and* $\mathrm{Spind}(A) = \{\omega_1, 2^\omega\}$? *Or in which 2^ω is arbitrarily large and* $\mathrm{Spind}(A) = [\omega_1, 2^\omega]_{\mathrm{card}}$?

References

[1] M. Holz, K. Steffens, E. Weitz, Introduction to Cardinal Arithmetic, Birkhäuser, Basel, 1999, 304pp.
[2] S. Koppelberg, General Theory of Boolean Algebras. Part I of Handbook of Boolean Algebras, North-Holland, Amsterdam 1989, 312pp.
[3] K. Kunen, Set Theory, North-Holland, Amsterdam, 1980 313pp.
[4] R. McKenzie, J.D. Monk, On some small cardinals for Boolean algebras, J. Symbolic Logic, submitted for publication.
[5] J.D. Monk Cardinal Invariants on Boolean Algebras, Birkhäuser, Basel, 1996, 298pp.
[6] J.D. Monk, Continuum cardinals generalized to Boolean algebras, J. Symbolic Logic (2001), to appear.
[7] S. Shelah, $CON(\mathfrak{u} > \mathfrak{i})$. Arch. Math. Logic 31 (1992) 433–443.
[8] S. Shelah, Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea?, preprint, publication no. 700, 2001.