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An atomless interval Boolean algebra A such that $\mathfrak{a}(A) < \mathfrak{t}(A)$

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Abstract. For any Boolean algebra A, $\mathfrak{a}(A)$ is the smallest cardinality of an infinite partition of unity in A. A *tower* in a Boolean algebra A is a subset X of A well-ordered by the Boolean ordering, with $1 \notin X$ but with $\sum X = 1$. $\mathfrak{t}(A)$ is the smallest cardinality of a tower of A. Given a linearly ordered set L with first element, the *interval algebra* of L is the algebra of subsets of L generated by the half-open intervals $[a, b)$. We prove that there is an atomless interval algebra A such that $\mathfrak{a}(A) < \mathfrak{t}(A)$.

This note is concerned with the generalization to Boolean algebras of two popular cardinals defined for the continuum. We give an example solving Problem 8 of Monk [2], which treats these cardinals and six others extensively. One result from that paper is needed here, but we recall it, along with crucial definitions, so that the statement and proof of the result can be understood independently.

A *partition* of a Boolean algebra A is a collection of pairwise disjoint nonzero elements of A with sum 1. $\mathfrak{a}(A)$ is the smallest cardinality of an infinite partition of A. A *tower* of A is a strictly increasing sequence of elements of A, each different from 1, with sum 1. $\mathfrak{t}(A)$ is the smallest cardinality of a tower in A.

Given an ordered set L with first element, the *interval algebra* on L, denoted by $\mathrm{Intalg}(L)$, is the algebra of subsets of L generated by all intervals $[a, b)$, where $a \in L$, $a < b$ and $b \in L \cup \{\infty\}$. If L is a dense linear order, then the *left character* of an element $x \in L$ is the smallest cardinality of a cofinal set of elements less than x; similarly for the *right character*. The *character* of x is the pair (κ, λ^*), where κ is the left character and λ is the right character. A *gap* in a linear order L is a pair (A, B) satisfying the following conditions:

(1) $A, B \subseteq L$, $A \neq \emptyset \neq B$;

(2) $A < B$ and $A \cup B = L$;

(3) A has no largest element, and B has no smallest element.

The *character* of a gap is defined in the natural way. A character of an element or a gap is *symmetric* iff it has the form (κ, κ^*).

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An η_α set is a linear order L such that if $A, B \subseteq L$, $A < B$, and $|A|, |B| < \aleph_\alpha$, then there is an element $x \in L$ such that $A < x < B$.

The following theorem, Proposition 41 of Monk [2], will be used below.

Theorem 1. *Suppose that* $A = \mathrm{Intalg}(L)$ *is atomless and* κ *is an uncountable regular cardinal. Then the following conditions are equivalent:*

(i) *There is a strictly increasing sequence* $\langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* A *with sum* 1.

(ii) *One of the following holds:*

  (a) *There is a* $c \in L$ *and a strictly decreasing sequence* $\langle b_\alpha : \alpha < \kappa \rangle$ *of elements of* L *coinitial with* c.

  (b) *There is a* $c \in L \cup \{\infty\}$ *and a strictly increasing sequence* $\langle b_\alpha : \alpha < \kappa \rangle$ *of elements of* L *cofinal in* c.

  (c) *There exist a strictly increasing sequence* $\langle b_\alpha : \alpha < \kappa \rangle$ *of elements of* L *and a strictly decreasing sequence* $\langle c_\alpha : \alpha < \kappa \rangle$ *of elements of* L *such that* $b_\alpha < c_\beta$ *for all* $\alpha, \beta < \kappa$ *and there is no element* d *of* L *such that* $b_\alpha < d < c_\beta$ *for all* $\alpha, \beta < \kappa$.

(iii) *There is a* $d \in L$ *and a strictly increasing sequence* $\langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* A *with sum* $[d, \infty)$.

The example in this note uses the following fact, which is a special case of Satz XVI and Satz XVII of Hausdorff [1]. To make this note more self-contained, we give a proof of this fact.

Theorem 2. *There is a linear order* L *with the following properties:*

(i) L *is dense without first or last elements.*

(ii) *The character of every gap of* L *is* (ω, ω_1^*) *or* (ω_1, ω^*).

(iii) *The character of every element of* L *is* (ω_1, ω_1^*).

Proof. We construct L as a certain subset of ${}^{\omega_1}2$, which is given the lexicographic order. If f and g are distinct elements of ${}^{\omega_1}2$, we denote by $\chi(f, g)$ the least $\alpha < \omega_1$ such that $f(\alpha) \neq g(\alpha)$. A function $f \in {}^{\omega_1}2$ is *eventually* 0 iff there is an $\alpha < \omega_1$ such that $f(\beta) = 0$ for all $\beta \geq \alpha$. Similarly we define *eventually* 1. Now we define

$$\begin{aligned} L = \{&f \in {}^{\omega_1}2 : f \text{ is not eventually 0, and is not eventually 1}\} \\ &\cup\ \{f \in {}^{\omega_1}2 : \text{there is an } \alpha \text{ such that } f(\alpha) = 1 \text{ and } f(\beta) = 0 \text{ for all } \beta > \alpha\}. \end{aligned}$$

Now we check the desired conditions.

No first element: Let $f \in L$. Then f is not constantly 0, so choose α such that $f(\alpha) = 1$. Define g by $g \restriction \alpha = f \restriction \alpha$, $g(\alpha) = 0$, $g(\alpha + 1) = 1$ and and $g(\beta) = 0$ for all $\beta > \alpha + 1$. Then $g \in L$ and $g < f$.

No last element: similarly.

Dense: Suppose that $f, g \in L$ and $f < g$. Let $\alpha = \chi(f,g)$. Since f is not eventually 1, choose $\beta > \alpha$ such that $f(\beta) = 0$. Now define h by $h \restriction \beta = f \restriction \beta$, $h(\beta) = 1$, and $h(\gamma) = 0$ for all $\gamma > \beta$. Then $h \in L$ and $f < h < g$.

The character of elements: First suppose that f is not eventually 0 and not eventually 1. Let $\langle \alpha_\xi : \xi < \omega_1 \rangle$ enumerate all of the ordinals α such that $f(\alpha) = 1$. For each $\xi < \omega_1$ define g^ξ by: $g^\xi \restriction \alpha_\xi = f \restriction \alpha_\xi$, $g^\xi(\alpha_\xi) = 0$, $g^\xi \restriction (\omega_1 \backslash (\alpha_\xi + 1)) = f \restriction (\omega_1 \backslash (\alpha_\xi + 1))$. Then $\langle g^\xi : \xi < \omega_1 \rangle$ is a strictly increasing sequence of elements of L converging to f. So the left character of f is ω_1.

The right character is treated similarly; so the right character of f is also ω_1.

Second, suppose that f is such that for some α we have $f(\alpha) = 1$ and $f(\beta) = 0$ for all $\beta > \alpha$. For each $\xi > \alpha$ define g^ξ by setting for each $\gamma < \omega_1$,

$$g^\xi(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma < \alpha, \\ 0 & \text{if } \alpha = \gamma, \\ 1 & \text{if } \alpha < \gamma \le \xi, \\ 0 & \text{if } \xi < \gamma. \end{cases}$$

Then $\langle g^\xi : \xi < \omega_1 \rangle$ is strictly increasing and converges to f, so the left character of f is ω_1.

For the right character, for each $\xi > \alpha$ define h^ξ by setting, for each $\gamma < \omega_1$,

$$h^\xi(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma < \xi, \\ 1 & \text{if } \gamma = \xi, \\ 0 & \text{if } \xi < \gamma. \end{cases}$$

Then $\langle h^\xi : \alpha < \xi < \omega_1 \rangle$ is a strictly decreasing sequence converging to f; so the right character is ω_1.

The character of gaps: Let (A, B) be a gap in L. Define f by recursion, setting for each $\alpha < \omega_1$,

$$f(\alpha) = \begin{cases} 1 & \text{if there is a } g \in A \text{ such that } f \restriction \alpha = g \restriction \alpha \text{ and } g(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $g \in A$, then $g < f$. To prove this, since A has no largest element it suffices to show that $g \le f$. Suppose that $f < g$. Let $\alpha = \chi(f,g)$. Then $f \restriction \alpha = g \restriction \alpha$, $f(\alpha) = 0$, and $g(\alpha) = 1$, contradicting the definition of f.

Also, if $h \in B$, then $f < h$. To prove this, since B has no smallest element it suffices to show that $f \le h$. Suppose that $h < f$. Let $\alpha = \chi(h,f)$. Then $f \restriction \alpha = h \restriction \alpha$, $h(\alpha) = 0$, and $f(\alpha) = 1$. By the definition of f, there is a $g \in A$ such that $f \restriction \alpha = g \restriction \alpha$ and $g(\alpha) = 1$. Thus $h < g$, contradicting $A < B$.

So we have proved

(1) $A < f < B$.

In particular, $f \notin L$. This gives rise to the following possibilities.

Case 1. f is eventually 0. Since $A \neq 0$, f is not the all 0 function. Choose α minimum such that $f(\beta) = 0$ for all $\beta \geq \alpha$. By the definition of L it follows that α is a limit ordinal. By the minimality of α, f takes on the value 1 cofinally below α. Let $\langle \beta_n : n \in \omega \rangle$ be strictly increasing and cofinal in α, with $f(\beta_n) = 1$ for all n. Now for each $n \in \omega$ define g^n by setting, for each $\gamma < \omega_1$,

$$g^n(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma < \beta_n, \\ 0 & \text{if } \gamma = \beta_n, \\ f(\gamma) & \text{if } \beta_n < \gamma < \alpha, \\ 1 & \text{if } \gamma = \alpha, \\ 0 & \text{if } \alpha < \gamma. \end{cases}$$

Clearly $\langle g^n : n \in \omega \rangle$ is a strictly increasing sequence of members of L converging to f. By (1), each g^n is in A. So the left character of our gap is ω.

For each $\xi \geq \alpha$ define h^ξ by setting, for each $\gamma < \omega_1$,

$$h^\xi(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma < \xi, \\ 1 & \text{if } \gamma = \xi, \\ 0 & \text{if } \gamma > \xi. \end{cases}$$

Then $\langle h^\xi : \alpha \leq \xi < \omega_1 \rangle$ is a strictly decreasing sequence of members of L converging to f. By (1), each h^ξ is in B. So the right character of our gap is ω_1.

Case 2. f is eventually 1. Since $B \neq \emptyset$, f is not the all 1 function. Choose α minimum such that $f(\beta) = 1$ for all $\beta \geq \alpha$. Assume that $\alpha = \beta + 1$ for some β. Thus $f(\beta) = 0$. Define g by $g \restriction \beta = f \restriction \beta$, $g(\beta) = 1$, and $g(\gamma) = 0$ for all $\gamma > \beta$. Then $g \in L$ and $f < g$, so $g \in B$ by (1). But every function smaller than g is $\leq f$, which contradicts (1) and the fact that B has no smallest element.

It follows that α is a limit ordinal. By the minimality of α, the function f takes on the value 0 cofinally below α. Let $\langle \beta_n : n \in \omega \rangle$ be strictly increasing and cofinal in α, with $f(\beta_n) = 0$ for all n. Define g^n by setting $g^n \restriction \beta_n = f \restriction \beta_n$, $g(\beta_n) = 1$, and $g(\gamma) = 0$ for all $\gamma > \beta_n$. Then $\langle g^n : n \in \omega \rangle$ is a strictly decreasing sequence converging to f. Hence the right character of our gap is ω.

A familiar argument shows that the left character is ω_1. □

Theorem 3. *There is an atomless interval algebra A such that $\mathfrak{a}(A) < \mathfrak{t}(A)$.*

Proof. We start with a linear order L with the properties given by Theorem 2. Below we use $A + B$ for ordered sets A, B to be the order formed from the disjoint union of A and B by putting each element of A less than each element of B. Now let $|L| = \aleph_\alpha$, and let M be an $\eta_{\alpha+1}$-set. The set $(1+M) \times \mathbb{Z} \times L$ is given the anti-lexicographic order, and we set

$$\begin{aligned} N &= M_1 + (1+M) \times \mathbb{Z} \times L + M_2, \\ A &= \mathrm{Intalg}(N), \end{aligned}$$

where M_1 and M_2 are copies of M. We claim that A is as desired. The first element of $1+M$ is denoted by 0_M.

First we show that $\mathfrak{a}(A) \le \aleph_\alpha$. In fact, choose $a < b$ in L, and let

$$\begin{aligned} X = \{\, [-\infty, (0_M, 0, a)) \,\} &\cup \{\, [(0_M, i, a), (0_M, i+1, a)) : i \in \omega \,\} \\ &\cup \{\, [(0_M, i, c), (0_M, i+1, c)) : i \in \mathbb{Z}, a < c < b \,\} \\ &\cup \{\, [(0_M, i, b), (0_M, i+1, b)) : i \in \mathbb{Z} \backslash \omega \,\} \\ &\cup \{\, [(0_M, 0, b), \infty) \,\}. \end{aligned}$$

Clearly X is a partition of A, so $\mathfrak{a}(A) \le \aleph_\alpha$.

To show that $\mathfrak{t}(A) > \aleph_\alpha$ we shall apply Theorem 1.

(1) Every element of N has character (κ, λ^*) with $\kappa, \lambda > \aleph_\alpha$.

For, this is clear if the element is in M_1 or M_2. Suppose that the element is in the middle part, and thus has the form (u, i, c). If $u \ne 0_M$, (1) is clear. For $u = 0_M$, the right character of (u, i, c) clearly is λ^* for some $\lambda > \aleph_\alpha$. The left character of $(0_M, i, c)$ is obtained from a cofinal subset of $\{(v, i-1, c) : v \in M\}$, and hence is κ for some $\kappa > \aleph_\alpha$. Thus (1) holds.

(2) $-\infty$ has character $(0, \lambda^*)$ with $\lambda > \aleph_\alpha$, and ∞ has character $(\kappa, 0)$ with $\kappa > \aleph_\alpha$.

This condition is clear.

(3) The symmetric gaps of N have character of the form (κ, κ^*) with $\kappa > \aleph_\alpha$.

To prove this, let (A, B) be a gap of N; see the beginning of the article for the conditions involved. We consider several cases.

Case 1. $M_1 \cap B \ne \emptyset$ or $M_2 \cap A \ne \emptyset$. Clearly the desired conclusion holds.

Case 2. $A = M_1$ or $B = M_2$. Clearly the gap is not symmetric in this case.

Case 3. There is a $c \in L$ such that $(u, i, c) \in A$ and $(v, i, c) \in B$ for some u, i, v. Clearly the desired conclusion holds.

Case 4. There is a $c \in L$ such that $(u, i, c) \in A$ and $(v, j, c) \in B$ for some u, i, v, j, but Case 3 does not hold. Therefore $i < j$. Let j be smallest such that $(w, j, c) \in B$ for some w. Hence $(x, j-1, c) \in A$ for some x. By the $\eta_{\alpha+1}$-property, the desired conclusion holds.

Case 5. There exist a $c \in L$ and u such that $(u, i, c) \in A$ for infinitely many positive i, and there do not exist v, j, d with $c < d$ such that $(v, j, d) \in A$. Then the character of (A, B) is (ω, ω_1^*) by (iii) of Lemma 1, so the character is not symmetric.

Case 6. There exist a $c \in L$ and u such that $(u, i, c) \in B$ for infinitely many negative i, and there do not exist v, j, d with $d < c$ such that $(v, j, d) \in B$. Similarly.

Case 7. The set $\{c : (u, i, c) \in A$ for some $u, i\}$ has no largest element, and the set $\{c : (u, i, c) \in B$ for some $u, i\}$ has no smallest element. Then by (ii) of Lemma 1, the character of the gap is not symmetric.

This finishes the proof of (3). By (1), (2), (3) and Theorem 1, $\mathfrak{t}(A) > \aleph_\alpha$. $\square$

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