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# CONTINUUM CARDINALS GENERALIZED TO BOOLEAN ALGEBRAS

J. DONALD MONK

A number of specific cardinal numbers have been defined in terms of $\mathscr{P}(\omega)/\text{fin}$ or ${}^{\omega}\omega$. Some have been generalized to higher cardinals, and some even to arbitrary Boolean algebras. Here we study eight of these cardinals, defining their generalizations to higher cardinals, and then defining them for Boolean algebras. We then attempt to completely describe their relationships within each of several important classes of Boolean algebras.

The generalizations to higher cardinals might involve several cardinals instead of just one as in the case of $\omega$. For example, the number $\mathfrak{a}$ associated with maximal almost disjoint families of infinite sets of integers can be generalized to talk about maximal subsets of $[\kappa]^{\mu}$ subject to the pairwise intersections having size less than $\nu$. (For this multiple generalization of $\mathfrak{a}$, see Monk [2001].) For brevity we do not consider such generalizations, restricting ourselves to just one cardinal. The set-theoretic generalizations then associate with each infinite cardinal $\kappa$ some other cardinal $\lambda$, defined as the minimum of cardinals with a certain property.

The generalizations to Boolean algebras assign to each Boolean algebra some cardinal $\lambda$, also defined as the minimum of cardinals with a certain property.

For the theory of the original "continuum" cardinal numbers, see Douwen [1984], Balcar and Simon [1989], and Vaughan [1990].

I am grateful to Mati Rubin for some conversations concerning these functions for superatomic algebras, and to Bohuslav Balcar for information concerning the function $\mathfrak{h}$.

The notation for set theory is standard. For Boolean algebras we follow Koppelberg [1989], but recall at the appropriate place any somewhat unusual notation.

## §1. Definitions.

Let $\kappa$ be an infinite cardinal.

(A) *Maximal almost disjoint families.* Let

$$\mathfrak{a}_{\kappa} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\kappa]^{\kappa}, \mathscr{A} \text{ maximal almost disjoint}, |\mathscr{A}| \geq \kappa\}.$$

A Boolean algebraic version is

$$\mathfrak{a}(A) = \min\{|X| : X \text{ is an infinite partition of unity of } A\}.$$

Both versions have been extensively studied; see Baumgartner [1976], Milner and Prikry [1987], Monk [1997], Monk [2001].

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Note that the side condition $|\mathscr{A}| \geq \kappa$ is not incorporated in the general Boolean algebraic version; there does not seem to be any natural way of doing this.

(B) *The pseudo-intersection number.* For $A, B \subseteq \kappa$ we define $A \subseteq^*_\kappa B$ iff $|A \backslash B| < \kappa$. If $\mathscr{F} \subseteq [\kappa]^\kappa$ and $A \in [\kappa]^\kappa$, we call $A$ a *pseudo-intersection* of $\mathscr{F}$ iff $A \subseteq^*_\kappa B$ for all $B \in \mathscr{F}$. $\mathscr{F} \subseteq [\kappa]^\kappa$ has the *strong finite intersection property*, sfip, if every finite subcollection of $\mathscr{F}$ has intersection of size $\kappa$. Let

$$\mathfrak{p}_\kappa = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\kappa]^\kappa \text{ has sfip but does not have a pseudo-intersection of size } \kappa\}.$$

A Boolean algebraic version of this function is

$$\mathfrak{p}(A) = \min\left\{|Y| : \sum Y = 1 \text{ and } \sum Y' \neq 1 \text{ for every finite } Y' \subseteq Y\right\};$$

this is briefly considered in Monk [1996].

(C) *The splitting number.* A set $\mathscr{S} \subseteq [\kappa]^\kappa$ is *splitting* if for every $A \in [\kappa]^\kappa$ there is an $S \in \mathscr{S}$ such that $|A \cap S| = |A \backslash S| = \kappa$. Let

$$\mathfrak{s}_\kappa = \min\{|\mathscr{S}| : \mathscr{S} \subseteq [\kappa]^\kappa \text{ is splitting}\}.$$

This generalization is studied in Zapletal [1997].

There is a clear Boolean algebraic version. Call $S \subseteq A$ a *splitting set* iff for all $a \in A^+$ there is an $s \in S$ such that $a \cdot s \neq 0 \neq a \cdot -s$. ($A^+$ is the set of all nonzero elements of $A$.)

$$\mathfrak{s}(A) = \min\{|S| : S \text{ is a splitting set for } A\}.$$

Clearly $\mathfrak{s}(A)$ is well-defined iff $A$ is atomless. And $\mathfrak{s}(A)$ is infinite in this case. In fact, if $A$ is atomless and $S$ is a finite splitting set for $A$, wlog $0, 1 \notin S$. Then $S$ generates a finite subalgebra $B$ of $A$, and no atom of $B$ is split, contradiction.

(D) *The tower number.* A set $\mathscr{T} \subseteq [\kappa]^\kappa$ is a *tower* iff it is inversely well-ordered by $\subseteq^*_\kappa$ and has no pseudointersection of size $\kappa$.

$$\mathfrak{t}_\kappa = \min\{|\mathscr{T}| : \mathscr{T} \subseteq [\kappa]^\kappa \text{ is a tower}\}.$$

An obvious Boolean algebraic version is this: a subset $T \subseteq A^+$ is a *tower* if it is well-ordered by $\geq$, and $\prod T = 0$.

$$\mathfrak{t}(A) = \min\{|T| : T \text{ is a tower in } A\}.$$

Clearly $\mathfrak{t}(A)$ is always regular.

Towers do not exist for all BAs, for example not for the algebra Finco$\kappa$ of finite and cofinite subsets of $\kappa$ if $\kappa$ is uncountable. Towers always exists if $A$ is atomless, and towers even exists for many superatomic BAs, for example for Intalg$\kappa$, the interval algebra on $\kappa$.

(E) *Height.* $B$ is $(\kappa, \infty)$-distributive iff every family $\langle P_\alpha : \alpha < \kappa\rangle$ of partitions of unity of $B$ has a common refinement to another partition of unity of $B$. Let

$$\mathfrak{h}(B) = \min\{\kappa : B \text{ is not } (\kappa, \infty) - \text{distributive}\}.$$

Of course this is well-defined iff there is some $\kappa$ such that $B$ is not $(\kappa, \infty)$-distributive. As is well-known, this is equivalent to $B$ not being atomic. For the general Boolean algebraic case see Balcar and Simon [1989].

(F) *The reaping number.* A subset $X$ of nonzero elements of a BA $A$ is *weakly dense* if for every $a \in A$ there is an $x \in X$ such that $x \leq a$ or $x \leq -a$. Let

$$\mathfrak{r}(A) = \min\{|X| : X \text{ is weakly dense in } A\}.$$

This function has been extensively studied in the general context of Boolean algebras. See, for example, Balcar and Simon [1992], Peterson [1998], Bozeman [1991], Monk [1996].

The following simple result about weakly dense sets should be noted.

PROPOSITION 1. (i) *If $A$ has an atom $a$, then $\{a\}$ is weakly dense, and hence $\mathfrak{r}(A) = 1$.*

(ii) *If $A$ is atomless, then $\mathfrak{r}(A)$ is infinite.*

PROOF. (i) is obvious. For (ii), suppose that $A$ is atomless, and $X \subseteq A$ is finite and weakly dense. Let $a_0, \ldots, a_{n-1}$ list all of the atoms of $\langle X \rangle$, and for each $i < n$ choose $0 < b_i < a_i$. Here $\langle X \rangle$ is the subalgebra of $A$ generated by $X$. Let $c = \sum_{i<n} b_i$. Choose $x \in X$ such that $x \leq c$ or $x \leq -c$. But $x \cdot a_i = 0$ or $a_i \leq x$ for every $i < n$, contradiction. □

(G) *Maximal independence number.* For any Boolean algebra $A$,

$$\mathfrak{i}(A) = \min\{|X| : X \subseteq A, X \text{ maximal independent}\}.$$

This function can have value 1. The situation is clarified by the following simple result.

PROPOSITION 2. (i) *If $A$ has at least four elements and has an atom $a$, then $\{a\}$ is a maximal independent set and so $\mathfrak{i}(A) = 1$.*

(ii) *If $A$ is atomless, then every maximal independent subset of $A$ is infinite.*

PROOF. (i) is obvious.

For (ii), it suffices to show that if $X$ is a finite nonempty independent set in $A$, then $X$ is not maximal. In fact, for each atom $a$ of $\langle X \rangle$ let $y_a$ be such that $0 < y_a < a$. Then set $x = \sum\{y_a : a \text{ an atom of } \langle X \rangle\}$. Clearly $X \cup \{x\}$ is independent. □

(H) *Ultrafilter number.* For any Boolean algebra $A$,

$$\mathfrak{u}(A) = \min\{|X| : X \text{ generates some nonprincipal ultrafilter on } A\}.$$

**§2. The algebra $\mathscr{P}(\omega)/\text{fin}$.** The original versions of these functions are obtained by taking $\kappa = \omega$, or by taking the Boolean algebra to be $\mathscr{P}(\omega)/\text{fin}$. In this section we survey the relationships between the functions in this case. It is enlightening to introduce in this regard two further functions, whose generalizations to Boolean algebras are not immediate.

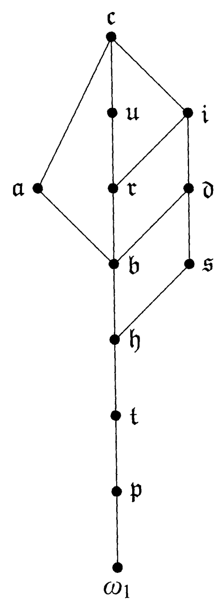
(I) *The boundedness number.* Let $\kappa$ be an infinite cardinal. For $f, g \in {}^\kappa\kappa$ we define $f \leq^*_\kappa g$ iff $|\{\alpha < \kappa : f(\alpha) > g(\alpha)\}| < \kappa$. Then we define

$$\mathfrak{b}_\kappa = \min\{|X| : X \subseteq {}^\kappa\kappa \text{ is unbounded under } \leq^*_\kappa\}.$$

(J) *The dominating number.*

$$\mathfrak{d}_\kappa = \min\{|X| : X \subseteq {}^\kappa\kappa \text{ is cofinal in } ({}^\kappa\kappa, \leq^*_\kappa)\}.$$

The diagram below indicates the relationships between the functions in the case $\kappa = \omega$. Note that the bottom of this diagram is $\omega_1$ and the top is $\mathfrak{c}$; so all the numbers are equal under CH. Hence results that the diagram has distinct nodes and no additional relationships are all consistency results; and some of the possibilities still constitute open problems. We summarize these consistency results and problems. The references give details and refer to the original papers.




**The functions for $\kappa = \omega$**

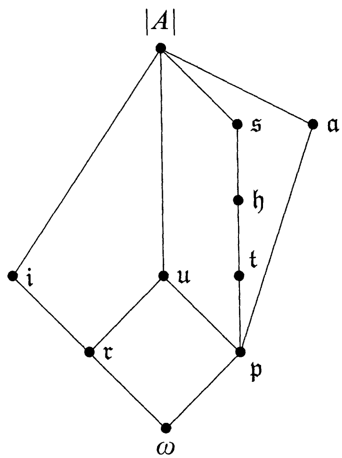
(1) $\mathfrak{t}$ is regular. (Douwen [1984, p. 116])
(2) $\mathfrak{p}$ is regular. (Douwen [1984, p. 116])
(3) $\mathfrak{p} = \omega_1 \Rightarrow \mathfrak{t} = \omega_1$. (Douwen [1984, p. 116])
(4) $\omega \leq \kappa < \mathfrak{t} \Rightarrow 2^\kappa = \mathfrak{c}$. (Douwen [1984, p. 116])
(5) $\mathrm{Con}(\omega_1 = \mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \mathfrak{s} = \mathfrak{t} = \mathfrak{p} < \mathfrak{c})$. (Douwen [1984, p. 127])
(6) $\mathrm{Con}(\omega_1 < \mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \mathfrak{s} = \mathfrak{t} = \mathfrak{p} < \mathfrak{c})$. (Douwen [1984, p. 127])
(7) $\mathrm{Con}(\omega_1 < \mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \mathfrak{s} = \mathfrak{t} = \mathfrak{p} = \mathfrak{c})$. (Douwen [1984, p. 127])
(8) $\mathrm{Con}(\omega_1 = \mathfrak{h} = \mathfrak{s} < \mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2)$. (Dow [1989, 2.5])
(9) $\mathrm{Con}(\omega_1 = \mathfrak{h} = \mathfrak{s} = \mathfrak{b} < \mathfrak{d} = \mathfrak{r} = \mathfrak{c} = \omega_2)$. (Dow [1989, 2.2] and Kunen [1980, proof of VIII2.6])
(10) $\mathrm{Con}(\mathfrak{u} < \mathfrak{s})$. (Blass and Shelah [1987])
(11) $\mathrm{Con}(\mathfrak{i} < \mathfrak{u})$. (Shelah [1992])
(12) $\mathrm{Con}(\mathfrak{t} < \mathfrak{h})$. (Dordal [1987])
(13) $\mathrm{Con}(\mathfrak{b} < \mathfrak{a})$. (Shelah [1984])
(14) $\mathrm{Con}(\mathfrak{s} < \mathfrak{b})$. (Balcar and Simon [1989])
(15) $\mathrm{Con}(\mathfrak{d} < \mathfrak{i})$. (Shelah [1990])
(16) $\mathrm{Con}(\mathfrak{i} < \mathfrak{c})$. (Kunen [1980, exc. VIIIA13])
(17) $\mathrm{Con}(\omega_1 = \mathfrak{h} = \mathfrak{s} = \mathfrak{b} = \mathfrak{d} < \mathfrak{r} = \mathfrak{c})$ (Dow [1989, 2.4] and Price [1982])
(18) $\mathrm{Con}(\mathfrak{a} < \mathfrak{r})$. (Kunen [1980, VIII2.3 and proof of VIII2.6])
(19) $\mathrm{Con}(\mathfrak{a} < \mathfrak{s})$. (Shelah [1984])
(20) $\mathrm{Con}(\mathfrak{u} < \mathfrak{a})$. (Shelah [2001])
(21) $\mathrm{Con}(\mathfrak{i} < \mathfrak{a})$. (Shelah [2001])

These results do not quite prove that the above diagram is exactly as indicated. For the readers convenience we indicate what they do show about the diagram, and what problems remain; these are consistency results and problems.

| | | | |
|---|---|---|---|
| $\omega_1 < \mathfrak{p}$: | (6) | $\mathfrak{d} < \mathfrak{i}$: | (15) |
| $\mathfrak{p} < \mathfrak{t}$: | problem (Vaughan [1990]) | $\mathfrak{u} < \mathfrak{c}$: | (10) |
| $\mathfrak{t} < \mathfrak{h}$: | (12) | $\mathfrak{i} < \mathfrak{c}$: | (16) |
| $\mathfrak{h} < \mathfrak{b}$: | (8) | $\mathfrak{a} < \mathfrak{r}$: | (18) |
| $\mathfrak{h} < \mathfrak{s}$: | (10) | $\mathfrak{a} < \mathfrak{s}$: | (19) |
| $\mathfrak{b} < \mathfrak{a}$: | (13) | $\mathfrak{u} < \mathfrak{a}$: | (20) |
| $\mathfrak{b} < \mathfrak{r}$: | (9) | $\mathfrak{u} < \mathfrak{s}$: | (10) |
| $\mathfrak{b} < \mathfrak{d}$: | (19) | $\mathfrak{i} < \mathfrak{a}$: | (21) |
| $\mathfrak{s} < \mathfrak{d}$: | (14) | $\mathfrak{i} < \mathfrak{u}$: | (11) |
| $\mathfrak{a} < \mathfrak{c}$: | (5) | $\mathfrak{d} < \mathfrak{r}$: | (17) |
| $\mathfrak{r} < \mathfrak{u}$: | (11) | $\mathfrak{s} < \mathfrak{b}$: | (14) |
| $\mathfrak{r} < \mathfrak{i}$: | (10) | | |

## Atomless Boolean algebras

The following diagram gives what we know about the general case of atomless Boolean algebras:




**The functions for atomless Boolean algebras**

We show that the diagram is exactly as indicated. The required examples are all constructed in ZFC, except the ones for $\mathfrak{u} < \mathfrak{i}$ and $\mathfrak{s} < \mathfrak{a}$, for which we refer to (10) and (14) above.

The inequalities $\mathfrak{p}(A) \le \mathfrak{t}(A)$, $\mathfrak{p}(A) \le \mathfrak{a}(A)$, and $\mathfrak{r}(A) \le \mathfrak{u}(A)$ are clear.

PROPOSITION 3. $\mathfrak{r}(A) \le \mathfrak{i}(A)$ *for every atomless BA* $A$.

PROOF. Let $X$ be a maximal independent subset of $A$ of size $\mathfrak{i}(A)$. Let $Y$ be the set of all monomials in members of $X$. We claim that $Y$ is weakly dense in $A$. For, let $a \in A^+$. If $a \in \langle X \rangle$, the desired conclusion is clear. Suppose that $a \notin \langle X \rangle$. Then $X \cup \{a\}$ is not independent, so there is a monomial $y$ in $X$ such that $a \cdot y = 0$ or $-a \cdot y = 0$, as desired. □

PROPOSITION 4. $\mathfrak{t}(A) \leq \mathfrak{h}(A)$ *for every BA A which is not atomic.*

PROOF. Let $\kappa = \mathfrak{h}(A)$. Let $\langle P_\alpha : \alpha < \kappa \rangle$ be a system of partitions of unity of $A$ which do not have a common refinement. By the minimality of $\mathfrak{h}(A)$, we may assume that $P_\beta$ is a refinement of $P_\alpha$ whenever $\alpha < \beta$. Let $Q$ be a maximal family of nonzero pairwise disjoint elements of $A$ such that for each $x \in Q$ and each $\alpha < \kappa$ there is a $y \in P_\alpha$ such that $x \leq y$. Then by the choice of the $P_\alpha$'s, there is some nonzero element $x$ such that $x \cdot y = 0$ for all $y \in Q$. We now define a decreasing sequence of elements $w_0, \ldots$ of $A$. Let $w_0 \in P_0$ be such that $x \cdot w_0 \neq 0$. Suppose that $w_\alpha \in P_\alpha$ has been defined so that $w_\alpha \cdot x \neq 0$. Choose $w_{\alpha+1} \in P_{\alpha+1}$ so that $w_\alpha \cdot x \cdot w_{\alpha+1} \neq 0$. Then $w_{\alpha+1} \leq w_\alpha$ since $P_{\alpha+1}$ refines $P_\alpha$. If $w_\alpha$ has been defined for all $\alpha < \beta$, with $\beta$ a limit ordinal, and if there is a nonzero $y$ such that $y \leq w_\alpha \cdot x$ for all $\alpha < \beta$, then there is a $z \in P_\beta$ such that $z \cdot y \neq 0$, and we let $w_\beta$ be such a $z$. Clearly then $w_\beta \cdot x \neq 0$. Note also that $w_\beta \leq w_\alpha$ for all $\alpha < \beta$, since $P_\beta$ refines $P_\alpha$. If there is no such $y$, then the construction stops and we have a tower of length $\leq \beta$.

Suppose that the construction does not stop at any limit level. Then

$$\prod_{\alpha<\kappa} w_\alpha \cdot x = 0,$$

since otherwise there would be some nonzero element $z \leq w_\alpha \cdot x$ for all $\alpha < \kappa$, and the maximality of $Q$ is contradicted. So we have a tower of length at most $\kappa$. $\square$

PROPOSITION 5. *If A is atomless, then* $\mathfrak{h}(A) \leq \mathfrak{s}(A)$.

PROOF. Assume that $\mathfrak{s}(A) < \mathfrak{h}(A)$. Let $S$ be a splitting set of size $\mathfrak{s}(A)$. Obviously we may assume that all members of $S$ are nonzero. Write $S = \{s_\alpha : \alpha < \kappa\}$, where $\kappa = \mathfrak{s}(A)$. For each $\alpha < \kappa$ let $P_\alpha$ be a partition of unity with $s_\alpha \in P_\alpha$. Let $Q$ be a common refinement of all $P_\alpha$. Take any $a \in Q$. Choose $\alpha < \kappa$ such that $a \cdot s_\alpha \neq 0 \neq a \cdot -s_\alpha$. Now $s_\alpha \in P_\alpha$, $a \cdot s_\alpha \neq 0$, and $Q$ refines $P_\alpha$, so $a \leq s_\alpha$, contradiction. $\square$

PROPOSITION 6. $\mathfrak{p}(A) \leq \mathfrak{u}(A)$ *for every BA A.*

PROOF. Suppose that $X$ generates a non-principal ultrafilter $F$, $|X| = \mathfrak{u}(A)$. Then $\prod X = 0$. In fact, suppose that $0 < a \leq x$ for all $x \in X$. If $a \in F$, then clearly $F$ is the principal filter generated by $a$, contradicting $F$ non-principal. If $-a \in F$, choose $x \in X$ such that $x \leq -a$. Then $a \leq -x$ and $a \leq x$, so $a = 0$, contradiction. Thus, indeed, $\prod X = 0$.

Since $\prod Y \neq 0$ for every finite subset $Y$ of $X$, this gives the inequality of the proposition. $\square$

Thus we have verified the relationships indicated in the diagram. We now consider various examples, mostly of familiar algebras. The examples aim to check that no other relationships can hold. From this point of view the following examples are crucial:

$\mathfrak{h}(A) < \mathfrak{s}(A)$. Example 14.
$\mathfrak{i}(A) < \mathfrak{p}(A)$. Example 18.
$\mathfrak{u}(A) < \mathfrak{i}(A)$. See (10); an example in ZFC is unknown.
$\mathfrak{u}(A) < \mathfrak{a}(A)$. Proposition 21.
$\mathfrak{s}(A) < \mathfrak{r}(A)$. Example 12.
$\mathfrak{s}(A) < \mathfrak{a}(A)$. See (14); an example in ZFC is unknown.
$\mathfrak{a}(A) < \mathfrak{r}(A)$. Example 12.
$\mathfrak{u}(A) < \mathfrak{t}(A)$. Example 20.
$\mathfrak{t}(A) < \mathfrak{h}(A)$. Example 16.
$\mathfrak{a}(A) < \mathfrak{t}(A)$. Example 20.

These examples work for other relations, according to the general diagram. For instance, example 12 also has $\mathfrak{s}(A) < |A|$ and $\mathfrak{a}(A) < \mathfrak{u}(A)$.

Before starting on the examples, we give some general properties of the functions, beginning with what happens to them under various kinds of products.

PROPOSITION 7. *Let $A$ and $B$ be atomless Boolean algebras.*

(i) *For $k \in \{\mathfrak{a}, \mathfrak{p}, \mathfrak{t}, \mathfrak{h}, \mathfrak{r}, \mathfrak{u}\}$ we have* $k(A \times B) = \min(k(A), k(B))$.
(ii) $\mathfrak{i}(A \times B) \leq \min(\mathfrak{i}(A), \mathfrak{i}(B))$.
(iii) $\mathfrak{s}(A \times B) = \max(\mathfrak{s}(A), \mathfrak{s}(B))$.

PROOF. The proofs of several parts of this proposition are similar. For illustration we treat $\mathfrak{h}$, $\mathfrak{i}$ and $\mathfrak{s}$.

If $X \subseteq A \times B$, let

$$\mathrm{pr}_0(X) = \{a \in A : \exists b \in B[(a,b) \in X]\}$$

and

$$\mathrm{pr}_1(X) = \{b \in B : \exists a \in A[(a,b) \in X\}.$$

If $X \subseteq A$, let $\mathrm{up}_0(X) = \{(x,0) : x \in X\}$, and if $Y \subseteq B$ let $\mathrm{up}_1(Y) = \{(0,y) : y \in Y\}$.

$\mathfrak{h}$: Suppose that $\langle P_\alpha : \alpha < \mathfrak{h}(A)\rangle$ is a system of partitions of unity of $A$ with no common refinement. For each $\alpha < \mathfrak{h}(A)$ let $Q_\alpha = \mathrm{up}_0(P_\alpha) \cup \{(0,1)\}$. Then each $Q_\alpha$ is a partition of unity in $A \times B$, and the $Q_\alpha$'s have no common refinement. Similarly for $B$, so this shows that $\mathfrak{h}(A \times B) \leq \min(\mathfrak{h}(A), \mathfrak{h}(b))$. Now suppose that $\langle R_\alpha : \alpha < \kappa\rangle$ is a system of partitions of unity of $A \times B$ with no common refinement, where $\kappa < \min(\mathfrak{h}(A), \mathfrak{h}(B))$. For each $\alpha < \kappa$ let $S_\alpha = \mathrm{pr}_0(R_\alpha)\backslash\{0\}$ and $T_\alpha = \mathrm{pr}_1(R_\alpha)\backslash\{0\}$. Each $S_\alpha$ is a partition of unity of $A$, and each $T_\alpha$ is a partition of unity of $B$. Let $U$ be a common refinement of the $S_\alpha$'s, and let $V$ be a common refinement of the $T_\alpha$'s. Define $W = \mathrm{up}_0(U) \cup \mathrm{up}_1(V)$. Then $W$ is a partition of unity in $A \times B$ which refines the $R_\alpha$'s, contradiction.

$\mathfrak{i}$: If $X$ is maximal independent in $A$, then $\mathrm{up}_0(X)$ is clearly independent in $A \times B$. Suppose that $(a,b) \in (A \times B)\backslash\mathrm{up}_0(X)$.

*Case 1.* $a \notin X$. Then $X \cup \{a\}$ is not independent. Hence there is a finite subset $F$ of $X$ and an $\varepsilon \in {}^{F\cup\{a\}}2$ such that $\prod_{y \in F\cup\{a\}} y^{\varepsilon(y)} = 0$. Choose $u \in X\backslash F$. Then

$$\prod_{y\in F}(y,0)^{\varepsilon(y)} \cdot (u,0) \cdot (a,b)^{\varepsilon(a)} = (0,0).$$

This shows that $\mathrm{up}_0(X) \cup \{(a,b)\}$ is not independent.

*Case* 2. $a \in X$. It follows that $b \neq 0$. Then $(a,0)\cdot(-a,-b) = (0,0)$ shows that $\mathrm{up}_0(X) \cup \{(a,b)\}$ is not independent.

$\mathfrak{s}$: If $X$ splits $A$ and $Y$ splits $B$, then $X \times Y$ splits $A \times B$. Hence

$$\mathfrak{s}(A \times B) \leq \max(\mathfrak{s}(A), \mathfrak{s}(B)).$$

Now suppose that $|Z| < \max(\mathfrak{s}(A), \mathfrak{s}(B))$ and $Z$ splits $A \times B$. Say by symmetry that $\max(\mathfrak{s}(A), \mathfrak{s}(B)) = \mathfrak{s}(A)$. Then $|\mathrm{pr}_0(Z)| < \mathfrak{s}(A)$, so there is some nonzero element $a$ of $A$ such that for all $(u,v) \in Z$, $a \cdot u = 0$ or $a \cdot -u = 0$. Hence for all $(u,v) \in Z$, $(a,0)\cdot(u,v) = (0,0)$ or $(a,0)\cdot -(u,v) = (0,0)$, contradiction. □

PROBLEM 1. *Is it true that for any atomless BAs $A$ and $B$ we have* $\mathfrak{i}(A \times B) = \min(\mathfrak{i}(A), \mathfrak{i}(B))$?

In the next proposition we consider the *weak product* $\prod^{\mathrm{w}}_{i\in I} A_i$ of Boolean algebras, consisting of all functions $f$ in the full product such that one of the two sets $\{i \in I : f(i) \neq 0\}$, $\{i \in I : f(i) \neq 1\}$ is finite.

PROPOSITION 8. *Suppose that $\langle A_i : i \in I\rangle$ is a system of atomless BAs, $I$ infinite. Let $B = \prod^{\mathrm{w}}_{i\in I} A_i$. Then*

(i) *For $k \in \{\mathfrak{a}, \mathfrak{p}, \mathfrak{u}\}$ we have* $k(B) = \min(|I|, \min_{i\in I} k(A_i))$.
(ii) *If $|I| = \omega$, then $\mathfrak{t}(B) = \omega$, while if $|I| > \omega$ then* $\mathfrak{t}(B) = \min_{i\in I} \mathfrak{t}(A_i)$.
(iii) $\mathfrak{h}(B) = \min_{i\in I} \mathfrak{h}(A_i)$.
(iv) *For $k \in \{\mathfrak{r}, \mathfrak{i}\}$ we have* $k(B) = \omega$.
(v) $\mathfrak{s}(B) = \max(|I|, \sup_{i\in I} \mathfrak{s}(A_i))$.

PROOF. If $i \in I$ and $a \in A_i$ we define $\mathrm{up}^i(a) \in \prod^{\mathrm{w}}_{i\in I} A_i$ by setting, for any $j \in I$,

$$\mathrm{up}^i(a)_j = \begin{cases} a & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\prod^{\mathrm{w}}_{i\in I} A_i \cong A_j \times \prod^{\mathrm{w}}_{i\in I\setminus\{j\}} A_i$ for all $j \in I$, by Proposition 7 we have $k(B) \leq \min_{i\in I} k(A_i)$ for each $k \in \{\mathfrak{a}, \mathfrak{p}, \mathfrak{t}, \mathfrak{h}, \mathfrak{r}, \mathfrak{u}, \mathfrak{i}\}$. Now we consider the functions one at a time.

$\mathfrak{a}$: Clearly $\mathfrak{a}(B) \leq |I|$. Now suppose that $X$ is an infinite partition of unity in $B$, with $|X| < \min(|I|, \min_{i\in I} k(A_i))$.

*Case 1.* There is an $x \in X$ such that $J \overset{\text{def}}{=} \{i \in I : x_i \neq 1\}$ is finite. Then $\{y \restriction J : y \in X\}$ is an infinite partition of unity in $\prod_{i\in J} A_i$, contradicting Proposition 7.

*Case 2.* For every $x \in X$ the set $\{i \in I : x_i \neq 0\}$ is finite. Since $|X| < |I|$, there is an $i \in I$ such that $x_i = 0$ for all $x \in X$, contradicting $\sum X = 1$.

$\mathfrak{p}$: Proof is very similar to that for $\mathfrak{a}$.

$\mathfrak{u}$: This follows easily from duality theory. The ultrafilters on $\prod^{\mathrm{w}}_{i\in I} A_i$ are essentially those of the $A_i$'s, plus the ultrafilter generated by $\{t_i : i \in I\}$, where

$$t_i(j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

$\mathfrak{t}$: Note that an equivalent definition of $\mathfrak{t}(A)$ is as follows: it is the least cardinal of a set $T$ of elements of $A$ such that $1 \notin T$, $T$ is well-ordered by $\leq$, and $\sum T = 1$.

Now if $|I| = \omega$, say that $I = \{i_m : m \in \omega\}$. For each $m \in \omega$ define $y^m \in B$ by setting, for any $n \in \omega$,

$$y^m_{i_n} = \begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$$

Clearly this gives a tower. So in this case, $\mathfrak{t}(B) = \omega$.

Now suppose that $|I| > \omega$. Suppose that $T$ is a tower in the above modified sense and $|T| < \min_{i \in I} \mathfrak{t}(A_i)$. We may assume that $|T|$ is regular.

*Case 1.* There is a $t \in T$ such that $J \overset{\text{def}}{=} \{i \in I : t_i \neq 1\}$ is finite. Then $\{y \restriction J : y \in T\}$ is a tower in $\prod_{i \in J} A_i$, contradicting Proposition 7.

*Case 2.* For every $t \in T$ the set $M_t \overset{\text{def}}{=} \{i \in I : t_i \neq 0\}$ is finite. If $|T| < |I|$, then $\left|\bigcup_{t \in T} M_t\right| < |I|$, and so there is an $i \in I$ such that $t_i = 0$ for all $t \in T$, contradicting $\sum T = 1$. So $|I| \leq |T|$, and hence $|T|$ is regular and uncountable. Hence there is a $t \in T$ such that $M_t = M_s$ whenever $t \leq s \in T$. This clearly contradicts $\sum T = 1$ again.

$\mathfrak{h}$: Suppose that $\kappa < \min_{i \in I} \mathfrak{h}(A_i)$ and $\langle P_\alpha : \alpha < \kappa \rangle$ is a system of partitions of $B$ with no common refinement. For each $i \in I$, let $Q^i_\alpha = \{x_i : x \in P_\alpha\} \backslash \{0\}$. Then $Q^i_\alpha$ is a partition of $A_i$. Let $R^i$ be a common refinement of all $Q^i_\alpha$. Let $S = \{\mathrm{up}^i(x) : i \in I, x \in R^i\}$. Then $S$ is a common refinement of all $P_\alpha$'s, contradiction.

$\mathfrak{r}$: Let $J$ be a countably infinite subset of $I$. We claim that $\{\mathrm{up}^i(1) : i \in J\}$ is weakly dense in $B$. For, let $x \in B$. Then $\{i \in I : x_i \neq 1\}$ is finite or $\{i \in I : -x_i \neq 1\}$ is finite. Hence there is an $i \in J$ such that $\mathrm{up}^i(1) \leq x$ or $\mathrm{up}^i(1) \leq -x$.

$\mathfrak{i}$: For convenience we suppose that $I = \kappa$, an infinite cardinal. For each $\alpha < \kappa$ let $\langle x_{\alpha.i} : i \in \omega \rangle$ be a system of independent elements of $A_\alpha$. We now define $y_n \in B$ for each $n \in \omega$: if $n \in \omega$ and $\alpha < \kappa$, then

$$y_n(\alpha) = \begin{cases} x_{\alpha.n-\alpha-1} & \text{if } \alpha < n, \\ 1 & \text{if } \alpha = n, \\ 0 & \text{if } n < \alpha. \end{cases}$$

We claim that $\langle y_n : n \in \omega \rangle$ is a maximal system of independent elements of $B$. For independence, suppose that $m \in \omega$ and $\varepsilon \in {}^m2$; we want to show that $z \overset{\text{def}}{=} \prod_{i<m} y_i^{\varepsilon(i)} \neq 0$. If $\varepsilon(i) = 0$ for all $i < m$, then

$$z(m) = \prod_{i<m} -y_i(m) = 1.$$

So, suppose that $\varepsilon(i) = 1$ for some $i < m$, and take the least such $i$. Then

$$\begin{aligned} z(i) &= \prod_{j<m} y_j(i) \\ &= \prod_{j<i} -y_j(i) \cdot y_i(i) \cdot \prod_{i<j<m} y_j^{\varepsilon(j)}(i) \\ &= \prod_{i<j<m} x^{\varepsilon(j)}_{i.j-i-1} \neq 0. \end{aligned}$$

So our system of elements is independent. Next, suppose that

$$w \in B^+ \backslash \{y_n : n \in \omega\};$$

we want to show that $\{y_n : n \in \omega\} \cup \{w\}$ is no longer independent. Wlog $\{\alpha < \kappa : w(\alpha) \neq 0\}$ is finite. If $w(\alpha) = 0$ for all $\alpha < \omega$, then $y_0 \cdot w = 0$, as desired. So assume that there is an $\alpha < \omega$ such that $w(\alpha) \neq 0$, and take the greatest such. Let $v = w \cdot \prod_{i \leq \alpha} -y_i \cdot y_{\alpha+1}$. We claim that $v = 0$ (as desired). To show this, let $\beta < \kappa$. If $\beta \leq \alpha$, then $v(\beta) \leq -y_\beta(\beta) = 0$. If $\beta = \alpha + 1$, then $v(\beta) \leq w(\beta) = 0$. Finally, if $\alpha + 1 < \beta$, then $v(\beta) \leq y_{\alpha+1}(\beta) = 0$.

$\mathfrak{s}$: For each $i \in I$ let $X_i$ be a splitting set for $A_i$ of size $\mathfrak{s}(A_i)$. Then

$$\{\mathrm{up}^i(x) : i \in I,\ x \in X_i\}$$

is clearly a splitting set for $B$, and this set has size $\max(|I|, \sup_{i \in I} \mathfrak{s}(A_i))$. Now suppose that $|Z| < \max(|I|, \sup_{i \in I} \mathfrak{s}(A_i))$ and $Z$ splits $B$. If $|Z| < |I|$, then there is an $i \in I$ such that $z_i = 0$ or $z_i = 1$ for all $z \in Z$, contradiction. So $|I| \leq |Z|$. Choose $i \in I$ such that $|Z| < \mathfrak{s}(A_i)$. Then $\{z_i : z \in Z\}$ splits $A_i$, contradiction. $\square$

PROPOSITION 9. *Suppose that $\langle A_i : i \in I\rangle$ is a system of atomless BAs, $I$ infinite. Let $B = \prod_{i \in I} A_i$. Then*

(i) *If $k \in \{\mathfrak{a}, \mathfrak{p}, \mathfrak{t}\}$, then $k(B) = \omega$.*
(ii) *If $k \in \{\mathfrak{u}, \mathfrak{i}\}$, then $k(B) \leq \min_{i \in I} k(A_i)$.*
(iii) *If $\kappa$ is the smallest cardinality of a subset of $\mathscr{P}(I)$ which generates a nonprincipal ultrafilter on $\mathscr{P}(I)$, and if $\kappa \leq \min_{i \in I} \mathfrak{u}(A_i)$, then $\kappa \leq \mathfrak{u}(B)$.*
(iv) *If $k \in \{\mathfrak{r}, \mathfrak{h}\}$, then $k(B) = \min_{i \in I} k(A_i)$.*
(v) *$\mathfrak{s}(B) = \sup_{i \in I} \mathfrak{s}(A_i)$.*

PROOF. $\mathfrak{a}$: write $I = \bigcup_{m \in \omega} J_m$, where each $J_m$ is nonempty and the $J_m$'s are pairwise disjoint. For each $m \in \omega$ define $\chi_m \in B$ by setting, for any $i \in I$,

$$\chi_m(i) = \begin{cases} 1 & \text{if } i \in J_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\chi_m : m \in \omega\}$ is a countably infinite partition of unity in $B$.

$\mathfrak{p} = \omega$ follows, while $\mathfrak{t}$ is treated similarly; see also Proposition 11.

(ii): Clear by Proposition 7.

(iii): Suppose not. Thus there is an ultrafilter $D$ on $B$ with a generating set $X$ such that $|X| < \kappa$. We may assume that $X$ is closed under multiplication. Let $E = \{y \subseteq I : \chi_y \in D\}$. Here $\chi_y$ is the characteristic function of $y$, equal to 1 on $y$ and to 0 on $I \backslash y$. So, $E$ is an ultrafilter on $\mathscr{P}(I)$. If $E$ is principal, then $D$ induces an ultrafilter on some $A_i$, and this contradicts $|X| < \kappa \leq \mathfrak{u}(A_i)$. So $E$ is nonprincipal. For each $x \in X$ let $y(x) = \{i \in I : x_i \neq 0\}$. Hence $y(x) \in E$ for each $x \in X$. So $\{y(x) : x \in X\}$ does not generate $E$. Hence we can choose $z \in E$ such that $y(x) \not\leq z$ for all $x \in X$. Hence $\chi_z \in D$, and $x \not\leq \chi_z$ for all $x \in X$, contradiction.

$\mathfrak{r}$: We have $\leq$ by Proposition 7. Now suppose that $X$ is weakly dense in $B$ and $|X| < \min_{i \in I} \mathfrak{r}(A_i)$. Then for each $i \in I$, the set $\{x_i : x \in X\} \backslash \{0\}$ is not weakly dense in $A_i$, and so there is an $a_i \in A_i$ such that for all $x \in X$, if $x_i \neq 0$, then $x_i \not\leq a_i$ and $x_i \not\leq -a_i$. Choose $x \in X$ such that $x \leq a$ or $x \leq -a$. Choose $i \in I$ such that $x_i \neq 0$. But then $x_i \leq a_i$ or $x_i \leq -a_i$, contradiction.

$\mathfrak{h}$: This is proved as with weak products.

$\mathfrak{s}$: For each $i \in I$ let $X_i$ be a splitting set for $A_i$ of size $\mathfrak{s}(A_i)$. Let $\kappa = \sup_{i\in I} \mathfrak{s}(A_i)$, and for each $i \in I$ let $\langle x^i_\alpha : \alpha < \kappa \rangle$ be an enumeration of $X_i$ (of course probably with repetitions). For each $\alpha < \kappa$ define $y^\alpha \in \prod_{i\in I} A_i$ by setting $y^\alpha_i = x^i_\alpha$. Let $Y = \{y^\alpha : \alpha < \kappa\}$. Clearly $Y$ splits $\prod_{i\in I} A_i$, and $|Y| \leq \kappa$.

Suppose that $Z$ splits $\prod_{i\in I} A_i$ and $|Z| < \kappa$. Choose $i \in I$ such that $|Z| < \mathfrak{s}(A_i)$. Then $\{z_i : z \in Z\}$ splits $A_i$, contradiction. $\square$

PROBLEM 2. *Give an exact expression for* $\mathfrak{u}(\prod_{i\in I} A_i)$ *and* $\mathfrak{i}(\prod_{i\in I} A_i)$.

We have not investigated the behaviour of our functions with respect to free products or ultraproducts.

The following result sheds some light on some of the examples below. Recall that the *cellularity* of a BA $A$ is the supremum $\mathrm{c}(A)$ of all cardinalities of disjoint subsets of $A$. Clearly $\mathfrak{t}(A) \leq \mathrm{c}(A)$ and $\mathfrak{a}(A) \leq \mathrm{c}(A)$.

PROPOSITION 10. *For any atomless BA $A$, if $\omega \leq \kappa < \mathfrak{t}(A)$, then $2^\kappa \leq \mathrm{c}(A)$.*

PROOF. We define elements $a_t$ of $A$ for each $t \in \bigcup_{\alpha\leq\kappa} {}^\alpha 2$ by recursion on $\alpha$. For $\alpha = 0$ and $t \in {}^\alpha 2$ we have $t = 0$; we set $a_0 = 1$. If $a_t$ has been defined for $t \in {}^\alpha 2$, split $a_t$ into two nonzero disjoint elements $a_{t\frown\langle 0\rangle}$ and $a_{t\frown\langle 1\rangle}$. If $\alpha$ is a limit ordinal $\leq \kappa$, $a_t$ has been defined for all $t \in \bigcup_{\beta<\alpha} {}^\beta 2$, and $s \in {}^\alpha 2$, let $a_s$ be a nonzero element $\leq a_{s\restriction\beta}$ for all $\beta < \alpha$. Clearly all of the elements $a_t$ for $t \in {}^\kappa 2$ are nonzero and pairwise disjoint, as desired. $\square$

PROPOSITION 11. *If $A$ is atomless, then $\mathfrak{p}(A) = \omega \Rightarrow \mathfrak{t}(A) = \omega$ and $\mathfrak{a}(A) = \omega$.*

PROOF. If $Y = \{a_i : i \in \omega\}$ satisfies the condition in the definition of $\mathfrak{p}(A)$, let $b_i = \sum_{j\leq i} a_j$ for all $i \in \omega$. Then $\langle b_i : i \in \omega\}$ is increasing with sum 1, and no $b_i$ is 1; this easily gives $\mathfrak{t}(A) = \omega$.

For the second statement, let $c_i = a_i \cdot \prod_{j<i} -a_j$. Clearly $\{c_i : i \in \omega\}$ is a partition of unity, and infinitely many $c_i$'s are nonzero. $\square$

For the next example and later too, we denote by $\mathrm{Fr}\kappa$ the free BA on $\kappa$ free generators.

EXAMPLE 12. *For $\kappa$ an infinite cardinal we have*

(i) $\mathfrak{s}(\mathrm{Fr}\kappa) = \omega$.
(ii) $\mathfrak{a}(\mathrm{Fr}\kappa) = \omega$.
(iii) $\mathfrak{r}(\mathrm{Fr}\kappa) = \kappa$.

PROOF. (i): Let $S$ be a denumerable set of free generators. Clearly $S$ is a splitting set.

(ii): Clear since $\mathrm{Fr}\kappa$ has ccc.

(iii): Suppose that $X$ is weakly dense and $|X| < \kappa$. For each $x \in X$ there is a finite set $Y_x$ of free generators such that $x \in \langle Y_x \rangle$. Let $y$ be a free generator not in any of the sets $Y_x$ for $x \in X$. Choose $x \in X$ such that $x \leq y$ or $x \leq -y$; this gives a contradiction. $\square$

In the next example and several others below, we use the interval algebra construction. Officially, according to Koppelberg [1989], this construction applies only to linear orders with a first element. When we mention linear orders without first element, we have in mind adjoining a new first element, which will usually be denoted by $-\infty$.

EXAMPLE 13. (i) $\mathfrak{s}(\text{Intalg}\mathbb{R}) = \omega$.
(ii) $\mathfrak{a}(\text{Intalg}\mathbb{R}) = \omega$.
(iii) $\mathfrak{u}(\text{Intalg}\mathbb{R}) = \omega$.
(iv) $\mathfrak{i}(\text{Intalg}\mathbb{R}) = \omega$.

PROOF. (i): $\{[r, s) : r < s \text{ rational}\}$ is clearly a splitting set.
(ii): Clear since $\text{Intalg}\mathbb{R}$ satisfies ccc.
(iii): $\{[-\infty, n) : n \in \mathbb{Z}\}$ clearly generates an ultrafilter.
(iv): Clear since $\text{Intalg}\mathbb{R}$ has countable independence. □

EXAMPLE 14. *Let* $A = \text{Intalg}(\kappa \times \mathbb{Q})$, *where* $\kappa$ *is an uncountable regular cardinal and the order on* $\kappa \times \mathbb{Q}$ *is lexicographic. Then*
(i) $\mathfrak{i}(A) = \omega$.
(ii) $\mathfrak{u}(A) = \omega$.
(iii) $\mathfrak{a}(A) = \omega$.
(iv) $\mathfrak{s}(A) = \kappa$.
(v) $\mathfrak{h}(A) = \omega$.

PROOF. (i): Clear since $A$ has no uncountable independent subset.
(ii): The set $\{[-\infty, (0, r)) : r \in \mathbb{Q}\}$ generates an ultrafilter.
(iii): The set

$$\{[-\infty, (0,0))\} \cup \{[(0,n), (0, n+1)) : n \in \omega\} \cup$$
$$\{[(1,n), (1, n+1)) : n \text{ a negative integer}\} \cup \{[(1,0), \infty)\}$$

is a denumerable partition of unity.

(iv): Suppose that $S$ is a splitting set, with $|S| < \kappa$. For each $\alpha < \kappa$ let $d_\alpha = [(\alpha, 0), (\alpha + 1, 0))$, and choose $a_\alpha \in S$ which splits $d_\alpha$. Then

$$\kappa = \bigcup_{s \in S} \{\alpha < \kappa : a_\alpha = s\},$$

so there is an $s \in S$ and an $M \in [\kappa]^\kappa$ such that $a_\alpha = s$ for all $\alpha \in M$. If $s$ contains an interval $[a, \infty)$, then it contains one of the $d_\alpha$'s, $\alpha \in M$. If it does not contain such an interval, then it is disjoint from some $d_\alpha$, $\alpha \in M$. This is a contradiction.

(v): For each positive integer $n$ let

$$P_n = \left\{ \left[ \left(\alpha, m + \frac{k}{n}\right), \left(\alpha, m + \frac{k+1}{n}\right) \right) : \alpha < \kappa, m \in \mathbb{Z}, 0 \leq k < n \right\}.$$

Thus $P_n$ is a partition of unity. Suppose that $Q$ is a common refinement. Choose $q \in Q$ such that $[(1,0), (1,1)) \cap q \neq 0$. So $q$ contains some interval $[a, b)$ such that $[(1,0), (1,1)) \cap [a, b) \neq 0$. Since $Q$ refines $P_1$, it follows that $a = (1, u)$ and $b = (1, v)$ for some $u, v$. Choose $n$ such that $\frac{1}{n} < v - u$. Then the fact that $Q$ refines $P_n$ leads to a contradiction. □

PROPOSITION 15. *Suppose that* $A$ *is complete and atomless. Then*
(i) $\mathfrak{a}(A) = \omega$.
(ii) $\mathfrak{t}(A) = \omega$.

PROOF. Clearly there is a denumerable partition of unity, and this gives rise to a tower of size $\omega$. □

We now recall the important construction of a complete BA starting with a partially ordered set $P$. For each $p \in P$ let $\mathscr{O}_p = \{q \in P : q \leq p\}$. These sets form a base for a topology on $P$, and we can consider the regular open algebra under this topology.

EXAMPLE 16. *Let $P$ be the partial order consisting of all functions mapping a countable subset of $\omega_1$ into 2, ordered by reverse inclusion. Let $A$ be the regular open algebra formed from $P$ in the natural way. Then*

(i) $\mathfrak{h}(A) = \omega_1$.
(ii) $\mathfrak{r}(A) = \omega_1$.

PROOF. For this partial order, the sets $\mathscr{O}_f$ are regular open.

(i): By Proposition 14.7 of Koppelberg [1989], $\mathfrak{h}(A) \geq \omega_1$. Now for each $\alpha < \omega_1$ let $h_\alpha = \{(\alpha, 0)\}$, and let $P_\alpha$ be the partition of unity $\{\mathscr{O}_{h_\alpha}, -\mathscr{O}_{h_\alpha}\}$. Suppose that $Q$ is a refinement of all $P_\alpha$. Pick any $a \in Q$, and pick any nonzero element $\mathscr{O}_p$ below $a$. Let $\alpha$ be an ordinal in $\omega_1$ which is not in the domain of $p$. Now $\mathscr{O}_p$ is below $\mathscr{O}_{h_\alpha}$ or $-\mathscr{O}_{h_\alpha}$, and this leads to a contradiction.

(ii): Suppose that $X \subseteq A$ is weakly dense and countable. Wlog for each $x \in X$ we can write $x = \mathscr{O}_{p_x}$. Let $M$ be the union of the domains of the functions $p_x$ for $x \in X$. Choose $\alpha \in \omega_1 \backslash M$, and define $\text{dmn}(r) = \{\alpha\}$, $r(\alpha) = 0$. Then for any $x \in X$, both $x \subseteq \mathscr{O}_r$ and $x \cap \mathscr{O}_r = 0$ are impossible, contradiction.

It remains to exhibit a weakly dense subset of size $\omega_1$. Let

$$X = \{f \in P : \exists\alpha < \omega_1[\text{dmn}(f) = \alpha \ \wedge \forall\xi < \alpha(f(\xi) = 0)]\}.$$

For any $g \in P$, if $g$ takes on only the value 0, then there is an $f \in X$ such that $g \subseteq f$, hence $\mathscr{O}_f \subseteq \mathscr{O}_g$. On the other hand, if $g$ takes on the value 1 somewhere, then there is an $f \in X$ such that $f$ and $g$ are incompatible, hence $\mathscr{O}_f \cap \mathscr{O}_g = 0$ and so $\mathscr{O}_f \subseteq -\mathscr{O}_g$. □

EXAMPLE 17. *Let $A$ be a denumerable atomless BA, let $\kappa$ be any infinite cardinal. Then $k(\prod^{\mathrm{w}}_{\alpha<\kappa} A) = \omega$ for $k \in \{\mathfrak{a}, \mathfrak{p}, \mathfrak{t}, \mathfrak{u}, \mathfrak{h}, \mathfrak{r}, \mathfrak{i}\}$, and $\mathfrak{s}(\prod^{\mathrm{w}}_{\alpha<\kappa} A_i) = \kappa$.*

PROOF. This is immediate from Proposition 8. □

For the next example we use the well-known notion of $\eta_\alpha$-sets. An *$\eta_\alpha$-set* is a linearly ordered set $L$ such that if $A$ and $B$ are subsets of $L$ of size less than $\aleph_\alpha$ and $a < b$ for all $a \in A$ and $b \in B$, then there is an $x \in L$ such that $a < x < b$ for all $a \in A$ and $b \in B$.

EXAMPLE 18. *Let $L$ be an $\eta_1$-set, and let $A = \text{Intalg}(L)$. Then*

(i) $\mathfrak{i}(A) = \omega$.
(ii) $\mathfrak{p}(A) \geq \omega_1$.
(iii) *If in addition $L$ is not an $\eta_2$-set, in particular if $|L| < 2^{\aleph_1}$, then $\mathfrak{p}(A) = \omega_1$.*

PROOF. (i) holds since independence is $\omega$. For (ii), suppose that $Y$ is a countable collection of elements of $A$ such that $\sum Y = 1$ while $\sum Y' \neq 1$ for every finite subset $Y'$ of $Y$. Let $L' = \{-\infty\} \cup L$. Wlog every element of $Y$ has the form $[a, b)$ with $a \in L'$ and $b \in L \cup \{\infty\}$.

(1) For every $a \in L'$ there is an $x \in Y$ such that $a \in x$.

In fact, otherwise the set $M \overset{\text{def}}{=} \{b : a < b \text{ and } [b, c) \in Y \text{ for some } c\}$ is a countable set of elements, and hence there is a $c$ such that $a < c$ and $c < b$ for all $b \in M$. But then $[a, c) \cap \sum Y = 0$, contradiction.

(2) There is an $a \in L'$ such that $[a, \infty) \in Y$.

In fact, suppose not. Let $N = \{b \in L' : [a,b) \in Y \text{ for some } a\}$. Then $N$ is a countable set, and so there exist $c, d$ such that $b < c < d$ for all $b \in N$. Hence $[c,d) \cap \sum Y = 0$, contradiction.

Now let $P = \{a : \text{there is a } b \text{ such that } [a,b) \in Y, \text{ and } [a,\infty) \text{ is contained in } \sum Y' \text{ for some finite } Y' \subseteq Y\}$. Then $-\infty \notin P$, by assumption. Hence $P$ is a countable set not having $-\infty$ as a member. Let

$$Q = \{x : \text{ there exists } [b,c) \in Y \text{ such that } b \notin P \text{ and } x \in \{b,c\}\}.$$

Then by (1), $-\infty \in Q$. If $x \in Q$ and $d \in P$, then $x < d$. In fact, choose $s, t, u$ such that $[d,s) \in Y$, $[d,\infty)$ is contained in a finite join of members of $Y$, $[t,u) \in Y$, $t \notin P$, and $x \in \{t,u\}$. If $d \le u$, then $t \in P$, contradiction. So $u < d$, and hence $x < d$.

Hence we can choose $d, e$ such that $b < d < e < c$ for all $b \in Q$ and $c \in P$. Choose $[u,v) \in Y$ such that $[d,e) \cap [u,v) \neq 0$. Then $u < e$, so $u \notin P$. Hence $v \in Q$, contradicting $d < v$.

Finally, for (iii), the additional assumption implies that there are sets $M, N$ of size at most $\omega_1$ such that $M < N$ and there is no element in between. Then $\{[a,b) : a \in M,\ b \in N\}$ shows that $\mathfrak{p}(A) \le \omega_1$. □

EXAMPLE 19. *Balcar and Simon* [1992] *have given an example in which* $\mathfrak{r}(A) < \mathfrak{u}(A)$.

For the next example we use the notion of $\kappa$-saturated Boolean algebra, a special case of the model-theoretic notion (see any modern logic textbook).

EXAMPLE 20. *Let $\kappa$ be an uncountable cardinal, let $A$ be atomless and $\kappa^+$-saturated, and let $B = \prod^{\mathrm{w}}_{\alpha<\kappa} A$. Then*

(i) $\mathfrak{t}(B) \ge \kappa^+$.
(ii) $\mathfrak{u}(B) \le \kappa$.
(iii) $\mathfrak{a}(B) = \mathfrak{p}(B) = \kappa$.
(iv) $\mathfrak{r}(B) = \mathfrak{i}(B) = \omega$.

PROOF. Immediate from Proposition 8. □

We call a system $\langle b_i : i \in I\rangle$ a *weakly infinite partition of unity* provided that $b_i \cdot b_j = 0$ for all distinct $i, j \in I$, $\sum_{i\in I} b_i = 1$, and $\{i \in I : b_i \neq 0\}$ is infinite. (Partitions are usually assumed to have nonzero elements, but we do not make this assumption.)

PROPOSITION 21. *There is an atomless BA $A$ such that*:

(i) $\mathfrak{t}(A) = \omega_1$.
(ii) $\mathfrak{u}(A) = \omega_1$.
(iii) $\mathfrak{a}(A) > \omega_1$.

To prove this we need two lemmas which may be of independent interest. Here we use the notation $\langle X\rangle^{\mathrm{fi}}$ for the filter generated by $X$, and $\langle X\rangle^{\mathrm{id}}$ for the ideal generated by $X$.

LEMMA 22. *Suppose that $A$ is a BA having a tower $\langle a_\alpha : \alpha < \omega_1\rangle$ such that $F \overset{\text{def}}{=} \langle\{a_\alpha : \alpha < \omega_1\}\rangle^{\mathrm{fi}}$ is an ultrafilter. Also suppose that $\langle b_\alpha : \alpha < \omega_1\rangle$ is a weakly infinite partition of unity in $A$.*

*Then $A$ has an extension $B$ in which $\langle a_\alpha : \alpha < \omega_1\rangle$ is still a tower, $\{a_\alpha : \alpha < \omega_1\}$ still filter-generates an ultrafilter, but $\langle b_\alpha : \alpha < \omega_1\rangle$ no longer has sum* 1.

PROOF. Let $A(x)$ be a free extension of $A$. For each $\beta < \omega_1$ let

$$I_\beta = \langle \{b_\alpha \cdot x : \alpha < \omega_1\} \cup \{a_\beta \cdot x\}\rangle^{\mathrm{id}}.$$

Thus clearly $A \cap I_\beta = \{0\}$.

(1) There is a $\beta < \omega_1$ such that $x \notin I_\beta$.

To prove (1) we take two cases.

*Case 1.* There is an $\alpha < \omega_1$ such that $b_\alpha \in F$. Say $a_\beta \leq b_\alpha$. Suppose that $x \in I_\beta$. Then we can write

$$x \leq b_{\alpha_0} \cdot x + \cdots + b_{\alpha_{m-1}} \cdot x + a_\beta \cdot x.$$

Choose $\gamma < \omega_1$ such that $\gamma \neq \alpha_0, \ldots, \alpha_{m-1}, \alpha$ and $b_\gamma \neq 0$. Then mapping $x$ to $b_\gamma$ and pointwise fixing $A$ yields $b_\gamma = 0$, contradiction.

*Case 2.* $-b_\alpha \in F$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$ let $\beta_\alpha$ be the least ordinal such that $a_{\beta_\alpha} \leq -b_\alpha$. Suppose that $\{\beta_\alpha : \alpha < \omega_1\}$ is bounded in $\omega_1$; say $\beta_\alpha < \gamma$ for all $\alpha < \omega_1$. Then $a_\gamma \leq -b_\alpha$ for all $\alpha < \omega_1$, contradiction. Thus $\{\beta_\alpha : \alpha < \omega_1\}$ is unbounded in $\omega_1$. Hence there is a strictly increasing sequence $\langle \alpha_\xi : \xi < \omega_1\rangle$ of countable ordinals such that $\beta_{\alpha_0} > 0$ and $\langle \beta_{\alpha_\xi} : \xi < \omega_1\rangle$ is strictly increasing. Note that $b_\alpha = 0 \Rightarrow \beta_\alpha = 0$. So $b_{\alpha_\xi} \neq 0$ for all $\xi < \omega_1$. Let

$$\Xi_\beta = \{\gamma < \omega_1 : b_\gamma \neq 0 \text{ and } a_\beta \cdot b_\gamma = 0\}$$

for all $\beta < \omega_1$. So $\beta < \delta < \omega_1 \Rightarrow \Xi_\beta \subseteq \Xi_\delta$. Now $\alpha_\xi \in \Xi_{\beta_{\alpha_\xi}}$ for all $\xi < \omega_1$. So $\Xi_{\beta_{\alpha_\omega}}$ is infinite. Now $x \notin I_{\beta_{\alpha_\omega}}$, for otherwise write

$$x \leq b_{\alpha_0} \cdot x + \cdots + b_{\alpha_{m-1}} \cdot x + a_{\beta_{\alpha_\omega}} \cdot x.$$

Choose $\gamma \in \Xi_{\beta_{\alpha_\omega}} \backslash \{\alpha_0, \ldots, \alpha_{m-1}\}$. Then the mapping $x \mapsto b_\gamma$, fixing $A$ pointwise, gives $b_\gamma = 0$, contradiction. Thus (1) holds in this case too.

Let $B = A(x)/I_\beta$ with $\beta$ as in (1). So $\langle b_\alpha : \alpha < \omega_1\rangle$, as a system of elements of $B$, no longer has sum 1. Next we claim that $\langle a_\alpha : \alpha < \omega_1\rangle$ is still a tower in $B$. For, suppose that $[c \cdot x + d \cdot -x] \leq [a_\gamma]$ for all $\gamma < \omega_1$. Then for all $\gamma < \omega_1$ we can write

(2) $(c \cdot x + d \cdot -x) \cdot -a_\gamma \leq b_{\alpha_{0\gamma}} \cdot x + \cdots + b_{\alpha_{m_\gamma - 1,\gamma}} \cdot x + a_\beta \cdot x;$

Then $x \mapsto 0$, fixing $A$ pointwise, shows that $d \leq a_\gamma$ for all $\gamma < \omega_1$, and hence $d = 0$. Then (2) for $\gamma = \beta$ gives

$$c \cdot x \cdot -a_\beta \leq b_{\alpha_{0\beta}} \cdot x + \cdots + b_{\alpha_{m_\beta - 1,\beta}} \cdot x + a_\beta \cdot x,$$

and hence $c \cdot x \leq b_{\alpha_{0\beta}} \cdot x + \cdots + b_{\alpha_{m_\beta - 1,\beta}} \cdot x + a_\beta \cdot x$. Hence $[c \cdot x] = 0$, as desired.

Clearly $\{a_\gamma : \gamma < \omega_1\}$ still generates an ultrafilter. □

PROPOSITION 23. *Suppose that $A$ is a BA having a tower $\langle a_\alpha : \alpha < \omega_1\rangle$ such that $F \overset{\text{def}}{=} \langle\{a_\alpha : \alpha < \omega_1\}\rangle^{\text{fi}}$ is an ultrafilter. Also suppose that $b \in A^+$. Then there is an extension $B$ of $A$ in which $\langle a_\alpha : \alpha < \omega_1\rangle$ is still a tower, $\{a_\alpha : \alpha < \omega_1\}$ still filter-generates an ultrafilter, and there is a $u \in B$ such that $0 < u < b$.*

PROOF. Let $A(y)$ be a free extension of $A$, and for each $\beta < \omega_1$ let

$$J_\beta = \langle\{y \cdot -b, a_\beta \cdot y\}\rangle^{\mathrm{id}}.$$

So $A \cap J_\beta = \{0\}$.

(1) There is a $\beta < \omega_1$ such that $y \notin J_\beta$.

For, choose $\beta < \omega_1$ such that $b \not\leq a_\beta$. Suppose that $y \in J_\beta$. Then

$$y \leq y \cdot -b + a_\beta \cdot y.$$

Mapping $y$ to $b$, pointwise fixing $A$, gives $b \leq a_\beta$, contradiction.

Take $\beta$ as in (1). Thus $0 < [y] \leq [b]$. Suppose that $[y] = [b]$. Then

$$b \cdot -y \leq y \cdot -b + a_\beta \cdot y.$$

Mapping $y$ to 0, pointwise fixing $A$, gives $b = 0$, a contradiction. Hence $[y] < [b]$.

Obviously $\{a_\alpha : \alpha < \omega_1\}$ still generates an ultrafilter.

Finally, suppose that $[c \cdot y + d \cdot -y] \leq [a_\alpha]$ for all $\alpha < \omega_1$. Thus

$$(c \cdot y + d \cdot -y) \cdot -a_\alpha \leq y \cdot -b + a_\beta \cdot y$$

for all $\alpha < \omega_1$. Mapping $y$ to 0, pointwise fixing $A$, shows that $d \leq a_\alpha$ for all $\alpha < \omega_1$, hence $d = 0$. So $c \cdot y \cdot -a_\beta \leq y \cdot -b + a_\beta \cdot y$, hence $c \cdot y \leq y \cdot -b + a_\beta \cdot y$ and so $[c \cdot y] = 0$, as desired. □

Proof of Proposition 21. We define $A_\alpha$ for $\alpha \leq \omega_2$ by induction. Let $A_0 = \mathrm{Intalg}\omega_1$, and $a_\alpha = [\alpha, \infty)$ for all $\alpha < \omega_1$. Let $\varphi(B)$ be the statement "$\langle a_\alpha : \alpha < \omega_1\rangle$ is a tower in $B$ and $\{a_\alpha : \alpha < \omega_1\}$ generates an ultrafilter of $B$". Thus $\varphi(A_0)$. Suppose that $A_\alpha$ has been defined so that $\varphi(A_\alpha)$ holds. Apply Lemma 22 many times to get an extension $B$ of $A_\alpha$ such that $\varphi(B)$ holds and every weakly infinite partition of unity of $A_\alpha$ is no longer such in $B$. Then apply Lemma 23 many times to get an extension $A_{\alpha+1}$ of $B$ such that $\varphi(A_{\alpha+1})$ holds and $A_{\alpha+1}$ is atomless. Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for $\alpha$ limit $\leq \omega_2$. Clearly $A_{\omega_2}$ is as desired. □

The following problems remain in our consideration of the relationships of the functions in atomless BAs.

Problem 3. *Can one construct in ZFC an atomless BA $A$ such that $\mathfrak{s}(A) < \mathfrak{a}(A)$?*

A weaker but still unsolved form of this problem is whether one can construct in ZFC an atomless BA $A$ such that $\mathfrak{h}(A) < \mathfrak{a}(A)$. See also Proposition 21.

Problem 4. *Can one construct in ZFC an atomless BA $A$ such that $\mathfrak{u}(A) < \mathfrak{i}(A)$?*

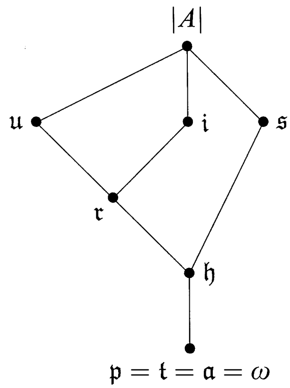
A weaker but still unsolved form of this problem is whether one can construct in ZFC an atomless BA $A$ such that $\mathfrak{r}(A) < \mathfrak{i}(A)$.

We finish this section by discussing the question whether any of our functions can have singular cardinals as values. We already noted in the definitional part that $\mathfrak{t}(A)$ is always regular. $\mathfrak{a}(A)$ can be singular, for example by Proposition 15 of Monk [2001]. $\mathfrak{p}(A)$ can be singular, by Example 20. $\mathfrak{s}(A)$ can be singular, by Example 17. And $\mathfrak{r}(A)$, $\mathfrak{i}(A)$, and $\mathfrak{u}(A)$ can be singular by Example 12. However, this is not true of $\mathfrak{h}(A)$:

Proposition 24. *For any atomless BA $A$, $\mathfrak{h}(A)$ is regular.*

Proof. Suppose to the contrary $\kappa \stackrel{\text{def}}{=} \mathfrak{h}(A)$ is singular. Let $\langle P_\alpha : \alpha < \kappa\rangle$ be a system of partitions of unity of $A$ with no common refinement. We may assume that $P_\beta$ refines $P_\alpha$ if $\alpha < \beta$. Let $\langle \lambda_\xi : \xi < \mathrm{cf}\kappa\rangle$ be an increasing system of infinite cardinals with supremum $\kappa$. Then $\langle P_{\lambda_\xi} : \xi < \mathrm{cf}\kappa\rangle$ has a common refinement; but clearly it is also a common refinement of $\langle P_\alpha : \alpha < \kappa\rangle$, contradiction. □

§3. **Complete Boolean algebras.** We begin with a diagram of relationships between the functions in this case.




**The functions for atomless complete Boolean algebras**

The following proposition shows that the relations in this diagram hold.

PROPOSITION 25. $\mathfrak{h}(A) \le \mathfrak{r}(A)$ *for any atomless complete BA* $A$.

PROOF. Suppose that $X$ is weakly dense and $|X| < \mathfrak{h}(A)$. For each $x \in X$ let $P_x = \{x, -x\}$, and let $Q$ be a common refinement of all of the $P_x$'s. For each $a \in Q$ choose nonzero disjoint elements $u_a, v_a \le a$. Let $y = \sum_{a\in Q} u_a$. Choose $x \in X$ such that $x \le y$ or $x \le -y$. Choose $a \in Q$ such that $a \cdot x \ne 0$. So $a \le x$. If $x \le y$, then $v_a \le y$; but clearly $v_a \cdot y = 0$, contradiction. If $x \le -y$, then $u_a \le -y$; but $u_a \le y$, contradiction. □

Again we need examples to show that the diagram really looks like this.

| | |
|---|---|
| $\omega < \mathfrak{h}(A)$: | example 16. |
| $\mathfrak{s}(A) < \mathfrak{r}(A)$: | Proposition 26 (for $\kappa$ uncountable). |
| $\mathfrak{i}(A) < \|A\|$: | Proposition 26 (for $\kappa < \kappa^\omega$). |
| $\mathfrak{s}(A) < \|A\|$: | Proposition 26 (for $\kappa$ uncountable). |
| $\mathfrak{u}(A) < \|A\|$: | Proposition 28. |
| $\mathfrak{i}(A) < \mathfrak{u}(A)$: | Proposition 27 (taking $A = \overline{\mathrm{Fr}\omega}$). |
| $\mathfrak{u}(A) < \mathfrak{s}(A)$: | Proposition 28. |
| $\mathfrak{i}(A) < \mathfrak{s}(A)$: | Proposition 28. |

This leaves one main question open: whether there is an atomless complete BA $A$ such that $\mathfrak{u}(A) < \mathfrak{i}(A)$.

PROPOSITION 26. $\kappa$ *be an infinite cardinal and let* $A = \overline{\mathrm{Fr}\kappa}$ *be the completion of* $\mathrm{Fr}\kappa$. *Then*

(i) $\mathfrak{s}(A) = \omega$.
(ii) $\mathfrak{i}(A) = \kappa$.
(iii) $\mathfrak{u}(A) \ge \kappa$.
(iv) $\mathfrak{r}(A) = \kappa$.

PROOF. Wlog $\kappa > \omega$. Let $X$ be a set of free generators of $\mathrm{Fr}\kappa$.

(i): Let $Y$ be a denumerable set of free generators. Suppose that $a \in A^+$. Choose a monomial $b$ over the set of free generators of $\mathrm{Fr}\kappa$ such that $b \le a$. Let $x \in Y$ not be among the free generators appearing in $b$. Then $b \cdot x \neq 0 \neq b \cdot -x$, and hence $a \cdot x \neq 0 \neq a \cdot -x$.

(ii): Clearly $\mathfrak{i}(A) \leq \kappa$. Now suppose that $Y$ is independent and $|Y| < \kappa$. For each $x \in Y$ write $x = \sum Y_x$, where $Y_x$ is a countable set of monomials in the free generators. Then let $u$ be a free generator not in the support of any member of $\bigcup_{x \in Y} Y_x$. Clearly then $Y \cup \{u\}$ is independent. This shows that $\mathfrak{i}(A) = \kappa$.

(iii): Suppose that $Y$ filter-generates $F$, where $F$ is an ultrafilter, $|Y| < \kappa$. For each $x \in X$ choose $\varepsilon(x) \in 2$ such that $x^{\varepsilon(x)} \in F$. Then choose $y_x \in Y$ such that $y_x \leq x^{\varepsilon(x)}$. Then there exist an infinite $Z \subseteq X$ and a $y \in Y$ such that $y \leq z^{\varepsilon(z)}$ for all $z \in Z$. Let $a$ be a monomial $\leq y$, and let $z \in Z$ not be in the support of $a$. Then $a \leq z^{\varepsilon(z)}$ is impossible.

(iv): Suppose that $Y$ is weakly dense in $A$, and $|Y| < \kappa$. For every $x \in X$ choose $y_x \in Y$ such that $y_x \leq x$ or $y_x \leq -x$. There is a $y \in Y$ such that $\{x \in X : y_x = y\}$ is uncountable. But $y$ has a countable support, so this easily gives a contradiction. □

Proposition 27. *$\mathfrak{u}(A) > \omega$ for every atomless complete BA $A$.*

Proof. Suppose that $F$ is an ultrafilter and $X$ is a countable set which generates $F$. Wlog $1 = x_0 > x_1 > \cdots$ with $X = \{x_0, x_1, \ldots\}$.

(1) $\prod X = 0$.

In fact, suppose that $\prod X \neq 0$. If $\prod X \in F$, say $x_i \leq \prod X$. So $\prod X = x_i$ and $F$ is principal, contradiction. If $-\prod X \in F$, say $x_i \leq -\prod X$. Then $\prod X = 0$, contradiction. So (1) holds.

Let $a_i = x_i \cdot -x_{i+1}$ for all $i \in \omega$. Note that $a_i \cdot a_j = 0$ for all distinct $i, j < \omega$.

(2) $\sum_{i \in \omega} a_i = 1$.

To prove (2), we first show by induction that $\sum_{i \leq m} a_i = -x_{m+1}$ for all $m \in \omega$. This is clear for $m = 0$. Assume that it is true for $m$. Then

$$\begin{aligned}\sum_{i \leq m+1} a_i &= -x_{m+1} + a_{m+1} = -x_{m+1} + x_{m+1} \cdot -x_{m+2} \\ &= -x_{m+1} + -x_{m+1} \cdot -x_{m+2} + x_{m+1} \cdot -x_{m+2} \\ &= -x_{m+1} + -x_{m+2} \\ &= -x_{m+2}.\end{aligned}$$

This finishes the inductive proof of $\sum_{i \leq m} a_i = -x_{m+1}$. Then (2) follows from (1).

Therefore $\sum_{i \in \omega} a_{2i}$ and $\sum_{i \in \omega} a_{2i+1}$ are complementary elements. If $\sum_{i \in \omega} a_{2i} \in F$, say $x_i \leq \sum_{i \in \omega} a_{2i}$. Wlog $i = 2j + 1$ for some $j$. So

$$a_{2j+1} = x_{2j+1} \cdot -x_{2j+2} \leq \sum_{i \in \omega} a_{2i},$$

contradiction. A similar contradiction is reached if $\sum_{i \in \omega} a_{2i+1} \in F$. □

Note that $\mathfrak{i}(A) < |A|$ for $A = \overline{\mathrm{Fr}\omega}$.

Proposition 28. *There is an atomless complete BA $A$ such that $\mathfrak{u}(A) < \mathfrak{s}(A)$ and $\mathfrak{i}(A) < \mathfrak{s}(A)$.*

Proof. Let $B = \overline{\mathrm{Fr}\omega}$. Note that $\mathfrak{u}(B) \leq 2^\omega$ and $\mathfrak{i}(B) = \omega$. Let $\kappa = 2^\omega$. Let $P$ be the set of all functions mapping an element of $[\kappa^+]^{\leq\kappa}$ into 2, ordered by reverse inclusion. Let $C$ be the regular open algebra of $P$. Then $\mathfrak{h}(C) \geq \kappa^+$ by Proposition 14.7 of Koppelberg [1989]. Hence $\mathfrak{s}(C) \geq \kappa^+$. Thus $B \times C$ is as desired. $\square$

Although Proposition 28 answers two natural questions, its proof is rather trivial. It would be more interesting to obtain such an example in which, additionally, $\mathfrak{s}(A \restriction a) = \mathfrak{s}(A)$ for every $a \in A^+$. See also Proposition 37.

It is natural to consider the relationship between $k(A)$ and $k(\overline{A})$ for our functions $k$:

PROPOSITION 29. *Suppose that $A$ is atomless. Then*

(i) $\mathfrak{s}(\overline{A}) \leq \mathfrak{s}(A)$.
(ii) $\mathfrak{s}(A) \leq \mathfrak{s}(\overline{A}) \cdot \mathrm{c}(A)$.
(iii) $\mathfrak{r}(A) \leq \mathfrak{r}(\overline{A})$.
(iv) $\mathfrak{h}(A) = \mathfrak{h}(\overline{A})$.

PROOF. (i): Suppose that $S$ splits $A$ and $|S| = \mathfrak{s}(A)$. Given $a \in \overline{A}^+$, choose $b \in A^+$ such that $b \leq a$. An element of $S$ that splits $b$ also splits $a$.

(ii): Let $S$ split $\overline{A}$, with $|S| = \mathfrak{s}(\overline{A})$. For each $s \in S$ there is a set $X_s \subseteq A^+$ such that $s = \sum X_s$ and $|X_s| \leq \mathrm{c}(A)$. Let $S' = \bigcup_{s \in S} X_s$. We claim that $S'$ splits $A$. For, let $a \in A$. Choose $s \in S$ such that $a \cdot s \neq 0 \neq a \cdot -s$. Choose $x \in X_s$ such that $a \cdot x \neq 0$. Clearly also $a \cdot -x \neq 0$.

(iii): Suppose that $X$ is weakly dense in $\overline{A}$. For every $a \in X$ choose $b_a \in A^+$ such that $b_a \leq a$. Suppose that $c \in A$. Choose $a \in X$ such that $a \leq c$ or $a \leq -c$. Then $b_a \leq c$ or $b_a \leq -c$.

(iv) First suppose that $\langle P_\alpha : \alpha < \mathfrak{h}(A)\rangle$ is a system of partitions of unity in $A$ with no common refinement. Suppose that $Q$ is a partition of unity in $\overline{A}$ which refines all the $P_\alpha$'s. For each $a \in Q$ let $X_a \subseteq A^+$ be disjoint and such that $a = \sum X_a$. Let $R = \bigcup_{a \in Q} X_a$. Then $R$ is a partition of unity in $A$ which refines all $P_\alpha$'s, contradiction. It follows that $\mathfrak{h}(\overline{A}) \leq \mathfrak{h}(A)$.

Second, suppose that $\langle S_\alpha : \alpha < \mathfrak{h}(\overline{A})\rangle$ is a system of partitions of unity in $\overline{A}$ with no common refinement. For each $x \in S_\alpha$ choose a disjoint $Y_{x\alpha} \subseteq A^+$ such that $x = \sum Y_{x\alpha}$. Let $S'_\alpha = \bigcup_{x \in S_\alpha} Y_{x\alpha}$. This is a partition of unity in $A$. If $T$ is a partition of unity in $A$ which refines all $S'_\alpha$'s, then it is also a partition of unity in $\overline{A}$ which refines all $S_\alpha$'s, contradiction. $\square$

EXAMPLE 30. *Let $\kappa$ be an uncountable regular cardinal, let $\kappa \times \mathbb{Q}$ be ordered lexicographically, and let $A = \mathrm{Intalg}(\kappa \times \mathbb{Q})$. Then $\mathfrak{s}(A) = \kappa$ but $\mathfrak{s}(\overline{A}) = \omega$.*

PROOF. $\mathfrak{s}(A) = \kappa$ by Example 14. Now let

$$X = \left\{ \sum_{\alpha<\kappa} [(\alpha, r), (\alpha, s)) : r < s \right\}.$$

We claim that $X$ splits $\overline{A}$. For, let $a \in \overline{A}^+$. Choose $[(\alpha, u), (\beta, v)) \leq a$. Take $r < s$ such that $u < r$ and, if $\beta = \alpha$, $s < v$. Then $\sum_{\gamma<\kappa}[(\gamma, r), (\gamma, s))$ splits $a$. $\square$

EXAMPLE 31. *There is a BA $A$ such that $\mathfrak{s}(A) < \mathfrak{s}(\overline{A}) \cdot \mathrm{c}(A)$.*

PROOF. Let $B = \mathrm{Intalg}(\kappa \times \mathbb{Q})$ as in Example 30, and let $A = {}^{\kappa^+}B$. Then $\mathfrak{s}(A) = \kappa$ and $\mathfrak{s}(\overline{A}) = \omega$ by Proposition 9, and $\mathrm{c}(A) = \kappa^+$. $\square$

EXAMPLE 32. *Let $A = \mathrm{Fr}\omega$. Then $\mathfrak{u}(A) = \omega < \mathfrak{u}(\overline{A})$.*

PROOF. Proposition 27. $\square$

PROPOSITION 33. (i) $\mathfrak{p}(\overline{A}) \leq \mathfrak{p}(A)$.
(ii) $\mathfrak{t}(\overline{A}) \leq \mathfrak{t}(A)$.
(iii) $\mathfrak{a}(\overline{A}) \leq \mathfrak{a}(A)$.
(iv) *There is an atomless BA $A$ in which each inequality in (i)–(iii) is proper.*

PROOF. (i)–(iii) hold since the left sides are equal to $\omega$. For an example as called for in (iv), see Example 18. □

COROLLARY 34. *If $L$ is an $\eta_1$-set, then* $\mathfrak{h}(\overline{\mathrm{Intalg}(L)}) > \omega$.

PROOF. The set $\{[x, y) : x, y \in L, x < y\}$ is $\omega_1$-closed and dense in $\overline{\mathrm{Intalg}(L)}$, so $\overline{\mathrm{Intalg}(L)}$ is $\omega$-distributive. □

COROLLARY 35. *There is a BA $A$ such that* $\mathfrak{r}(A) < \mathfrak{r}(\overline{A})$ *and* $\mathfrak{i}(A) < \mathfrak{i}(\overline{A})$.

Concerning the relationships between $k(A)$ and $k(\overline{A})$, the following problems appear to be open.

PROBLEM 5. *Is there a BA $A$ such that* $\mathfrak{i}(\overline{A}) < \mathfrak{i}(A)$?

PROBLEM 6. *Is there a BA $A$ such that* $\mathfrak{u}(\overline{A}) < \mathfrak{u}(A)$?

PROPOSITION 36. $\mathfrak{i}(\mathscr{P}(\omega)) \geq \omega_1$.

PROOF. Suppose that $\langle a_i : i \in \omega\rangle$ is independent. Let

$$\mathscr{F} = \{(F, \varepsilon) : F \in [\omega]^{<\omega}, \varepsilon \in {}^F2\}.$$

Say $\mathscr{F} = \{(F_\alpha, \varepsilon_\alpha) : \alpha < \omega\}$. Choose distinct

$$y_\alpha, z_\alpha \in \bigcap_{i \in F_\alpha} a_\alpha^{\varepsilon_\alpha(i)} \backslash (\{y_\beta : \beta < \alpha\} \cup \{z_\beta : \beta < \alpha\}).$$

Let $b = \{y_\alpha : \alpha < \omega\}$. Given $\bigcap_{i\in F} a_i^{\varepsilon(i)}$, choose $\alpha$ such that $(F, \varepsilon) = (F_\alpha, \varepsilon_\alpha)$. Then $y_\alpha \in \bigcap_{i\in F} a_i^{\varepsilon(i)} \cap b$ and $z_\alpha \in \bigcap_{i\in F} a_i^{\varepsilon(i)} \cap -b$. □

The next result is relevant to Proposition 28. As far as the inequality $\mathfrak{u}(A) < |A|$ is concerned it gives a more interesting construction that that in the proof of Proposition 28.

PROPOSITION 37. *Suppose that $\kappa$ is an uncountable regular cardinal and $\kappa^{<\kappa} = \kappa$. Then there is an atomless complete BA $A$ such that $|A \restriction a| = 2^\kappa$ for each $a \in A^+$, while $A$ has an ultrafilter $F$ such that $\chi(F) \leq \kappa$.*

PROOF. Let $A$ be the completion of the BA

$$B \stackrel{\text{def}}{=} \oplus_{\alpha<\kappa}\left(\prod_{\alpha<\kappa}^{\mathrm{w}} \mathrm{Fr}\,\omega\right).$$

Clearly for each $b \in B^+$ there is a disjoint family $X \subseteq B \restriction b$ such that $|X| = \kappa$. Hence $|A \restriction a| = 2^\kappa$ for each $a \in A^+$. By Theorem 2.5 of Balcar and Simon [1989] $A$ has an ultrafilter $F$ such that $\chi(F) \leq \kappa$. □

For atomless complete Boolean algebras the following problem remains open. Here even consistency results would be interesting.

PROBLEM 7. *Is there an atomless complete BA $A$ such that* $\mathfrak{u}(A) < \mathfrak{i}(A)$?

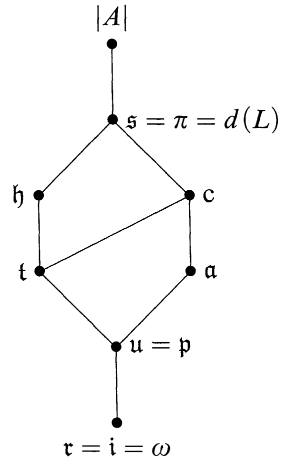
A weaker unsolved problem is whether there is a complete BA $A$ such that $\mathfrak{r}(A) < \mathfrak{i}(A)$.

**§4. Atomless interval algebras.** See the diagram below. Here we add some standard cardinal functions:

$\pi(A)$ $= \min\{X : X$ is a dense subset of $A\}$, where $X$ is *dense* provided that $X \subseteq A^+$ and for every $a \in A^+$ there is an $x \in X$ such that $x \leq a$.

d(L) $= \min\{|D| : D$ is a dense subset of $L\}$, where $L$ is a dense linear order. Here *dense* is taken in the usual sense for linear orders.

In the diagram, the functions are applied to an atomless interval algebra $A = \mathrm{Intalg}(L)$.




**The functions for atomless interval algebras**

First we give some results which prove that the relationships indicated in this diagram hold.

PROPOSITION 38. *If $A$ is an atomless interval algebra, then* $\mathrm{c}(A) \leq \mathfrak{s}(A)$.

PROOF. Suppose that $X$ is an infinite pairwise disjoint set; wlog each member of $X$ has the form $[a, b)$. Let $S$ be a splitting set. We want to show that $|X| \leq |S|$.

(1) Wlog each member of $S$ has the form $[a, b)$.

For, let $a \in A^+$. Choose $x \in S$ such that $a \cap x \neq 0 \neq a \backslash x$. Write

$$x = [b_0, c_0) \cup \ldots \cup [b_i, c_i).$$

Say $a \cap [b_i, c_i) \neq 0$. Now $[b_i, c_i) \subseteq x$, so $a \backslash x \subseteq a \backslash [b_i, c_i)$. Thus (1) holds.

(2) If $[c, d) \in S$ splits $[a, b) \in X$, then $a < c < b$ or $a < d < b$.

For, $[a, b) \cap [c, d) \neq 0$, so

(3) $\max(a, c) < \min(b, d)$

(4) $a < c$ or $d < b$.

For, if $c \leq a$ and $b \leq d$ then $[a, b) \backslash [c, d) = 0$, contradiction. Thus (4) holds.

From (3), $c < b$ and $a < d$. So (2) follows from (4).

By (2), each $[c, d) \in S$ splits at most two members of $X$. For each $x \in X$ choose $f(x) \in S$ which splits $x$. For every $s \in S$ there are at most two $x \in X$ such that $f(x) = s$. This shows that $|X| \leq |S|$, as desired. □

PROPOSITION 39. *Let $A = \mathrm{Intalg}L$, $A$ atomless. Then* $\pi(A) = \mathfrak{s}(A) = \mathrm{d(L)}$.

PROOF. 1) $\mathfrak{s}(A) \leq \mathrm{d(L)}$: Let $D$ be dense in $L$ with $|D| = \mathrm{d(L)}$. Let

$$S = \{[a, b) : a, b \in D, a < b\}.$$

Suppose $x \in A^+$. Say

$$x = [c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1}).$$

Choose $a, b \in D$ with $c_0 < a < b < d_0$. Clearly $[a, b)$ splits $x$.

2) $\mathrm{d(L)} \leq \pi(\mathrm{A})$: Let $X \subseteq A$ be dense, with $|X| = \pi(A)$. Wlog $x = [a_x, b_x)$ for all $x \in X$. Choose $a_x < c_x < b_x$. Let $D = \{c_x : x \in X\}$. Clearly $D$ is dense in $L$.

3) $\pi(A) \leq \mathrm{d(L)}$: Let $D \subseteq L$ be dense, $|D| = \mathrm{d(L)}$. Define

$$Y = \{[a, b) : a, b \in D, a < b\}.$$

Given $[c, d)$, choose $a, b \in D$ such that $c < a < b < d$. Then $[a, b) \subseteq [c, d)$.

4) $\mathrm{d(L)} \leq \mathfrak{s}(\mathrm{A})$: Let $S$ split $A$, $|S| = \mathfrak{s}(A)$. Let

$$D = \{a : a \text{ is a left or right endpoint of a summand of some element of } S\}.$$

Suppose that $c < d$. Let $x \in S$ split $[c, d)$. Then there is an $a \in D$ such that $c < a < d$, since otherwise $x \cap [c, d) = 0$ or $[c, d) \backslash x = 0$. □

The following proposition is well-known. It will not actually be used later, but it motivates the more complicated Proposition 41.

PROPOSITION 40. *Suppose that $A = \mathrm{Intalg}L$ is atomless. If $\langle a_\alpha : \alpha < \kappa \rangle$ is a strictly increasing sequence of elements of $A$ with $\kappa$ regular, then in $L$ there is a strictly increasing or strictly decreasing sequence of order type $\kappa$.*

PROOF. Wlog $\kappa$ is uncountable. Write

$$a_\alpha = [b_0^\alpha, c_0^\alpha) \cup \ldots \cup [b_{m_\alpha - 1}^\alpha, c_{m_\alpha - 1}^\alpha).$$

Wlog $m_\alpha = m$ does not depend on $\alpha$. We then proceed by induction on $m$. The case $m = 1$ is obvious. Now assume that $m > 1$. For every $\alpha > 0$ there is an $f(\alpha) < m$ such that

$$b_{f(\alpha)}^\alpha \leq b_0^0 < c_0^0 \leq c_{f(\alpha)}^\alpha.$$

If $0 < \alpha < \beta$, clearly $b_{f(\beta)}^\beta \leq b_{f(\alpha)}^\alpha$ and $c_{f(\alpha)}^\alpha \leq c_{f(\beta)}^\beta$. So wlog $\langle b_{f(\alpha)}^\alpha : \alpha < \kappa \rangle$ and $\langle c_{f(\alpha)}^\alpha : \alpha < \kappa \rangle$ are constant beyond some $\delta < \kappa$. Then the induction hypothesis applies to $\langle a_\alpha \backslash [b_{f(\alpha)}^\alpha, c_{f(\alpha)}^\alpha) : \alpha \in (\delta, \kappa) \rangle$. □

PROPOSITION 41. *Suppose that $A = \mathrm{Intalg}(L)$ is atomless and $\kappa$ is an uncountable regular cardinal. Then the following conditions are equivalent*:

(i) *There is a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of $A$ with sum* 1.

(ii) *One of the following holds*:

  (a) *There is a $c \in L$ and a strictly decreasing sequence $\langle b_\alpha : \alpha < \kappa \rangle$ of elements of $L$ coinitial with $c$.*

  (b) *There is a $c \in L \cup \{\infty\}$ and a strictly increasing sequence $\langle b_\alpha : \alpha < \kappa \rangle$ of elements of $L$ cofinal in $c$.*

  (c) *There exist a strictly increasing sequence $\langle b_\alpha : \alpha < \kappa \rangle$ of elements of $L$ and a strictly decreasing sequence $\langle c_\alpha : \alpha < \kappa \rangle$ of elements of $L$ such that $b_\alpha < c_\beta$ for all $\alpha, \beta < \kappa$ and there is no element $d$ of $L$ such that $b_\alpha < d < c_\beta$ for all $\alpha, \beta < \kappa$.*

(iii) *There is a* $d \in L$ *and a strictly increasing sequence* $\langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* $A$ *with sum* $[d, \infty)$.

PROOF. (i)$\Rightarrow$(iii): take $d = 0$, the smallest element of $L$.
(iii)$\Rightarrow$(i): for each $\alpha < \kappa$, let $b_\alpha = [0, d) \cup a_\alpha$.
(ii)$\Rightarrow$(i): Assume (a). Then $\{[c, b_\alpha) : \alpha < \kappa\}$ is a tower of size $\kappa$.
Assume (b). Then $\{[b_\alpha, c) : \alpha < \kappa\}$ is a tower of size $\kappa$.
Assume (c). Then $\{[b_\alpha, c_\alpha) : \alpha < \kappa\}$ is a tower of size $\kappa$.
(iii)$\Rightarrow$(ii): Write

$$a_\alpha = [b_0^\alpha, c_0^\alpha) \cup \ldots \cup [b_{m_\alpha - 1}^\alpha, c_{m_\alpha - 1}^\alpha).$$

Wlog $m_\alpha = m$ does not depend on $\alpha$. We then proceed by induction on $m$. The case $m = 1$ is clear, implying (a) or (b). Now assume that $m > 1$. Wlog $\langle b_0^\alpha : \alpha < \kappa \rangle$ is eventually equal to $d$; otherwise we get (a). If $\langle c_0^\alpha : \alpha < \kappa \rangle$ is eventually constant, the inductive hypothesis applies, since if $c_0^\alpha = e$ and $b_0^\alpha = d$ for $\alpha \geq \delta$, then $\langle a_\alpha \backslash [b_0^\alpha, c_0^\alpha) : \delta \leq \alpha < \kappa \rangle$ is strictly increasing with sum $[e, \infty)$. If $\langle c_0^\alpha : \alpha < \kappa \rangle$ is not eventually constant but is cofinal in some element $e \in L \cup \{\infty\}$, then (b) holds. So assume that $\langle c_0^\alpha : \alpha < \kappa \rangle$ is not eventually constant and is not cofinal in any element $d \in L \cup \{\infty\}$. Suppose that $\langle b_1^\alpha : \alpha < \kappa \rangle$ is eventually constant, say to an element $e$. Then there is an $f < e$ such that $c_0^\alpha < f$ for all $\alpha < \kappa$. But then $[f, e) \cap a_\alpha = 0$ for all $\alpha < \kappa$, contradiction. Thus $\langle b_1^\alpha : \alpha < \kappa \rangle$ is not eventually constant. If it is coinitial in some element $e$, a contradiction is reached again. Clearly then (c) must hold. □

COROLLARY 42. $\mathfrak{u}(A) \leq \mathfrak{t}(A)$ *for* $A$ *an atomless interval algebra.*

PROOF. This is immediate from the description of the character of ultrafilters in interval algebras given in Monk [1996, p. 188]. □

PROPOSITION 43. $\mathfrak{u}(A) = \mathfrak{p}(A)$ *for any atomless interval algebra* $A$.

PROOF. Since $\mathfrak{p}(A) \leq \mathfrak{u}(A)$ for any atomless BA $A$, it suffices to show that $\mathfrak{u}(A) \leq \mathfrak{p}(A)$ for any atomless interval algebra $A$. Let $A = \text{Intalg}(L)$, where $L$ is dense. For each ultrafilter $U$ on $A$ let $T_U = \{v \in L : [0, v) \in U\}$. This is the terminal segment of $L$ associated with $U$. Suppose that $\sum X = 1$ and $\sum F \neq 1$ for every finite $F \subseteq X$; we show that $\mathfrak{u}(A) \leq |X|$. Wlog each member of $X$ has the form $[u, v)$, with $u \in L$ and $v \in L \cup \{\infty\}$. Let $U$ be any ultrafilter containing $\{a : -a \in X\}$. Define

$$M = \{w : \exists v([v, w) \in X \text{ and } [w, \infty) \in U)\},$$
$$N = \{v : \exists w([v, w) \in X \text{ and } [0, v) \in U)\}.$$

Clearly then $M \subseteq L \backslash T_U$ and $N \subseteq T_U$.

*Case 1.* $\text{cf}(L \backslash T_U) \geq \text{ci}(T_U)$, where $\text{ci}(T_U)$ is the coinitiality of $T_U$. Thus $\chi(U) = \text{cf}(L \backslash T_U)$. So it suffices to show that $M$ is cofinal in $L \backslash T_U$. Suppose, to the contrary, that $x \in L \backslash T_U$ and $w < x$ for all $w \in M$. Choose $y$ with $x < y \in L \backslash T_U$. Choose $a \in X$ such that $a \cap [x, y) \neq 0$. Say $a = [u, v)$. So $\max(u, x) < \min(v, y)$. In particular, $u < y$, so $u \in L \backslash T_U$, hence $[0, u) \notin U$, and so $-a \cap [u, \infty) = [v, \infty) \in U$. It follows that $v \in M$, hence $v < x$, contradiction.

*Case 2.* $\text{cf}(L \backslash T_U) < \text{ci}(T_U)$. Similarly. □

This checks that the relations given in the diagram hold. To check that there are no further relations, the following examples are needed:

$\omega < \mathfrak{p}(A)$: example 18.
$\mathfrak{a}(A) < \mathrm{c}(A)$: example 14.
$\mathrm{c}(A) < \mathfrak{s}(A)$: a Souslin line.
$\mathfrak{t}(A) < \mathfrak{h}(A)$: example 44.
$\mathfrak{h}(A) < \mathfrak{s}(A)$: example 14.
$\mathfrak{a}(A) < \mathfrak{s}(A)$: example 14.
$\mathfrak{t}(A) < \mathfrak{a}(A)$: example 44.
$\mathfrak{a}(A) < \mathfrak{t}(A)$: No example is known.
$\mathfrak{h}(A) < \mathfrak{a}(A)$: No example is known.

An example with $\mathrm{c}(A) < \mathfrak{s}(A)$ really does require a Souslin line of some cardinality, as is well-known. Otherwise all of these examples are constructed in ZFC.

EXAMPLE 44. *Let $L$ be an $\eta_2$-set with no first or last element, and let $M \stackrel{\text{def}}{=} \omega_1 \times L$ be lexicographically ordered. Define $A = \text{Intalg}M$. Then*

(i) $\mathfrak{h}(A) \geq \omega_2$.
(ii) $\mathfrak{s}(A) \geq \omega_2$.
(iii) $\mathfrak{a}(A) \geq \omega_2$.
(iv) $\mathfrak{p}(A) = \omega_1$.
(v) $\mathfrak{t}(A) = \omega_1$.

PROOF. (iii): Suppose that $X$ is an infinite partition of unity and $|X| < \omega_2$. We may assume that each member of $X$ is a half-open interval. By the property of $\eta_2$-sets we have

(1) If $[(\beta, s), (\beta, t)) \in X$ and $[(\beta, t), \infty) \notin X$, then there is an $a$ such that $[(\beta, t), a) \in X$.
(2) For every $\beta < \omega_1$ the set $\{x \in X : \exists s, t[x = [(\beta, s), (\beta, t))]\}$ is finite.

For, suppose that the indicated set is infinite. Let

$$M = \{s : [(\beta, s), (\beta, t)) \in X \text{ for some } t\}.$$

*Case 1.* $M$ is well-ordered. Let $s_0, s_1, \ldots$ be the first $\omega$ members of $M$. Choose $v, w$ so that $s_i < v < w < t$ for all $i \in \omega$ and all $t$ such that $[(\gamma, t), a) \in X$ for some $a$ and $\gamma$, and $s_i < t$ for all $i \in \omega$. Then $[(\beta, v), (\beta, w)) \cap x = 0$ for all $x \in X$, contradiction.

*Case 2.* $M$ is not well-ordered. One gets a similar contradiction.

So, (2) holds. Hence $N \stackrel{\text{def}}{=} \{\alpha : [(\alpha, s), a) \in X \text{ for some } s, a\}$ is infinite. Let $\alpha_0 < \alpha_1 < \cdots$ be the first $\omega$ members of $N$. Let $\beta = \sup_{i \in \omega} \alpha_i$. Let $v < w$ be members of $L$. Then there is an $x \in X$ such that $[(\beta, v), (\beta, w)) \cap x \neq 0$. Then $x$ has the form $[(\beta, s), a)$. Choose $m < n$ in $L$ such that $n < t$ whenever $[(\beta, t), b) \in X$ for some $t, b$. Then $[(\beta, m), (\beta, n)) \cap y = 0$ for all $y \in X$, contradiction.

(iv): Fix $u \in L$. Let $P = \{[-\infty, (\beta, u)) : \beta < \omega_1\}$. Given a nonzero $a \in A$, there is a nonzero $[(\alpha, v), (\beta, w))$ below $a$. Then $[\infty, (\beta + 1, u)) \cap a \neq 0$. This shows that $\sum P = 1$. Clearly $\sum F \neq 1$ for every finite $F \subseteq P$. Hence $\mathfrak{p}(A) \leq \omega_1$; equality follows from (iii) and Proposition 11.

(i): By (iv), $\mathfrak{h}(A) \geq \omega_1$. Let $P_\alpha$ be a partition of unity for each $\alpha < \omega_1$. We may assume that no member of $P_\alpha$ has a summand of the form $[a, \infty)$ or $(-\infty, a)$. Let $Q$ be maximal pairwise disjoint such that $\forall a \in Q \forall \alpha < \omega_1 \exists b \in P_\alpha (a \subseteq b)$. It suffices now to take $a = [(\gamma, v), (\gamma, w))$ and show that $a \cap b \neq 0$ for some $b \in Q$. For every $\alpha < \omega_1$ choose $b_\alpha \in P_\alpha$ such that $a \cap b_\alpha \neq 0$. Say $[(\delta_\alpha, s_\alpha), (\varepsilon_\alpha, t_\alpha))$ is a summand of $b_\alpha$ such that $a \cap [(\delta_\alpha, s_\alpha), (\varepsilon_\alpha, t_\alpha)) \neq 0$. Say $(\gamma, u_\alpha) \in a \cap [(\delta_\alpha, s_\alpha), (\varepsilon_\alpha, t_\alpha))$. Thus for all $\alpha < \omega_1$ we have

(1) $v \leq u_\alpha < w$.
(2) $\delta_\alpha < \gamma$ or ($\delta_\alpha = \gamma$ and $s_\alpha \leq u_\alpha$).
(3) $\gamma < \varepsilon_\alpha$ or ($\gamma = \varepsilon_\alpha$ and $u_\alpha < t_\alpha$).

Define

$$m_\alpha = \begin{cases} s_\alpha & \text{if } \delta_\alpha = \gamma, \\ u_\alpha & \text{if } \delta_\alpha < \gamma; \end{cases}$$

$$n_\alpha = \begin{cases} \min(t_\alpha, w) & \text{if } \gamma = \varepsilon_\alpha, \\ w & \text{if } \gamma < \varepsilon_\alpha. \end{cases}$$

Then $m_\alpha \leq u_\alpha < n_\alpha \leq w$ for all $\alpha$. Choose $r, s$ so that $u_\alpha < r < s < n_\alpha$ for all $\alpha$; they exist sin $L$ is an $\eta_2$-set. Then

(4) $[(\gamma, r), (\gamma, s)) \subseteq a$.

For, $v \leq u_\alpha < r$ and $s < n_\alpha \leq w$, so (4) follows.

Finally, for any $\alpha < \omega_1$,

(5) $[(\gamma, r), (\gamma, s)) \subseteq [(\delta_\alpha, s_\alpha), (\varepsilon_\alpha, t_\alpha))$.

For, if $\delta_\alpha < \gamma$, clearly $(\delta_\alpha, s_\alpha) < (\gamma, r)$. If $\delta_\alpha = \gamma$, then $s_\alpha = m_\alpha \leq u_\alpha < r$, so $(\delta_\alpha, s_\alpha) < (\gamma, r)$. And if $\gamma < \varepsilon_\alpha$, clearly $(\gamma, s) < (\varepsilon_\alpha, t_\alpha)$. If $\gamma = \varepsilon_\alpha$ then $(\gamma, s) < (\gamma, n_\alpha) \leq (\gamma, t_\alpha)$, as desired.

From (4) and (5) the desired result follows.

(ii) follows from (i).

(v): See the proof of (iv). □

Example 45. *There is an atomless interval algebra $A$ such that* $\mathfrak{u}(A) < \mathfrak{t}(A)$.

Proof. Let $L$ be an $\eta_2$-set, and define $M$ to be the disjoint union

$$(\omega \times L) \cup (\omega_1^* \times L),$$

where each product is ordered lexicographically and each member of $\omega \times L$ is less than each member of $\omega_1^* \times L$. Notationally, members of the right-hand union are denoted by $(\alpha^*, u)$. Clearly

(1) Each element $(\alpha, u)$ has character $(\geq \omega_2, \geq \omega_2)$.

To determine $\mathfrak{u}(A)$ we need to consider the characters of terminal segments; see Monk [1996][page 188]. Clearly

(2) $\omega_1^* \times L$ has character $(\omega, \omega_1)$.

Now suppose that $T$ is a terminal segment different from $\omega_1^* \times L$.

*Case 1.* $T \subseteq \omega_1^* \times L$. If $T$ is empty, it has character $(\geq \omega_2, 1)$. Suppose that $T \neq 0$, and let $\beta = \sup\{\alpha : \exists u[(\alpha^*, u) \in T]\}$.

*Subcase 1.1.* There is a $u$ such that $(\beta^*, u) \in T$. Then the character of $T$ is $(\geq \omega_2, \geq \omega_2)$.

*Subcase 1.2.* For all $u$, $(\beta^*, u) \notin T$. Then $\beta$ is a limit ordinal, and the character of $T$ is $(\geq \omega_2, \omega)$.

*Case 2.* $T \cap (\omega \times L) \neq 0$. If $T = M$, then the character of $T$ is $(1, \geq \omega_2)$. Otherwise its character is $(\geq \omega_2, \geq \omega_2)$.

From this and Proposition 41 it follows that $\mathfrak{t}(A) \geq \omega_2$ and $\mathfrak{u}(A) \leq \omega_1$. □

PROBLEM 8. *Is there an atomless interval algebra $A$ such that $\mathfrak{a}(A) < \mathfrak{t}(A)$?*

PROBLEM 9. *Is there an atomless interval algebra $A$ such that $\mathfrak{h}(A) < \mathfrak{a}(A)$?*

**§5. Superatomic Boolean algebras.** As mentioned in the definitions of the functions, some of them are not defined for atomic algebras. So in considering them for superatomic algebras, we omit $\mathfrak{h}$, $\mathfrak{r}$, $\mathfrak{i}$, and $\mathfrak{s}$. Note also that $\mathfrak{t}(A)$ is not always defined. A variant of $\mathfrak{s}$ seems worth considering. A set $S$ *weakly splits* $A$ provided that for every $a \in A$ such that $A \upharpoonright a$ is infinite, there is an $s \in S$ such that $a \cdot s \neq 0 \neq a \cdot -s$. Then we let $\mathfrak{s}'(A)$ be the smallest size of a weakly splitting set. This coincides with $\mathfrak{s}(A)$ in case $A$ is atomless.

See below for the diagram for superatomic Boolean algebras. Of course the diagram, as far as $\mathfrak{t}$ is concerned, is to be understood as valid only when it is defined. Again we try to show that the diagram is exactly as indicated.

PROPOSITION 46. *Suppose that $\mathfrak{p}(A) < \mathfrak{u}(A)$. Then $A$ is not superatomic.*

PROOF. Suppose that $\prod X = 0$ while $\prod F \neq 0$ for all finite $F \subseteq X$, and $|X| < \mathfrak{u}(A)$. Let $U = \langle\{\prod F : F \in [X]^{<\omega}\}\rangle^{\text{fi}}$.

(1) There is no atom in $U$.

For, suppose that $a \in U$ is an atom. Choose $F \in [X]^{<\omega}$ such that $\prod F \leq a$. Choose $y \in X$ such that $a \leq -y$ (possible since $\sum_{y \in X} -y = 1$). Then $\prod F \cdot y = 0$, contradiction.

Thus (1) holds. It follows that $U$ is not an ultrafilter. Choose $d_0$ such that $d_0, -d_0 \notin U$. Thus

$$\forall F \in [X]^{<\omega}\left[\prod F \cdot d_0 \neq 0 \neq \prod F \cdot -d_0\right].$$

Now suppose that we have defined $d_0, \dots, d_{m-1}$ so that

$$\forall \varepsilon \in {}^m2 \forall F \in [X]^{<\omega}\left[\prod F \cdot \prod_{i<m} d_i^{\varepsilon(i)} \neq 0\right].$$

For each $\varepsilon \in {}^m2$ let

$$V_\varepsilon = \left\langle \left\{\prod F : F \in [X]^{<\omega}\right\} \cup \left\{\prod_{i<m} d_i^{\varepsilon(i)}\right\}\right\rangle^{\text{fi}}.$$

(2) There is no atom in $V_\varepsilon$.

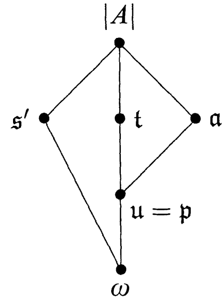
The proof is very similar to that of (1). Because of (2), $V_\varepsilon$ is not an ultrafilter, and so there is a $y_\varepsilon$ such that

$$\forall F \in [X]^{<\omega}\left[\prod F \cdot \prod_{i<m} d_i^{\varepsilon(i)} \cdot y_\varepsilon \neq 0 \neq \prod F \cdot \prod_{i<m} d_i^{\varepsilon(i)} \cdot -y_\varepsilon\right].$$

Let $d_m = \sum_{\varepsilon \in {}^m2}\left(\prod_{i<m} d_i^{\varepsilon(i)} \cdot y_\varepsilon\right)$. It is easy to see that $\prod_{i \leq m} d_i^{\varepsilon(i)} \neq 0$ for all $\varepsilon \in {}^{m+1}2$.

By this construction, $\{d_i : i < \omega\}$ is an infinite independent set, and hence $A$ is not superatomic. □

This proposition shows that the relations in the diagram hold.




**The functions for superatomic Boolean algebras**

To see that no other relations hold, we need some examples:

$\mathfrak{s}'(A) < \mathfrak{p}(A)$: example 47.
$\mathfrak{t}(A) < \mathfrak{s}'(A)$: example 48.
$\mathfrak{a}(A) < \mathfrak{s}'(A)$: example 48.
$\mathfrak{t}(A) < \mathfrak{a}(A)$: example 49.
$\mathfrak{a}(A) < \mathfrak{t}(A)$: example 49.

EXAMPLE 47. *Let* $A = \text{Finco}\kappa$. *Then*:

(i) $\mathfrak{s}'(A) = \omega$.
(ii) $\mathfrak{p}(A) = \kappa$.
(iii) $\mathfrak{t}(A)$ *is defined iff* $\kappa = \omega$.

PROOF. (i): An infinite set of atoms weakly splits $A$.

(ii): If $|Y| = \mathfrak{p}(A)$ and $Y$ satisfies the conditions for $\mathfrak{p}$, then all members of $Y$ are finite, and so $|Y| = \kappa$.

(iii): Obvious. □

For the next example, recall the notation $\langle X\rangle^A$ for the subalgebra of $A$ generated by $X$.

EXAMPLE 48. *Let $X$ be a partition of $\omega_1$ into $\omega_1$ sets each of size $\omega$. Let* $A = \langle X \cup \{\{\alpha\} : \alpha < \omega_1\}\rangle^{\mathscr{P}(\omega_1)}$. *Then*

(i) $\mathfrak{a}(A) = \omega$.
(ii) $\mathfrak{t}(A) = \omega$.
(iii) $\mathfrak{u}(A) = \omega$.
(iv) $\mathfrak{s}'(A) = \omega_1$.

PROOF. For (i), we can choose $x \in X$ and take the partition

$$\{\omega_1 \backslash x\} \cup \{\{\alpha\} : \alpha \in x\}.$$

For (ii), again take any $x \in X$; write $x = \{\alpha_i : i < \omega\}$, and take the tower $\{x \backslash \{\alpha_i : i < m\} : m \in \omega\}$. Now (iii) follows from (ii).

To check (iv) the following fact is useful:

FACT. *The elements of $A$ have one of the following two forms*:

$$F \cup \left(\bigcup G \backslash H\right) \qquad \text{with } F, H \in [\omega_1]^{<\omega} \text{ and } G \in [X]^{<\omega},$$

$$F \cup \left(\omega_1 \backslash \left(\bigcup G \cup H\right)\right) \qquad \text{with } F, H \in [\omega_1]^{<\omega} \text{ and } G \in [X]^{<\omega}.$$

It is straightforward to check this fact, by showing that the collection of indicated elements is closed under $\cap$ and $-$. We call $x$ *of type* 1 or 2 depending on whether the first or the second possibility given in the fact is true. For $x = F \cup \left(\bigcup G \backslash H\right)$ we may assume that $H \subseteq \bigcup G$ and $F \cap \bigcup G = 0$. For $x = F \cup \left(\omega_1 \backslash \left(\bigcup G \cup H\right)\right)$ we may assume that $H \cap \bigcup G = 0$ and $F \subseteq \bigcup G$.

(1) If $F_1 \cup \left(\bigcup G_1 \backslash H_1\right) \subseteq F_2 \cup \left(\bigcup G_2 \backslash H_2\right)$ with the above assumptions, then $G_1 \subseteq G_2$.

In fact, suppose that $z \in G_1 \backslash G_2$. Choose $\alpha \in z \backslash (H_1 \cup F_2)$. Then $\alpha \in F_1 \cup \left(\bigcup G_1 \backslash H_1\right)$ but $\alpha \notin F_2 \cup \left(\bigcup G_2 \backslash H_2\right)$.

(2) If $F_1 \cup \left(\omega_1 \backslash \left(\bigcup G_1 \cup H_1\right)\right) \subseteq F_2 \cup \left(\omega_1 \backslash \left(\bigcup G_2 \cup H_2\right)\right)$, then $G_2 \subseteq G_1$.

For, suppose that $G_2 \nsubseteq G_1$; choose $z \in G_2 \backslash G_1$. Then choose $\alpha \in z \backslash (H_1 \cup F_2)$. Thus $\alpha \in F_1 \cup \left(\omega_1 \backslash \left(\bigcup G_1 \cup H_1\right)\right)$ but $\alpha \notin F_2 \cup \left(\omega_1 \backslash \left(\bigcup G_2 \cup H_2\right)\right)$.

Now suppose that $S$ weakly splits $A$ and $S$ is countable. Let $Y$ be the set of all $z \in X$ satisfying one of the following conditions:

(a) $z \in G$ for some $G$ occurring in a representation of an $s \in S$ as of type 1 or 2.
(b) $z \cap F \neq 0$ for some $F$ occurring in a representation of an $s \in S$ as of type 1 or 2.
(c) $z \cap H \neq 0$ for some $H$ occurring in a representation of an $s \in S$ as of type 1 or 2.

Thus $Y$ is a countable set, so we can choose $z \in X \backslash Y$. Let $s \in S$ split $z$.

*Case 1.* $s = F \cup \left(\bigcup G \backslash H\right)$ with the above assumptions. Then $s \cap z \neq 0$ implies that $F \cap z \neq 0$ or $(\bigcup G \backslash H) \cap z \neq 0$. But $F \cap z \neq 0$ implies that $z \in Y$, and $(\bigcup G \backslash H) \cap z \neq 0$ implies that $z \in G$ and hence $z \in Y$, contradiction.

*Case 2.* $s = F \cup \left(\omega_1 \backslash \left(\bigcup G \cup H\right)\right)$ with the above assumptions. Since $z \backslash s \neq 0$, we have $z \cap \left(\bigcup G \cup H\right) \neq 0$, contradiction. □

The idea of the following example is due to Mati Rubin.

EXAMPLE 49. *Let $\kappa$ and $\lambda$ be uncountable regular cardinals. Then there is a superatomic BA $A$ such that*

(i) $\mathfrak{t}(A) = \kappa$
(ii) $\mathfrak{a}(A) = \lambda$.
(iii) $\mathfrak{s}'(A) \leq \kappa$.

PROOF. We define a poset $P$. Its elements are denoted by $a_\alpha$ for $\alpha < \kappa$ and $x_{\alpha\beta}$ for limit $\alpha < \kappa$ and $\beta < \lambda$. All of these elements are assumed to be distinct. The ordering is defined by the following conditions:

$$\begin{aligned} a_\alpha < a_\beta &\quad \text{iff} \quad \alpha < \beta \\ a_\alpha < x_{\gamma\beta} &\quad \text{iff} \quad \alpha < \gamma \\ x_{\gamma\beta} < a_\delta &\quad \text{iff} \quad \gamma \leq \delta \\ x_{\gamma\beta} < x_{\delta\varepsilon} &\quad \text{iff} \quad \gamma < \delta. \end{aligned}$$

Note that each element $a_\alpha$ is comparable with all elements of $P$. Let $A$ be the subalgebra of $\mathscr{P}(P)$ generated by all sets $[b, c)$ with $c$ not an $x_{\alpha\beta}$. Note that $[x_{\gamma\beta}, a_\gamma) = \{x_{\gamma\beta}\}$, so $\{x_{\gamma\beta}\} \in A$, for all limit $\gamma < \kappa$ and all $\beta < \lambda$. Also, $\{a_\gamma\} = [a_\gamma, a_{\gamma+1})$ for all $\gamma < \kappa$. Note that $A/I_{\mathrm{at}}$ is isomorphic to $\mathrm{Intalg}(\kappa)$, so $A$ is superatomic. ($I_{\mathrm{at}}$ is the ideal generated by the atoms.)

(1) Every element of $A$ can be written in the form

$$x = F \cup ([c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1}))\backslash G,$$

where the following conditions hold:
(a) $F$ and $G$ are finite sets of $x_{\alpha\beta}$'s.
(b) no $d_i$ is an $x_{\alpha\beta}$, and no $c_i$ is an $x_{\alpha\beta}$.
(c) $c_0 < d_0 < c_1 < \cdots < c_{m_1} < d_{m-1} \leq \infty$.
(d) $G \subseteq [c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1})$
(e) $F \cap ([c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1})) = 0$.

To prove this, first note that each generator of $A$ has this form; note that $[x_{\gamma\beta}, y) = \{x_{\gamma\beta}\} \cup [a_\gamma, y)$ for any $y$. Clearly then it suffices to show that the set of elements of this form is closed under unions and complements.

That the set is closed under unions follows from the following arithmetic law holding in any BA:

$$a \cdot -g + b \cdot -h = (a + b) \cdot -(g \cdot -b + h \cdot -a + g \cdot h).$$

Closure under complementation is more complicated. By the simple arithmetic law $-(f + b \cdot -g) = g \cdot -f + -b \cdot -f$, it suffices to show that the complement of the big sum which is a part of $x$ again is of the form in (1). In fact, clearly

$$\begin{aligned}&-([c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1}))\\ &\qquad = [a_0, c_0) \cup [d_0, c_1) \cup \ldots \cup [d_{m-2}, c_{m-1}) \cup [d_{m-1}, \infty).\end{aligned}$$

This completes the proof of (1).

Now note that the following is a partition of unity:

$$\{\{x_{\omega\beta}\} : \beta < \lambda\} \cup \{\{a_i\} : i < \omega\} \cup \{[a_\omega, \infty)\}.$$

Suppose that $\mathscr{P}$ is an infinite partition of unity with $|\mathscr{P}| < \lambda$. By (1) we may assume that if $x \in \mathscr{P}$, then $x$ either has the form $F_x$, a finite set of $x_{\alpha\beta}$'s, or $[b_x, c_x)\backslash G_x$, $G_x$ a finite set of $x_{\alpha\beta}$'s, with $b_x, c_x$ not $x_{\alpha\beta}$'s.

We now define $\alpha_0, \ldots$ and $G_0, \ldots$ by induction. Let $\alpha_0 = 0$. Now $\{a_0\} \cap x \neq 0$ for some $x \in \mathscr{P}$. By the assumption on the form of the elements of $P$, we can write $x = [a_0, c)\backslash G$ for some $c$ which is not an $x_{\alpha\beta}$. Now $c \neq \infty$, since otherwise $\mathscr{P}$ would be finite. Say $c = a_{\alpha_1}$, and let $G_0 = G$.

Suppose that $\alpha_0, \ldots, \alpha_i$ and $G_0, \ldots, G_{i-1}$ have been defined so that $\alpha_0 < \cdots < \alpha_i$, $G_0, \ldots, G_{i-1}$ are finite sets of $x_{\alpha\beta}$'s, and

$$[a_{\alpha_0}, a_{\alpha_1})\backslash G_0, \ldots, [a_{\alpha_{i-1}} a_{\alpha_i})\backslash G_{i-1} \in \mathscr{P}.$$

Then by an argument similar to the above, there is an $\alpha_{i+1} > \alpha_i$ and a finite set $G_i$ of $x_{\alpha\beta}$'s such that $[a_{\alpha_i}, a_{\alpha_{i+1}})\backslash G_i \in \mathscr{P}$.

Let $\beta = \sup_{i\in\omega} \alpha_i$. Thus $\beta < \kappa$ since $\mathrm{cf}\kappa > \omega$. By an argument above, there are $c, G$ such that $[a_\beta, c)\backslash G \in \mathscr{P}$. Now if $\delta < \lambda$, then $\{x_{\alpha_\beta\delta}\} \cap x \neq 0$ for some $x \in \mathscr{P}$. The element $x$ cannot have the form $[a_\delta, c)\backslash G$, so $x$ is a finite set of $x_{\mu\nu}$'s. Since $|\mathscr{P}| < \lambda$, not all $x_{\mu\nu}$'s are in $\bigcup\mathscr{P}$, contradiction. This shows that $\mathfrak{a}(A) = \lambda$.

Note that $\{[a_\alpha, \infty) : \alpha < \kappa\}$ is a tower. Suppose that $\langle y_\alpha : \alpha < \mu\rangle$ is strictly increasing with sum 1, and $\omega \leq \mu < \kappa$ with $\mu$ regular; we want to get a contradiction. Write

$$y_\alpha = F_\alpha \cup ([c_0^\alpha, d_0^\alpha) \cup \ldots \cup [c_{m_\alpha-1}^\alpha, d_{m_\alpha-1}^\alpha))\backslash G_\alpha,$$

with obvious assumptions. If $d^\alpha_{m_\alpha-1} \neq \infty$ for all $\alpha < \mu$, say $d^\alpha_{m_\alpha-1} = a_{\xi_\alpha}$ for all $\alpha < \mu$. Hence, since $\mu < \kappa$ and $\kappa$ is regular, there is an $\eta < \kappa$ such that $\xi_\alpha < \eta$ for all $\alpha < \mu$. Then $\{a_\eta\} \cdot \sum_{\alpha<\mu} y_\alpha = 0$, contradiction. So wlog $d^\alpha_{m_\alpha-1} = \infty$ for all $\alpha < \mu$.

Next we claim that if $\alpha < \beta < \mu$, then $c^\alpha_{m_\alpha-1} \geq c^\beta_{m_\beta-1}$. For, suppose that $c^\alpha_{m_\alpha-1} < c^\beta_{m_\beta-1}$. If $m_\beta = 1$, or $m_\beta > 1$ and $d_{m_\beta-2} \leq c^\alpha_{m_\alpha-1}$, then $c^\alpha_{m_\alpha-1} \in y_\alpha \backslash y_\beta$, contradiction. If $m_\beta > 1$ and $d_{m_\beta-2} > c^\alpha_{m_\alpha-1}$, then $d_{m_\beta-2} \in y_\alpha \backslash y_\beta$, contradiction. So our claim holds.

Now $c^0_{m_0-1} \geq c^1_{m_1-1} \geq \cdots$, so $\langle c^\alpha_{m_\alpha-1} : \alpha < \mu \rangle$ is eventually constant, say equal to $e$. Wlog $c^\alpha_{m_\alpha-1} = e$ for all $\alpha < \mu$. Hence $m_\alpha > 1$ for all $\alpha < \mu$. Then for all $\alpha < \mu$ we have $d^\alpha_{m_\alpha-2} < e$. Say $d^\alpha_{m_\alpha-2} = a_{\rho_\alpha}$ and $e = a_\sigma$. Let $\tau = \sup_{\alpha<\mu} \rho_\alpha$. If $\tau < \sigma$, then $\{a_\tau\} \cdot \sum_{\alpha<\mu} y_\alpha = 0$, contradiction. So $\tau = \sigma$.

Let $Y = \{x_{\sigma\varepsilon} : \varepsilon < \lambda\}$. Now for any $\alpha < \mu$,

$$(*) \qquad Y \cap ([c^\alpha_0, d^\alpha_0) \cup \ldots \cup [c^\alpha_{m_\alpha-2} d^\alpha_{m_\alpha-1}) \cup [e, \infty)) = 0,$$

so $Y = \bigcup_{\alpha<\mu}(Y \cap F_\alpha)$. If $\alpha < \beta < \mu$, then $Y \cap F_\alpha \subseteq y_\beta$, and so by $(*)$ for $\beta$ we must have $Y \cap F_\alpha \subseteq F_\beta$. Since $Y$ is infinite, $\langle Y \cap F_\alpha : \alpha < \mu \rangle$ is not eventually constant. It follows that $\mu = \omega$ and $Y$ is countable, contradiction. □

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DEPARTMENT OF MATHEMATICS
395 UCB
UNIVERSITY OF COLORADO
BOULDER, COLORADO 80303-0395, USA
*E-mail*: Don.Monk@colorado.edu