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GENERALIZED FREE PRODUCTS

BY

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Abstract. A subalgebra B of the direct product $\prod_{i\in I} A_i$ of Boolean algebras is *finitely closed* if it contains along with any element f any other member of the product differing at most at finitely many places from f. Given such a B, let $B^\star$ be the set of all members of B which are nonzero at each coordinate. The *generalized free product* corresponding to B is the subalgebra of the regular open algebra with the poset topology on $B^\star$ generated by the natural basic open sets. Properties of this product are developed. The full regular open algebra is also treated.

A natural construction in the theory of partially ordered sets, particularly as considered in constructing generic extensions of models of set theory, is the product construction. If we apply this construction to Boolean algebras, it is natural to delete the zero elements in the factors; we then obtain a product which is no longer a Boolean algebra, but which can be embedded in one. When considering two Boolean algebras, this gives the well known and important construction of the free product. Applied to an infinite system of Boolean algebras the construction no longer coincides with the infinite free product. It gives a new construction of Boolean algebras, one that has evidently not been studied in general. The particular case of products of copies of $(\mathscr{P}(\omega)/\mathrm{fin})\setminus\{0\}$ has been studied; see, e.g., Spinas [96].

The purpose of this article is to develop the elementary properties of this construction for general Boolean algebras, mainly for incomplete generalized free products. Beginning the study of cardinal invariants for such generalized free products, we give some results on cellularity. Complete generalized free products are also discussed, and a simple application to a Boolean algebraic formulation of the Easton theorem for sets is given.

1. Definition and simple properties. For any function f, any element i of its domain, and any object a, $\mathcal{S}(f,i,a)$ is the function c with the same domain as f such that, for any element x of that domain,

$$c(x)=\begin{cases} f(x) & \text{if } x\neq i,\\ a & \text{if } x=i.\end{cases}$$

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Let $\langle A_i : i \in I\rangle$ be a system of BAs each with more than one element. A subalgebra B of $\prod_{i\in I} A_i$ is *finitely closed* provided that the following condition holds:

($\star$) For every $b \in B$, $i \in I$, and $a \in A_i$, the function $\mathcal{S}(b, i, a)$ is also in B.

Examples of finitely closed subalgebras of $\prod_{i\in I} A_i$ are $\prod_{i\in I} A_i$ itself, the weak product $\prod^{\mathrm{w}}_{i\in I} A_i$ consisting of all functions which are either 0 except for finitely many places or 1 except for finitely many places, and, more generally, for each infinite cardinal κ, the subalgebra

$$\Big\{b \in \prod_{i\in I} A_i : |\{i \in I : b_i \neq 0\}| < \kappa \text{ or } |\{i \in I : b_i \neq 1\}| < \kappa\Big\}.$$

It is also clear that any finitely closed subalgebra of $\prod_{i\in I} A_i$ contains $\prod^{\mathrm{w}}_{i\in I} A_i$. And note that if B is a finitely closed subalgebra of $\prod_{i\in I} A_i$, then $\{f\restriction J : f \in B\}$ is a finitely closed subalgebra of $\prod_{j\in J} A_i$ for any $J \subseteq I$.

Now let B be a finitely closed subalgebra of $\prod_{i\in I} A_i$. We define

$$B^\star = \{b \in B : \forall i \in I\ (b_i \neq 0)\}.$$

$B^\star$ is partially ordered by: $b \le c$ iff $\forall i \in I\ (b_i \le c_i)$. For each $b \in B^\star$ define

$$\mathscr{O}_b = \{x \in B^\star : x \le b\}.$$

These sets form a base for a topology on $B^\star$.

LEMMA 1.1. *$\mathscr{O}_b$ is regular open for every $b \in B^\star$.*

Proof. Note that $\operatorname{cl}\mathscr{O}_b = \{x \in B^\star : \mathscr{O}_x \cap \mathscr{O}_b \neq \emptyset\} = \{x \in B^\star : x$ and b are compatible$\}$. Now suppose that $y \in \operatorname{int}\operatorname{cl}\mathscr{O}_b$. Then for every $w \le y$, w and b are compatible. Suppose that $y \not\le b$. Choose $i \in I$ such that $y_i \not\le b_i$. Then $\mathcal{S}(y, i, y_i \cdot -b_i) \in B^\star$, $\mathcal{S}(y, i, y_i \cdot -b_i) \le y$, but $\mathcal{S}(y, i, y_i \cdot -b_i)$ and b are not compatible, contradiction. ∎

Now we define the *B-generalized free product* of the system $\langle A_i : i \in I\rangle$ to be the subalgebra of $\mathrm{RO}(B^\star)$ generated by all of the sets $\mathscr{O}_b$, $b \in B^\star$; this subalgebra is denoted by $\bigoplus^B_{i\in I} A_i$. Suppose that B is a finitely closed subalgebra of $\prod_{i\in I} A_i$, $i \in I$, and $a \in A_i$. Then we define

$$(g(i,a))_j = \begin{cases} a & \text{if } j = i,\\ 1 & \text{otherwise.}\end{cases}$$

Thus $g(i,a) \in B$. Now we define $f_i(a) = \mathscr{O}_{g(i,a)}$ for $a \neq 0$, and $f_i(0) = 0$. This defines $f_i : A_i \to \bigoplus^B_{i\in I} A_i$.

PROPOSITION 1.2. *f_i is an isomorphism of A_i into $\bigoplus^B_{i\in I} A_i$.*

Proof. Suppose that $a_0, a_1 \in A$; we show that $f_i(a_0 + a_1) = f_i(a_0) + f_i(a_1)$. If one of a_0, a_1 is 0, this is clear, so assume that both are nonzero. We want to show that $\mathscr{O}_{g(i,a_0+a_1)} = \operatorname{int}\operatorname{cl}(\mathscr{O}_{g(i,a_0)} \cup \mathscr{O}_{g(i,a_1)})$. Clearly $\mathscr{O}_{g(i,a_0)}$

$\cup \mathscr{O}_{g(i,a_1)} \subseteq \mathscr{O}_{g(i,a_0+a_1)}$, and hence $\operatorname{int}\operatorname{cl}(\mathscr{O}_{g(i,a_0)} \cup \mathscr{O}_{g(i,a_1)}) \subseteq \mathscr{O}_{g(i,a_0+a_1)}$. Now suppose that $x \in \mathscr{O}_{g(i,a_0+a_1)}$ and $y \le x$; we want to show that $\mathscr{O}_y \cap (\mathscr{O}_{g(i,a_0)} \cup \mathscr{O}_{g(i,a_1)}) \ne \emptyset$. Suppose that $\mathscr{O}_y \cap \mathscr{O}_{g(i,a_0)} = \emptyset$. Clearly then $y_i \cdot a_0 = 0$. Since $y \le x \in \mathscr{O}_{g(i,a_0+a_1)}$, it follows that $y_i \le a_0 + a_1$, so $y_i \le a_1$. So $y \in \mathscr{O}_{g(i,a_1)}$, as desired. Thus f_i preserves $+$.

To show that f_i preserves $-$, note first that $f_i(1) = 1$, and hence it suffices to take $a \in A_i$ such that $0 < a < 1$ and show that $f_i(-a) = -f_i(a)$. Now $-f_i(a) = -\mathscr{O}_{g(i,a)} = \operatorname{int}(B^\star \setminus \mathscr{O}_{g(i,a)})$. Clearly $f_i(-a) = \mathscr{O}_{g(i,-a)} \subseteq B^\star \setminus \mathscr{O}_{g(i,a)}$, and hence $f_i(-a) \subseteq -f_i(a)$. Now suppose that $x \in -f_i(a)$. If $x_i \cdot a \ne 0$, then clearly $\mathscr{O}_x \cap \mathscr{O}_{g(i,a)} \ne 0$, contradiction. So $x_i \cdot a = 0$, hence $x \in \mathscr{O}_{g(i,-a)} = f_i(-a)$, as desired. So f is a homomorphism. Clearly it is one-one. ■

Proposition 1.3. *If $b \in B^\star$, then $\mathscr{O}_b = \bigcap_{i\in I} f_i(b_i) = \prod_{i\in I} f_i(b_i)$.*

Proof. Clearly $b \le g(i, b_i)$, so $\mathscr{O}_b \subseteq f_i(b_i)$, for each $i \in I$. If $y \in f_i(b_i)$ for all $i \in I$, then $y \le g(i, b_i)$ for all $i \in I$, hence $y \le b$ and so $y \in \mathscr{O}_b$. ■

Corollary 1.4. *$\langle f_i[A_i] : i \in I\rangle$ is an independent system of subalgebras of $\bigoplus^B_{i\in I} A_i$.* ■

Proposition 1.5. *If $B = \prod^{\mathrm{w}}_{i\in I} A_i$, then $\bigoplus^B_{i\in I} A_i \cong \bigoplus_{i\in I} A_i$.*

Proof. By Handbook 11.4 it suffices to show that $\bigoplus^B_{i\in I} A_i$ is generated by $\bigcup_{i\in I} f_i[A_i]$. Take any $b \in B^\star$. Then $F := \{i \in I : b_i \ne 1\}$ is finite. Hence $\bigcap_{i\in I} f_i(b_i) = \bigcap_{i\in F} f_i(b_i)$. The desired conclusion now follows from Proposition 1.3. ■

Proposition 1.6. *If $b \ne c$, then $\mathscr{O}_b \ne \mathscr{O}_c$.*

Proof. Say $b \not\le c$. Then $b \in \mathscr{O}_b \setminus \mathscr{O}_c$. ■

Proposition 1.7. $-\mathscr{O}_b = \{x \in B^\star : \exists i \in I\ (x_i \le -b_i)\}$.

Proof. To prove this, first recall that $-\mathscr{O}_b = \operatorname{int}(B^\star \setminus \mathscr{O}_b)$. If $x_i \le -b_i$ for some $i \in I$, then $\mathscr{O}_x \cap \mathscr{O}_b = 0$, and so $x \in \operatorname{int}(B^\star \setminus \mathscr{O}_b)$. Now suppose that $x_i \cdot b_i \ne 0$ for all $i \in I$. Clearly then $\mathscr{O}_x \cap \mathscr{O}_b \ne 0$, and so $x \notin \operatorname{int}(B^\star \setminus \mathscr{O}_b)$. ■

Proposition 1.8. *$B^\star$ is order-isomorphic to a dense generating set of $\bigoplus^B_{i\in I} A_i$. Moreover, $b \le c$ iff $\mathscr{O}_b \subseteq \mathscr{O}_c$.*

Proof. The second statement is obvious, and it immediately implies the first statement. ■

Proposition 1.9. (i) *$\mathscr{O}_b \cdot \mathscr{O}_c = \mathscr{O}_b \cap \mathscr{O}_c$.*
(ii) *$\mathscr{O}_b \cdot \mathscr{O}_c \ne 0$ iff $\forall i \in I\ [b_i \cdot c_i \ne 0]$.*
(iii) *If $\mathscr{O}_b \cdot \mathscr{O}_c \ne 0$, then $\mathscr{O}_b \cdot \mathscr{O}_c = \mathscr{O}_{b\cdot c}$.* ■

Proposition 1.10. *Suppose that m is a positive integer and $b, c^0, \ldots, c^{m-1} \in B^\star$. Then the following conditions are equivalent:*

(i) $\mathscr{O}_b \subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}}$.

(ii) $\forall w \in B^\star\ (w \le b \Rightarrow \exists i < m\ (w \cdot c^i \in B^\star))$.

(iii) *For all* $j < m$ *and all* $i \in I$, *if* $b_i \cdot -c^j_i \neq 0$, *then* $\mathscr{O}_{\mathcal{S}(b,i,b_i\cdot -c^j_i)} \subseteq \sum_{k<m,\, k\neq j} \mathscr{O}_{c^k}$.

Proof. Note that $\mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}} = \operatorname{int}\operatorname{cl}(\mathscr{O}_{c^0} \cup \ldots \cup \mathscr{O}_{c^{m-1}})$. Hence

$$\begin{aligned}\mathscr{O}_b \subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}} \quad &\text{iff} \quad \mathscr{O}_b \subseteq \operatorname{cl}(\mathscr{O}_{c^0} \cup \ldots \cup \mathscr{O}_{c^{m-1}})\\ &\text{iff} \quad \forall w \in B^\star\ (w \le b\ \Rightarrow \exists i < m\ (w \cdot c^i \in B^\star)).\end{aligned}$$

It follows that (i) and (ii) are equivalent.

For (i)$\Rightarrow$(iii), suppose that (iii) fails. We then obtain $j < m$ and $i \in I$ such that $b_i \cdot -c^j_i \neq 0$ and $\mathscr{O}_{\mathcal{S}(b,i,b_i\cdot -c^j_i)} \not\subseteq \sum_{k<m,\, k\neq j} \mathscr{O}_{c^k}$. This means by (ii) that there is an $s \le \mathcal{S}(b,i,b_i \cdot -c^j_i)$ such that $\mathscr{O}_s \cap \mathscr{O}_{c^k} = 0$ for all $k < m$ for which $k \neq j$. But also clearly $\mathscr{O}_s \cap \mathscr{O}_{c^j} = 0$, contradiction.

(iii)$\Rightarrow$(i). Suppose that $\mathscr{O}_b \not\subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}}$. Then by (ii) there is an $s \in \mathscr{O}_b$ such that $\mathscr{O}_s \cap \mathscr{O}_{c^j} = 0$ for all $j < m$. Choose $i \in I$ such that $s_i \cdot c^0_i = 0$. Then $b_i \cdot -c^0_i \neq 0$, so by (iii), $\mathscr{O}_{\mathcal{S}(b,i,b_i\cdot -c^0_i)} \subseteq \sum_{k<m,\, k\neq 0} \mathscr{O}_{c^k}$. But $s \le \mathcal{S}(b,i,b_i \cdot -c^0_i)$, contradiction. ∎

Corollary 1.11. *Suppose that* m *is a natural number*, $b, c^0, \ldots, c^{m-1} \in B^\star$, *and* $\mathscr{O}_b \subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}}$. *Then* $m > 0$ *and* $b \le c^0 + \ldots + c^{m-1}$.

Proof. Since $\mathscr{O}_b \neq 0$, it follows that $m > 0$. Now suppose that $b \not\le c^0 + \ldots + c^{m-1}$. Choose i such that $u := b_i \cdot -c^0_i \cdot \ldots \cdot -c^{m-1}_i \neq 0$. Then $\mathcal{S}(b,i,u) \le b$ and $\forall j < m\ (\mathcal{S}(b,i,u) \cdot c^i \notin B^\star)$, contradicting Proposition 1.10. ∎

Proposition 1.12. *Suppose that* m *is a positive integer and* $b, c^0, \ldots, c^{m-1} \in B^\star$. *For each* $\varepsilon \in {}^m I$ *define* $d^\varepsilon \in B$ *by setting, for each* $i \in I$,

$$d^\varepsilon_i = b_i \cdot \prod_{j<m,\, \varepsilon(j)=i} -c^j_i.$$

Then

$$\mathscr{O}_b \cap -\mathscr{O}_{c^0} \cap \ldots \cap -\mathscr{O}_{c^{m-1}} = \bigcup_{\varepsilon \in {}^m I,\, d^\varepsilon \in B^\star} \mathscr{O}_{d^\varepsilon}.$$

Proof. Suppose $w \in \mathscr{O}_b \cap -\mathscr{O}_{c^0} \cap \ldots \cap \mathscr{O}_{c^{m-1}}$. By Proposition 1.7, for each $j < m$ choose $\varepsilon(j) \in I$ such that $w_{\varepsilon(j)} \le -c^j_{\varepsilon(j)}$. Clearly then $w \le d^\varepsilon$.

Conversely, if $w \in d^\varepsilon$, it is clear that $w \in \mathscr{O}_b \cap -\mathscr{O}_{c^0} \cap \ldots \cap -\mathscr{O}_{c^{m-1}}$. ∎

Corollary 1.13. *Suppose that* m *is a positive integer and* $b, c^0, \ldots, c^{m-1} \in B^\star$. *Then* $\mathscr{O}_b \le \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{m-1}}$ *iff*

$$\forall \varepsilon \in {}^m I\ \exists i \in I\ \Big[b_i \le \sum_{j<m,\, \varepsilon(j)=i} c^j_i\Big]. \ \blacksquare$$

So much for the elementary arithmetic of generalized free products. Now we turn to elementary algebraic results, specifically to universal mapping properties.

PROPOSITION 1.14. *Suppose that B and C are finitely closed subalgebras of $\prod_{i\in I} A_i$, and $B \le C$. Then $\bigoplus_{i\in I}^{B} A_i$ can be isomorphically embedded in $\bigoplus_{i\in I}^{C} A_i$. In fact, the mapping f such that $f(\mathscr{O}_b^B) = \mathscr{O}_b^C$ for all $b \in B^\star$ can be extended to an isomorphism into.*

Proof. Let $b^0, \ldots, b^{m-1}, c^0, \ldots, c^{n-1} \in B^\star$. It suffices to show that

$$\begin{aligned}&\mathscr{O}_{b^0}^{\mathrm{B}} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}}^{\mathrm{B}} \cdot -\mathscr{O}_{c^0}^{\mathrm{B}} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}}^{\mathrm{B}} = 0\\ &\qquad\qquad\text{iff}\quad \mathscr{O}_{b^0}^{\mathrm{C}} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}}^{\mathrm{C}} \cdot -\mathscr{O}_{c^0}^{\mathrm{C}} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}}^{\mathrm{C}} = 0.\end{aligned}$$

We may assume that $m > 0$ (put $b^0 = 1$ otherwise). Now

$$\mathscr{O}_{b^0}^{\mathrm{B}} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}}^{\mathrm{B}} = \begin{cases} 0 & \text{if } (b^0 \cdot \ldots \cdot b^{m-1})_i = 0 \text{ for some } i \in I,\\ \mathscr{O}_{b^0 \cdot \ldots \cdot b^{m-1}}^{\mathrm{B}} & \text{otherwise.}\end{cases}$$

and similarly for $\mathscr{O}_{b^0}^{\mathrm{C}} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}}^{\mathrm{C}}$, so we may assume that $m = 1$. Clearly then $n > 0$.

Next, using Corollary 1.13 we have

$$\begin{aligned}\mathscr{O}_{b^0}^{B} \cdot -\mathscr{O}_{c^0}^{B} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}}^{B} = 0 \quad &\text{iff}\quad \forall \varepsilon \in {}^nI\ \exists i \in I\ \Big[b_i^0 \le \sum_{j<n,\, \varepsilon(j)=i} c_i^j\Big]\\ &\text{iff}\quad \mathscr{O}_{b^0}^{C} \cdot -\mathscr{O}_{c^0}^{C} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}}^{C} = 0. \ \blacksquare\end{aligned}$$

The following proposition abstractly characterizes generalized free products.

THEOREM 1.15. *Let $\langle A_i : i \in I\rangle$ be a system of BAs, and let B be a finitely closed subalgebra of $\prod_{i\in I} A_i$. Then for any BA C, the following conditions are equivalent:*

(i) $C \cong \bigoplus_{i\in I}^{B} A_i$.

(ii) *There exist embeddings f_i of A_i into C with the following properties:*

(a) *For all $b \in B^\star$, $\prod_{i\in I}^{C} f_i(b_i)$ exists and is nonzero;*

(b) *$\{\prod_{i\in I}^{C} f_i(b_i) : b \in B^\star\}$ is a dense generating set for C.*

Proof. (i)⇒(ii) by Propositions 1.2, 1.3.

(ii)⇒(i). Assume (ii). Define $F(\mathscr{O}_b) = \prod_{i\in I}^{C} b_i$ for any $b \in B^\star$. It suffices now to show that if $b^0, \ldots, b^{m-1}, c^0, \ldots, c^{n-1} \in B^\star$, then

$$\begin{aligned}&\mathscr{O}_{b^0} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}} \cdot -\mathscr{O}_{c^0} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}} = 0\\ &\quad\text{iff}\quad \prod_{i\in I} f_i(b_i^0) \cdot \ldots \cdot \prod_{i\in I} f_i(b_i^{m-1}) \cdot -\prod_{i\in I} f_i(c_i^0) \cdot \ldots \cdot -\prod_{i\in I} f_i(c_i^{n-1}) = 0.\end{aligned}$$

As in the proof of Proposition 1.14 we may assume that $m = 1$. Then $n > 0$. So what we want to prove is that

$$\mathscr{O}_b \subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{n-1}} \quad \text{iff} \quad \prod_{i\in I} f_i(b_i) \le \prod_{i\in I} f_i(c_i^0) + \ldots + \prod_{i\in I} f_i(c_i^{n-1}).$$

We have

$$\begin{aligned}
&\prod_{i\in I} f_i(b_i) \le \prod_{i\in I} f_i(c_i^0) + \ldots + \prod_{i\in I} f_i(c_i^{n-1}) \\
&\qquad \text{iff} \quad \prod_{i\in I} f_i(b_i) \cdot \sum_{i\in I} -f_i(c_i^0) \cdot \ldots \cdot \sum_{i\in I} -f_i(c_i^{n-1}) = 0 \\
&\qquad \text{iff} \quad \sum_{\varepsilon\in {}^nI} \Big(\prod_{i\in I} f_i(b_i) \cdot \prod_{j\in n} -f_{\varepsilon(j)}(c^j_{\varepsilon(j)}) \Big) = 0 \\
&\qquad \text{iff} \quad \sum_{\varepsilon\in {}^nI} \Big(\prod_{i\in I} f_i(b_i) \cdot \prod_{j\in n} f_{\varepsilon(j)}(-c^j_{\varepsilon(j)}) \Big) = 0.
\end{aligned}$$

Now we claim that this last equality is equivalent to saying

$$(*) \qquad \forall \varepsilon \in {}^nI \; \exists i \in I \; \Big[f_i(b_i) \cdot \prod_{j<n,\, \varepsilon(j)=i} f_{\varepsilon(j)}(-c_i^j) = 0 \Big].$$

In fact, the latter condition clearly implies the indicated equality. Conversely, suppose that for some $\varepsilon \in {}^nI$ it is the case that

$$\forall i \in I \; \Big[f_i(b_i) \cdot \prod_{j<n,\, \varepsilon(j)=i} f_{\varepsilon(j)}(-c_i^j) \ne 0 \Big].$$

Then, since f_i is an embedding,

$$\forall i \in I \; \Big[b_i \cdot \prod_{j<n,\, \varepsilon(j)=i} -c_i^j \ne 0 \Big].$$

So if we define a new element e by

$$e_i = b_i \cdot \prod_{j<n,\, \varepsilon(j)=i} -c_i^j$$

for all $i \in I$, then $e \in B^\star$, and so by (a), $\prod_{i\in I} f_i(e_i) \ne 0$. But this means that

$$\prod_{i\in I} f_i(b_i) \cdot \prod_{j\in n} f_{\varepsilon(j)}(-c^j_{\varepsilon(j)}) \ne 0,$$

so that the indicated equality fails. Thus our equivalence is true, and hence

$$\prod_{i\in I} f_i(b_i) \leq \prod_{i\in I} f_i(c_i^0) + \ldots + \prod_{i\in I} f_i(c_i^{n-1})$$
$$\text{iff} \quad \forall \varepsilon \in {}^nI\ \exists i \in I\ \Big[b_i \cdot \prod_{j<n,\, \varepsilon(j)=i} -c_i^j = 0\Big]$$
$$\text{iff} \quad \mathscr{O}_b \subseteq \mathscr{O}_{c^0} + \ldots + \mathscr{O}_{c^{n-1}}.$$

Here we have used the hypothesis (ii)(a) and Corollary 1.13. ∎

Corollary 1.16. *Suppose that B is a finitely closed subalgebra of $\prod_{i\in I} A_i$, and $J \subseteq I$. Let $B_J = \{f{\restriction}J : f \in B\}$ and $B_{I\setminus J} = \{f{\restriction}(I \setminus J) : f \in B\}$. Assume also*

$$(*) \qquad B = \{u^\frown v : u \in B_J \text{ and } v \in B_{I\setminus J}\}.$$

Then

$$\bigoplus_{i\in I}^{B} A_i \cong \Big(\bigoplus_{i\in J}^{B_J} A_i\Big) \oplus \Big(\bigoplus_{i\in I\setminus J}^{B_{I\setminus J}} A_i\Big).$$

Proof. For brevity let $C = \bigoplus_{i\in J}^{B_J} A_i$ and $D = \bigoplus_{i\in I\setminus J}^{B_{I\setminus J}} A_i$. We consider C and D as subalgebras of $E := C \oplus D$. For each $i \in J$ let f_i be the isomorphism of A_i into C defined before Proposition 1.2, and for each $i \in I\setminus J$ let g_i be the isomorphism of A_i into D given there. We intend to check the conditions of Theorem 1.15 in order to show that $E \cong \bigoplus_{i\in I}^{B} A_i$. To check 1.15(a), suppose that $b \in B^\star$. Let $c_0 = \prod_{i\in J}^{C} f_i(b_i)$; this product exists and is nonzero by 1.15 for C. Similarly, let $c_1 = \prod_{i\in I\setminus J}^{D} g_i(b_i)$; it is nonzero. Thus the member $c_0 \cdot c_1$ of E is nonzero. We claim that it is the product in E of all members of

$$(**) \qquad \{f_i(b_i) : i \in J\} \cup \{g_i(b_i) : i \in I \setminus J\}.$$

To check this, first we have $c_0 \cdot c_1 \leq c_0 \leq f_i(b_i)$ for all $i \in I$ by the definition of c_0. Similarly, $c_0 \cdot c_1 \leq g_i(b_i)$ for all $i \in I \setminus J$. Now suppose that $e \in E$ and e is a lower bound for all members of the set $(**)$. Write

$$e = \sum_{i<m} u_i \cdot v_i,$$

where each $u_i \in C$ and $v_i \in D$. From $u_i \cdot v_i \leq f_j(b_j)$ we infer that $u_i \leq f_j(b_j)$ by the basic property of free products, for each $i < m$ and each $j \in J$. So u_i is a lower bound for $\{f_j(b_j) : j \in J\}$, so $u_i \leq c_0$. Similarly, $v_i \leq c_1$ for each $i < m$. Hence $e \leq c_0 \cdot c_1$. This establishes our claim. Hence 1.15(a) holds for E.

To prove 1.15(b), given a nonzero $e \in E$, choose nonzero $d_0 \in C$ and $d_1 \in D$ such that $d_0 \cdot d_1 \leq e$. Then by 1.15(b) for C and D we can find $u \in B_J^\star$ and $v \in B_{I\setminus J}^\star$ such that $\prod_{i\in J}^{C} f_i(u_i) \leq d_0$ and $\prod_{i\in I\setminus J}^{D} g_i(v_i) \leq d_1$. Let $b = u^\frown v$; then $b \in B^\star$ by $(*)$, and by the above the product in E of all

$f_i(u_i)$ and $g_j(v_j)$ for $i \in J$, $j \in I \setminus J$ is $\leq e$. So the indicated elements are dense in E. Clearly they generate E. ■

The two most important special cases of $\bigoplus_{i\in I}^B A_i$ are the one in which $B = \prod_{i\in I}^{\mathrm{w}} A_i$, where $\bigoplus_{i\in I}^B A_i$ is isomorphic to the ordinary free product by Proposition 1.5, and the one in which $B = \prod_{i\in I} A_i$. We denote the latter by $\bigoplus_{i\in I}^{\pi} A_i$. It is the notion mainly studied here.

The following universal property of generalized free products generalizes the one for usual free products.

THEOREM 1.17. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, and B is a finitely closed subalgebra of $\prod_{i\in I} A_i$. Let C be any BA, and suppose that $h_i : A_i \to C$ is a homomorphism for every $i \in I$ such that for any $b \in B^\star$, the product $\prod_{i\in I} h_i(b_i)$ exists. Then there is a homomorphism $k : \bigoplus_{i\in I}^B A_i \to C$ such that $k(\mathscr{O}_b) = \prod_{i\in I} h_i(b_i)$ for all $b \in B^\star$.*

Proof. For any $b \in B^\star$ let $k(\mathscr{O}_b) = \prod_{i\in I} h_i(b_i)$. We want to show that k extends to a homomorphism from $\bigoplus_{i\in I}^B A_i$ into C. To this end, suppose that

$$\mathscr{O}_{b^0} \cdot \ldots \cdot \mathscr{O}_{b^{m-1}} \cdot -\mathscr{O}_{c^0} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}} = 0;$$

we want to show that

$$\prod_{i\in I} h_i(b_i^0) \cdot \ldots \cdot \prod_{i\in I} h_i(b_i^{m-1}) \cdot -\prod_{i\in I} h_i(c_i^0) \cdot \ldots \cdot -\prod_{i\in I} h_i(c_i^{n-1}) = 0.$$

As in the proof of Proposition 1.14, we may assume that $m = 1$; so we drop the superscript ^{0} on b^0. Then it is clear that $n > 0$. Now suppose that $\prod_{i\in I} h_i(b_i) \cdot -\prod_{i\in I} h_i(c_i^0) \cdot \ldots \cdot -\prod_{i\in I} h_i(c_i^{n-1}) \neq 0$. Then there exist $i_0, \ldots, i_{n-1} \in I$ such that $\prod_{i\in I} h_i(b_i) \cdot -h_{i_0}(c_{i_0}^0) \cdot -h_{i_{n-1}}(c_{i_{n-1}}^{n-1}) \neq 0$. We now define

$$w_i = b_i \cdot \prod\{-c_{i_k}^k : i_k = i\}$$

for every $i \in I$. Then

$$\prod_{i\in I} h_i(w_i) = \prod_{i\in I} h_i(b_i) \cdot -h_{i_0}(c_{i_0}^0) \cdot \ldots \cdot -h_{i_{n-1}}(c_{i_{n-1}}^{n-1}) \neq 0,$$

and hence $w \in B^\star$. But by Proposition 1.7 we have $w \in \mathscr{O}_b \cdot -\mathscr{O}_{c^0} \cdot \ldots \cdot -\mathscr{O}_{c^{n-1}}$, contradiction.

It follows that k can be extended to a homomorphism. ■

PROPOSITION 1.18. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, and B is a finitely closed subalgebra of $\prod_{i\in I} A_i$. Then $\bigoplus_{i\in I} A_i$ is a retract of $\bigoplus_{i\in I}^B A_i$.*

Proof. By 1.14 let g be the isomorphism of $\bigoplus_{i\in I} A_i$ into $\bigoplus^B_{i\in I} A_i$ such that $g(\mathscr{O}_b) = \mathscr{O}^B_b$ for all $b \in C^\star$, where $C = \prod^{\mathrm{w}}_{i\in I} A_i$. Let $f_i : A_i \to \bigoplus_{i\in I} A_i$ be as before 1.2.

(1) If $b \in B^\star$ and $\{i \in I : b_i \neq 1\}$ is infinite, then $\prod_{i\in I} f_i(b_i) = 0$.

For, suppose that $\prod_{i\in I} f_i(b_i) \neq 0$. Then there is a $c \in C^\star$ such that $\mathscr{O}_c \subseteq f_i(b_i)$ for all $i \in I$. Choose $i \in I$ such that $c_i = 1$ and $b_i \neq 1$. Then $\mathscr{O}_c \subseteq f_i(b_i) = \mathscr{O}_{g(i,b_i)}$, so by 1.8, $1 \leq b_i$, contradiction. Thus (1) holds.

By (1) and 1.17 let k be a homomorphism from $\bigoplus^B_{i\in I} A_i$ into $\bigoplus_{i\in I} A_i$ such that $k(\mathscr{O}^B_b) = \prod_{i\in I} f_i(b_i)$ for all $b \in B^\star$. Then for any $b \in C^\star$,

$$k(g(\mathscr{O}_b)) = k(\mathscr{O}^B_b) = \prod_{i\in I} f_i(b_i) = \mathscr{O}_b,$$

and so $k \circ g$ is the identity on $\bigoplus_{i\in I} A_i$, as desired. ■

PROPOSITION 1.19. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, and B is a finitely closed subalgebra of $\prod_{i\in I} A_i$. Let $\langle F_i : i \in I\rangle$ be a system consisting of an ultrafilter F_i on A_i for each $i \in I$. Then there is a homomorphism k from $\bigoplus^B_{i\in I} A_i$ into B such that $(\prod_{i\in I} F_i) \cap B$ is a subset of* $\mathrm{rng}(k)$.

Proof. For each $i \in I$ we define $h_i : A_i \to \prod_{k\in I} A_k$ as follows: for any $a \in A_i$ and $k \in I$,

$$(h_i(a))_k = \begin{cases} a & \text{if } i = k, \\ 1 & \text{if } i \neq k \text{ and } a \in F_i, \\ 0 & \text{if } i \neq k \text{ and } a \notin F_i. \end{cases}$$

Clearly h_i is a homomorphism from A_i into $\prod^{\mathrm{w}}_{i\in I} A_i$, for each $i \in I$.

Now we check the condition of Theorem 1.17. Suppose that $b \in B^\star$. Define $c \in \prod_{k\in I} A_k$ as follows: for any $k \in I$,

$$c_k = \begin{cases} 0 & \text{if } -b_i \in F_i \text{ for some } i \neq k, \\ b_k & \text{otherwise.} \end{cases}$$

We claim that $c = \prod_{i\in I} h_i(b_i)$ in B. First take any $i \in I$. To show that $c \leq h_i(b_i)$, take any $k \in I$; we want to show that $c_k \leq (h_i(b_i))_k$. This is clear if $-b_j \in F_j$ for some $j \neq k$, so assume that $b_j \in F_j$ for all $j \neq k$. Then

$$(h_i(b_i))_k = \begin{cases} b_i & \text{if } i = k, \\ 1 & \text{if } i \neq k. \end{cases}$$

Since $c_k = b_k$, it follows that $c_k \leq (h_i(b_i))_k$. Thus c is a lower bound for all of the $h_i(b_i)$'s.

Now suppose that d is any lower bound for the $h_i(b_i)$'s. To show that $d \leq c$, take any $i \in I$. If $b_k \in F_k$ for all $k \neq i$, then $d_i \leq (h_i(b_i))_i = b_i = c_i$. If $b_k \notin F_k$ for some $k \neq i$, then $d_i \leq (h_k(b_k))_i = 0 \leq c_i$.

So we have established that $c = \prod_{i\in I} h_i(b_i)$. Hence we can apply Theorem 1.17 to obtain a homomorphism $k : \bigoplus^B_{i\in I} A_i \to B$ such that $k(\mathscr{O}_b) = \prod_{i\in I} h_i(b_i)$ for all $b \in B^\star$. If $b \in (\prod_{i\in I} F_i) \cap B$, then c, as defined above, is equal to b, and so $k(\mathscr{O}_b) = b$. ■

PROPOSITION 1.20. (i) *If b_i is an atom for all $i \in I$, then $\mathscr{O}_b$ is an atom of $\bigoplus^\pi_{i\in I} A_i$.*

(ii) *If A_i is atomic for all $i \in I$, then $\bigoplus^\pi_{i\in I} A_i$ is atomic.* ■

2. Duality. Let B be a finitely closed subalgebra of $\prod_{i\in I} A_i$. Suppose that $F = \langle F_i : i \in I\rangle$ is a system consisting of an ultrafilter F_i on A_i for each $i \in I$. Then

$$\{\mathscr{O}_b : \forall i \in I\ (b_i \in F_i)\} \cup \{-\mathscr{O}_b : \exists i \in I\ (b_i \notin F_i)\}$$

has fip (and hence filter-generates an ultrafilter). In fact, suppose that

$$\mathscr{O}_{b^0} \cap \ldots \cap \mathscr{O}_{b^{m-1}} \cap -\mathscr{O}_{c^0} \cap \ldots \cap -\mathscr{O}_{c^{n-1}} = 0,$$

where $f^i \in F_i$ and $c^i \notin F_i$ for all $i \in I$. Then $b^0 \cdot \ldots \cdot b^{m-1} \in B^\star$. For each $j < n$ choose $i_j \in I$ such that $c^j_{i_j} \notin F_i$. Now define

$$x_i = b^0_i \cdot \ldots \cdot b^{m-1}_i \cdot \prod_{j<n,\, i_j=i} -c^j_{i_j}$$

for each $i \in I$. Since $x_i \in F_i$ for each $i \in I$, we have $x \in B^\star$. And

$$x \in \mathscr{O}_{b^0} \cap \ldots \cap \mathscr{O}_{b^{m-1}} \cap -\mathscr{O}_{c^0} \cap \ldots \cap -\mathscr{O}_{c^{n-1}},$$

contradiction. This shows that the indicated set has the fip, and we let U_F be the associated ultrafilter.

Now conversely, let G be an ultrafilter on $\bigoplus^B_{i\in I} A_i$, and let $i \in I$. Clearly, $\{a \in A^+_i : f(i,a) \in G\}$ has fip. We let K^G_i be an ultrafilter containing this set. Let $K^G = \langle K^G_i : i \in I\rangle$.

Suppose now that $F = \langle F_i : i \in I\rangle$ is a system consisting of an ultrafilter F_i on A_i for each $i \in I$. We claim that $K^{U_F} = F$. For, let $i \in I$. We show that $K^{U_F}_i \subseteq F_i$ (hence they are equal). Let $a \in K^{U_F}_i$. Then $f(i,a) \in U_F$, and hence $a \in F_i$.

From this it follows that U is one-one.

We claim that U is continuous with respect to the box topology when $B = \prod_{i\in I} A_i$. For, suppose that $F \in U^{-1}[\mathcal{S}(\mathscr{O}_b)]$. Thus $U_F \in \mathcal{S}(\mathscr{O}_b)$, so $\mathscr{O}_b \in U_F$. Hence $\forall i \in I\ (b_i \in F_i)$. We claim that $F \in \prod_{i\in I} \mathcal{S}(b_i) \subseteq U^{-1}[\mathcal{S}(\mathscr{O}_b)]$. For, suppose that $H \in \prod_{i\in I} \mathcal{S}(b_i)$. Then $\forall i \in I\ (b_i \in H_i)$, so $\mathscr{O}_b \in U_H$ and $H \in U^{-1}[\mathcal{S}(\mathscr{O}_b)]$, as desired.

It is not true in general that $U_{K^G} = G$ for G an ultrafilter on $\bigoplus^B_{i\in I} A_i$. For example, for each $i \in \omega$ let A_i be the free BA on free generators

$z_0, z_1, \ldots$, and let $B = \prod_{i\in\omega} A_i$. Then z itself is a member of $B^\star$, and by Proposition 1.7,

$$-\mathscr{O}_z = \{x \in B^\star : \exists i \in \omega(x_i \leq -z_i)\}.$$

Clearly now $\{f(i, z_i) : i \in \omega\} \cup \{-\mathscr{O}_z\}$ has fip, and so is included in an ultrafilter G on $\bigoplus_{i\in I}^B A_i$. For any $i \in \omega$ we have $z_i \in K_i^G$, so $\mathscr{O}_z \in U_{K^G}$. This shows that $G \neq U_{K^G}$.

On the other hand, *if* $B = \prod_{i\in I}^{\mathrm{w}} A_i$, *then always* $U_{K^G} = G$, *and the Stone topology on* $\bigoplus_{i\in I}^B A_i$ *corresponds to the product topology on* $\prod_{i\in I} \mathrm{Ult}(A_i)$, *as one would expect.*

To prove this, suppose that G is an ultrafilter on $\bigoplus_{i\in I}^B A_i$. If $\mathscr{O}_b \in G$, then $\forall i \in I\ [b_i \in K_i^G]$, and so $\mathscr{O}_b \in U_{K^G}$. On the other hand, suppose that $-\mathscr{O}_b \in G$. Let $F = \{i \in I : b_i \neq 1\}$. So F is finite. By Proposition 1.3, $\mathscr{O}_b = \prod_{i\in F} f_i(b_i)$. It follows that there is an $i \in F$ such that $f_i(b_i) \notin G$. Hence $b_i \notin K_i^G$. Hence $-\mathscr{O}_b \in U_{K^G}$. Thus we have shown that $U_{K^G} = G$.

To finish proving our italicized statement it suffices to show that K is continuous. To do this it suffices to take any $i \in I$, any $a \in A_i$, and any $G \in K^{-1}[\{x \in \prod_{j\in I} \mathrm{Ult}(A_j) : x_i \in \mathcal{S}(a)\}]$ and find an open set U in $\mathrm{Ult}(\bigoplus_{j\in I}^B A_j)$ such that

$$G \in U \subseteq K^{-1}\left[\left\{x \in \prod_{j\in I} \mathrm{Ult}(A_j) : x_i \in \mathcal{S}(a)\right\}\right].$$

Let $U = \mathcal{S}(f(i, a))$. Now $K_i^G \in \mathcal{S}(a)$, so $a \in K_i^G$ and hence $f(i, a) \in G$ and $G \in U$. Now suppose that $H \in U$. Then $f(i, a) \in H$, $a \in K_i^H$, $K_i^H \in \mathcal{S}(a)$, and hence

$$H \in K^{-1}\left[\left\{x \in \prod_{j\in I} \mathrm{Ult}(A_j) : x_i \in \mathcal{S}(a)\right\}\right],$$

as desired.

Thus these facts do not actually characterize the Stone spaces. We now give such a characterization. A *suitable set* is a subset C of $B^\star$ with the following property: for every finite subset F of C and every finite subset G of $B^\star \setminus C$ there is a $j \in {}^G I$ such that for all $i \in I$,

$$\prod_{c\in F} c_i \cdot \prod_{b\in G,\, j(b)=i} -b_i \neq 0.$$

If U is an ultrafilter on $\bigoplus_{i\in I}^B A_i$, let $\mathscr{C}^U = \{b : \mathscr{O}_b \in U\}$. Then $\mathscr{C}^U$ is suitable. In fact, suppose that F is a finite subset of $\mathscr{C}^U$ and G is a finite subset of $B^\star \setminus \mathscr{C}^U$. Hence $\mathscr{O}_b \in U$ for all $b \in F$, and $-\mathscr{O}_b \in U$ for all $b \in G$.

Therefore,

$$\bigcap_{b\in F}\mathscr{O}_b \cap \bigcap_{b\in G} -\mathscr{O}_b \neq 0.$$

Choose x in this intersection. Thus $x \in B^\star$. Moreover, $x \in \mathscr{O}_c$ for all $c \in F$, so $x \leq c$ for all $c \in F$. For each $b \in G$ choose $j(b) \in I$ such that $x_{j(b)} \leq -b_{j(b)}$, by Proposition 1.7. Thus for all $i \in I$,

$$x_i \leq \prod_{c\in F} c_i \cdot \prod_{b\in G,\, j(b)=i} -b_i,$$

as desired. So, we have shown that $\mathscr{C}^U$ is suitable.

Conversely, suppose that C is suitable. Then clearly the set $\{\mathscr{O}_b : b \in C\} \cup \{-\mathscr{O}_b : b \in B^\star \setminus C\}$ has fip, and hence determines an ultrafilter V^C.

If C is suitable, clearly $\mathscr{C}^{V^C} = C$. And if U is an ultrafilter on $\bigoplus^B_{i\in I} A_i$, clearly $V^{\mathscr{C}^U} = U$. Thus we have a one-one correspondence between ultrafilters on $\bigoplus^B_{i\in I} A_i$ and suitable subsets of $B^\star$.

The Stone topology on suitable sets is given by the basis consisting of the following set for each $a \in \bigoplus^B_{i\in I} A_i$:

$$\mathcal{S}'(a) = \Big\{C : C \text{ is suitable and there exist } F \subseteq C \text{ and } G \subseteq B^\star \setminus C \\ \text{such that } \bigcap_{b\in F}\mathscr{O}_b \cap \bigcap_{b\in G} -\mathscr{O}_b \subseteq a\Big\}.$$

This is proved as follows: for any suitable set C,

$$\begin{aligned} C \in \mathscr{C}[\mathcal{S}(a)] \quad &\text{iff} \quad \exists U \in \mathcal{S}(a)\ (C = \mathscr{C}^U) \\ &\text{iff} \quad \exists U\ (U \text{ is an ultrafilter, } a \in U, \text{ and } C = \mathscr{C}^U) \\ &\text{iff} \quad \exists U\ (U \text{ is an ultrafilter, } a \in U, \text{ and } V^C = U) \\ &\text{iff} \quad a \in V^C \\ &\text{iff} \quad C \in \mathcal{S}'(a). \end{aligned}$$

3. Cellularity. Recall that $\mathrm{c}A$ is the supremum of cardinalities of disjoint subsets of A, while $\mathrm{c}'A$ is the least infinite cardinal greater than all such cardinalities. Two related notions are the set $\mathrm{PT}(A)$ of cardinalities of partitions of unity of A, and $\mathfrak{a}(A)$, the least infinite member of $\mathrm{PT}(A)$.

PROPOSITION 3.1. *If $\kappa_i \in \mathrm{PT}(A_i)$ for each $i \in I$, then $\prod_{i\in I}\kappa_i \in \mathrm{PT}(\bigoplus^\pi_{i\in I} A_i)$.*

Proof. For each $i \in I$ let X_i be a partition of unity in A_i such that $|X_i| = \kappa_i$. It suffices to show that

$$Y := \Big\{\mathscr{O}_b : b \in \prod_{i\in I} X_i\Big\}$$

is a partition of unity in $\bigoplus_{i\in I}^{\pi} A_i$. Clearly Y is a collection of nonzero pairwise disjoint elements. Suppose that $\mathscr{O}_c$ is given. For each $i \in I$ there is a $b_i \in X_i$ such that $c_i \cdot b_i \neq 0$. Then $\mathscr{O}_c \cdot \mathscr{O}_b \neq 0$, as desired. ∎

Proposition 3.2. *Assume that A_i has at least four elements for all $i \in I$, I infinite. Then $\omega \in \mathrm{PT}(\bigoplus_{i\in I}^{\pi} A_i)$, and hence $\mathfrak{a}(\bigoplus_{i\in I}^{\pi} A_i) = \omega$.*

Proof. Let $B = \prod_{i\in I} A_i$. Let f be a one-one function mapping ω into I. For each $i \in I$ let a_i be an element of A_i such that $0 < a_i < 1$. Now for each $i \in \omega$ we define $b^i \in B$ by setting, for each $j \in I$,

$$b^i_j = \begin{cases} -a_j & \text{if } j \in \mathrm{rng}(f) \text{ and } f^{-1}(j) < i, \\ a_i & \text{if } j \in \mathrm{rng}(f) \text{ and } f^{-1}(j) = i, \\ 1 & \text{otherwise.} \end{cases}$$

We also define $b^\infty \in B$ by setting, for each $j \in I$,

$$b^\omega_j = \begin{cases} -a_j & \text{if } j \in \mathrm{rng}(f), \\ 1 & \text{otherwise.} \end{cases}$$

Thus $b^i \in B^\star$ for each $i \in \omega+1$. If $i < j < \omega$, then $b^i_{f(i)} \cdot b^j_{f(i)} = a_{f(i)} \cdot -a_{f(i)} = 0$. Hence $\mathscr{O}_{b^i} \cdot \mathscr{O}_{b^j} = 0$. And if $i < \omega$, then $b^i_{f(i)} \cdot b^\omega_{f(i)} = a_{f(i)} \cdot -a_{f(i)} = 0$, and hence $\mathscr{O}_{b^i} \cdot \mathscr{O}_{b^\omega} = 0$. Now suppose that $c \in B^\star$. If $c_j \leq -a_j$ for all $j \in \mathrm{rng}(f)$, then $c \leq b^\omega$, and hence $\mathscr{O}_c \subseteq \mathscr{O}_{b^\omega}$. Suppose that $c_j \cdot a_j \neq 0$ for some $j \in \mathrm{rng}(f)$. Choose i minimum such that $c_{f(i)} \cdot a_{f(i)} \neq 0$. Then $\mathscr{O}_c \cdot \mathscr{O}_{b^i} \neq 0$. ∎

Now we begin the discussion of $\mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$ itself. If I is finite, so that we are dealing with the ordinary free product, the situation has been thoroughly treated by Todorčević and Shelah; see, e.g., Monk [96]. For example, there is an atomless BA C such that $\mathrm{c}(C \oplus C) > \mathrm{c}(C)$.

For infinite index sets I the situation is different: rather than $\sup_{i\in I} \mathrm{c}(A_i)$, which is the natural thing to compare $\mathrm{c}(\bigoplus_{i\in I} A_i)$ with, the product $\prod_{i\in I} \mathrm{c}(A_i)$ turns out to be what should be compared with $\mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$.

Proposition 3.3. *If $\kappa_i < \mathrm{c}'A_i$ for all $i \in I$, then $\prod_{i\in I} \kappa_i < \mathrm{c}'(\bigoplus_{i\in I}^{\pi} A_i)$.*

Proof. For each $i \in I$ let Y_i be a disjoint subset of A_i of size κ_i. Clearly $\{\mathscr{O}_b : b \in \prod_{i\in I} Y_i\}$ is a disjoint subset of $\bigoplus_{i\in I}^{\pi} A_i$. ∎

Corollary 3.4. *If $\mathrm{c}A_i$ is attained for each $i \in I$, then $\prod_{i\in I} \mathrm{c}A_i \leq \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$.* ∎

Corollary 3.5. $\mathrm{c}(A_j) \leq \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$ *for each $j \in I$.* ∎

Proposition 3.6. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each of size at least four, with I infinite, and $\mathrm{c}A_i$ is attained and is at most equal to $|I|$, for all $i \in I$. Then $\mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i) = \prod_{i\in I} \mathrm{c}(A_i)$, and it is attained.*

Proof. The inequality $\geq$, and the fact that there is a disjoint set of size $\prod_{i\in I} \mathrm{c}(A_i)$, are true by Proposition 3.3. Now suppose that $X \subseteq \bigoplus_{i\in I}^{\pi} A_i$,

X is disjoint, and $|X| > \prod_{i\in I} \mathrm{c}(A_i)$. Without loss of generality $X = \{\mathscr{O}_b : b \in Y\}$, where $Y \subseteq B^\star$. Then

$$[Y]^2 = \bigcup_{i\in I} \{\{x,y\} : x,y \in Y,\ x \neq y,\ x_i \cdot y_i = 0\}.$$

Note that $2^{|I|} \leq \prod_{i\in I} \mathrm{c}A_i$. Hence by the Erdős–Rado theorem, there exist $Z \in [Y]^{|I|^+}$ and $i \in I$ such that for any two distinct $x, y \in Z$ we have $x_i \cdot y_i = 0$. This gives a disjoint subset of A_i of size $|I|^+$, contradiction. ∎

For the next proposition, recall that for any BA B, the cardinal number $\pi(B)$ is the smallest cardinality of a dense subset of B.

PROPOSITION 3.7. *Let $\langle A_i : i \in I\rangle$ be a system of BAs each with at least four elements, I infinite. Then* $\mathrm{c}(\bigoplus^\pi_{i\in I} A_i) \leq \prod_{i\in I} \pi A_i$.

Proof. Suppose that $X \subseteq \bigoplus^\pi_{i\in I} A_i$ is pairwise disjoint. We may assume that $X = \{\mathscr{O}_b : b \in Y\}$. For each $i \in I$ let Z_i be a subset of $A_i \setminus \{0\}$ which is dense in A_i, with $|Z_i| = \pi A_i$. For each $b \in Y$ and $i \in I$ choose $c_b(i) \in Z_i$ such that $c_b(i) \leq b(i)$. Now if $b, b' \in Y$ and $b \neq b'$, then $\mathscr{O}_b \cap \mathscr{O}_{b'} = 0$, and hence there is an $i \in I$ such that $b_i \cdot b'_i = 0$; so $c_b(i) \cdot c_{b'}(i) = 0$ and hence $c_b \neq c_{b'}$. Each c_b is in $\prod_{i\in I} Z_i$, and hence $|X| \leq \prod_{i\in I} \pi A_i$. ∎

COROLLARY 3.8. *Let $\langle A_i : i \in I\rangle$ be a system of atomic BAs each with at least four elements, I infinite. Then* $\mathrm{c}(\bigoplus^\pi_{i\in I} A_i) = \prod_{i\in I} \mathrm{c}A_i$, *with cellularity attained.* ∎

EXAMPLE 3.9. There is a system $\langle A_i : i \in I\rangle$ such that $\prod_{i\in I} \mathrm{c}A_i < \mathrm{c}(\bigoplus^\pi_{i\in I} A_i)$.

This example is just a slight adaptation of an example of Shelah concerning cellularity in ultraproducts. (The example is based on a method of Todorčević.) It depends on the following theorem of Shelah (Theorem 3.22 in Monk [96]):

Let $\lambda = \theta^+$ with θ an infinite cardinal. Then there is a $d : [\lambda]^2 \to \omega$ such that for all $m, n \in \omega$, if $\langle \zeta_i : i < \lambda\rangle$ is a system of n-tuples of members of λ such that $\zeta^1_i < \ldots < \zeta^n_i$ for all $i < \lambda$ and $\zeta^n_i < \zeta^1_j$ for $i < j < \lambda$, then there exist $i, j \in \lambda$ with $i < j$ such that $d\{\zeta^k_i, \zeta^i_j\} \geq m$ for all $k, l = 1, \ldots, n$.

We now describe the construction of some BAs and their properties found in the proof of Theorem 3.23 of Monk [96], also due to Shelah. Take λ, θ, and d as indicated. Also, take any $n \in \omega$. Let C_n be freely generated by $\langle x^n_\alpha : \alpha < \lambda\rangle$. Let I_n be the ideal in C_n generated by the set $\{x^n_\alpha \cdot x^n_\beta : \alpha < \beta < \lambda \text{ and } d\{\alpha, \beta\} \leq n\}$. Let $B_n = C_n/I_n$, and let $y^n_\alpha = x^n_\alpha/I_n$ for each $\alpha < \lambda$. It is shown in the indicated proof that each B_n satisfies the λ-cc, and that each y^n_α is nonzero.

We claim that $\bigoplus_{n\in\omega}^{\pi} B_n$ has a disjoint subset of size λ. Namely, let $b_\alpha = \langle y_\alpha^n : n \in \omega\rangle$ for each $\alpha < \lambda$. Then $\langle \mathscr{O}_{b_\alpha} : \alpha < \lambda\rangle$ is the desired family. For, suppose that $\alpha < \beta < \lambda$. With $n = d\{\alpha, \beta\}$, we have $y_\alpha^n \cdot y_\beta^n = 0$, and hence $\mathscr{O}_{b_\alpha} \cap \mathscr{O}_{b_\beta} = 0$, as desired.

Now, taking any infinite cardinal κ and letting $\theta = 2^\kappa$ and $\lambda = \theta^+$ in this construction we get the desired example: each B_n has cellularity at most 2^κ, hence $\prod_{n\in\omega} \mathrm{c}B_n \le 2^\kappa$, while $\mathrm{c}(\bigoplus_{n\in\omega}^{\pi} B_n) \ge (2^\kappa)^+$. ∎

The following question appears to be open:

Problem 1. *Is there a system $\langle A_i : i \in I\rangle$ of BAs such that $\prod_{i\in I} \mathrm{c}A_i > \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$?*

With regard to this problem, the above results imply that an example of such a system would necessarily have infinitely many A_i's with $\mathrm{c}(A_i)$ not attained (therefore inaccessible by the Erdős–Tarski theorem). In fact, if the set $J := \{i \in I : \mathrm{c}(A_i)$ is not attained$\}$ is finite, then by Proposition 3.3, $\prod_{i\in I\setminus J} \mathrm{c}(A_i) \le \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$, and by Corollary 3.5, $\prod_{i\in J} \mathrm{c}(A_j) \le \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$, so that $\prod_{i\in I} \mathrm{c}A_i \le \mathrm{c}(\bigoplus_{i\in I}^{\pi} A_i)$.

4. Complete generalized free products. We call the algebras $\mathrm{RO}(B^\star)$ *complete generalized free products*.

Theorem 4.1. *If A_i is complete, then the embedding f_i defined before Proposition* 1.2 *is a complete embedding.*

Proof. Let $X \subseteq A_i$. Obviously $\sum_{x\in X} f_i(x) \le f_i(\sum X)$. Suppose that $f_i(\sum X) \cdot -\sum_{x\in X} f_i(x) \ne 0$. Choose $b \in B^\star$ such that $\mathscr{O}_b \le f_i(\sum X) \cdot -\sum_{x\in X} f_i(x) \ne 0$. Then $\mathscr{O}_b \subseteq \mathscr{O}_{g(i,\sum X)}$, so $b \le g(i, \sum X)$, and hence $b_i \le \sum X$. On the other hand, $\mathscr{O}_b \cap \mathscr{O}_{g(i,x)} = 0$ for each $x \in X$, so $b_i \cdot x = 0$ for all $x \in X$, contradiction. ∎

Corollary 4.2. *Suppose that B and C are finitely closed subalgebras of $\prod_{i\in I} A_i$, and $B \le C$. Then $\mathrm{RO}(B^\star)$ is isomorphicallly embedded into $\mathrm{RO}(C^\star)$ by a mapping extending the one sending each set $\mathscr{O}_b^B$ to $\mathscr{O}_b^C$. In case B is a dense subalgebra of C, the embedding is complete.*

Proof. It is immediate from Proposition 1.14 and Sikorski's extension theorem that the indicated mapping f exists and is an isomorphism into.

Now assume that B is a dense subalgebra of C. Then define f as follows: for any $x \in \mathrm{RO}(B^\star)$,

$$(*) \qquad f(x) = \sum\{\mathscr{O}_b^C : b \in B^\star \text{ and } \mathscr{O}_b^B \subseteq x\}.$$

In fact, for f defined this way, it is clear that $f(\mathscr{O}_b^B) = \mathscr{O}_b^C$ for all $b \in B^\star$. To show that f preserves $\cdot$, suppose that $x, y \in \mathrm{RO}(B^*)$. Thus

$$f(x \cdot y) = f(x \cap y) = \sum\{\mathscr{O}_b^C : b \in B^\star \text{ and } \mathscr{O}_b^B \subseteq x \cap y\}$$

and

$$\begin{aligned}
&f(x) \cdot f(y) \\
&\quad = \Big(\sum\{\mathscr{O}^C_b : b \in B^\star \text{ and } \mathscr{O}^B_b \subseteq x\}\Big) \cdot \Big(\sum\{\mathscr{O}^C_b : b \in B^\star \text{ and } \mathscr{O}^B_b \subseteq y\}\Big) \\
&\quad = \sum\{\mathscr{O}^C_b \cap \mathscr{O}^C_d : b, d \in B^\star \text{ and } \mathscr{O}_b \subseteq x \text{ and } \mathscr{O}_d \subseteq y\}.
\end{aligned}$$

Clearly then $f(x \cdot y) \subseteq f(x) \cdot f(y)$. For the converse, it suffices to take any $b, d \in B^\star$ such that $\mathscr{O}^C_b \cap \mathscr{O}^C_d \neq 0$, $\mathscr{O}^B_b \subseteq x$, and $\mathscr{O}^B_d \subseteq y$ and show that $\mathscr{O}^C_b \cap \mathscr{O}^C_d \subseteq f(x \cdot y)$. Thus $b \cdot d \in B^\star$ and $\mathscr{O}^B_{b\cdot d} \subseteq x \cap y$, and hence

$$\mathscr{O}^C_b \cap \mathscr{O}^C_d = \mathscr{O}^C_{b\cdot d} \subseteq f(x \cdot y).$$

So, f preserves $\cdot$.

To show that f preserves $-$, let $x \in \mathrm{RO}(B^\star)$. Suppose that $\mathscr{O}^B_b \subseteq x$, $\mathscr{O}^B_d \subseteq -x$, and $\mathscr{O}^C_b \cap \mathscr{O}^C_d \neq 0$. Then $b\cdot d \in B^\star$ and $\mathscr{O}^B_{b\cdot d} \subseteq x \cdot -x$, contradiction. Hence $f(x) \cdot f(-x) = 0$. To show that $f(x) + f(-x) = B^\star$, it suffices to show that $f(x) \cup f(-x)$ is dense in $B^\star$. To this end, take any $\mathscr{O}^C_c$.

Case 1: $\mathscr{O}^C_c \cap x \neq 0$. Choose $d \in C^\star$ such that $\mathscr{O}^C_d \subseteq \mathscr{O}^C_c \cap x$. By the denseness, choose $b \in B^\star$ such that $b \leq d$. Then $\mathscr{O}^C_b \subseteq f(x)$, and hence $\mathscr{O}^C_c \cap f(x) \neq 0$.

Case 2: $\mathscr{O}^C_c \subseteq C^\star \setminus x$. Again choose $b \in B^\star$ such that $b \leq c$. Then $\mathscr{O}^B_b \subseteq -x$, and hence $\mathscr{O}^C_c \cap f(-x) \neq 0$. Thus we have proved $(*)$.

To show that f is a complete embedding, suppose that $X \subseteq \mathrm{RO}(B)$. Clearly $\sum_{x\in X} f(x) \leq f(\sum X)$. Suppose that $f(\sum X) \cdot -\sum_{x\in X} f(x) \neq 0$, and choose $c \in C^\star$ such that $\mathscr{O}_c \subseteq f(\sum X) \cdot -\sum_{x\in X} f(x)$. Then there is a $b \in B^\star$ such that $b \leq c$. Since

$$f\Big(\sum X\Big) = \sum\Big\{\mathscr{O}^C_b : b \in B^\star \wedge \mathscr{O}^B_b \subseteq \sum X\Big\},$$

there is a $b' \in B^\star$ such that $\mathscr{O}^C_b \cap \mathscr{O}^C_{b'} \neq 0$ and $\mathscr{O}^B_{b'} \subseteq \sum X$. It follows that b'', the pointwise infimum of b and b', is in $B^\star$. Thus $\mathscr{O}^B_{b''} \subseteq \mathscr{O}^B_{b'} \subseteq \sum X$. So there is an $x \in X$ such that $\mathscr{O}^B_{b''} \cap x \neq 0$. Then there is a $b''' \in B^\star$ such that $\mathscr{O}^B_{b''} \cap \mathscr{O}^B_{b'''} \neq 0$ and $\mathscr{O}^B_{b'''} \subseteq x$. Let b^{iv} be the pointwise infimum of b'' and b'''. Then $\mathscr{O}^C_{b^{\mathrm{iv}}} \subseteq f(x)$. But $\mathscr{O}^C_c \cap f(x) = 0$, and $\mathscr{O}^C_{b^{\mathrm{iv}}} \subseteq \mathscr{O}^C_c$, contradiction. ■

Corollary 4.3. *Let $\langle A_i : i \in I\rangle$ be a system of BAs, and let B be a finitely closed subalgebra of $\prod_{i\in I} A_i$. Then for any complete BA C, the following conditions are equivalent*:

(i) $C \cong \mathrm{RO}(B^\star)$.

(ii) *There exist embeddings f_i of A_i into C with the following properties*:

 (a) *For all $b \in B^\star$, $\prod^C_{i\in I} f_i(b_i) \neq 0$;*

 (b) *$\{\prod^C_{i\in I} f_i(b_i) : b \in B^\star\}$ is dense in C.*

Proof. By Theorem 1.15. ■

COROLLARY 4.4. *Suppose B is a finitely closed subalgebra of $\prod_{i\in I} A_i$, and $J \subseteq I$. Let $B_J = \{f\restriction J : f \in B\}$ and $B_{I\setminus J} = \{f\restriction(I \setminus J) : f \in B\}$. Assume also*

$$(*)\qquad B = \{u^\frown v : u \in B_J \text{ and } v \in B_{I\setminus J}\}.$$

Then $\mathrm{RO}(B^\star) \cong \overline{\mathrm{RO}(B_J^\star) \oplus \mathrm{RO}(B_{I\setminus J}^\star)}$.

Proof. By 1.16 and 4.3. ■

PROPOSITION 4.5. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, and B is a finitely closed subalgebra of $\prod_{i\in I} A_i$. Then $\overline{\bigoplus_{i\in I} A_i}$ is a retract of* $\mathrm{RO}(B^\star)$.

Proof. We use the notation of the proof of Proposition 1.18. By Sikorski's extension theorem, let g^+ and k^+ be extensions of g, k to homomorphisms from $\overline{\bigoplus_{i\in I} A_i}$ to $\mathrm{RO}(B^\star)$ and from $\mathrm{RO}(B^\star)$ to $\overline{\bigoplus_{i\in I} A_i}$ respectively. Then

$$(*)\qquad a \le k^+(g^+(a)) \quad \text{for any } a \in \overline{\bigoplus_{i\in I} A_i}.$$

In fact, if $\mathscr{O}_b \subseteq a$ with $b \in C^\star$, then $\mathscr{O}_b = k(g(\mathscr{O}_b)) = k^+(g^+(\mathscr{O}_b) \le k^+(g^+(a))$. Since $a = \sum\{\mathscr{O}_b : b \in C^\star,\ \mathscr{O}_b \subseteq a\}$, the condition $(*)$ follows.

From $(*)$ we also get $-a \le k^+(g^+(-a)) = -k^+(g^+(a))$, so $a = k^+(g^+(a))$ for all $a \in \overline{\bigoplus_{i\in I} A_i}$. ■

5. On Easton's theorem. As an illustration of using the methods of this paper, we indicate the connection between forcing and complete BAs connected to Easton's theorem (for sets, not proper classes). We follow the notation of Kunen [80].

The basic forcing topology for posets, used in our main definitions, runs as follows. If P is a poset, the sets $\{q : q \le p\}$, for p a member of P, form a base for the topology.

Here we apply this to the sets $\mathrm{Fn}(\kappa, \lambda, \mu)$ defined in Kunen [80], where the order is reverse inclusion.

Suppose that E is an Easton function, as on page 263 of Kunen's book. Let $I = \mathrm{dmn}(E)$. For each $\kappa \in I$ let $A_{E\kappa} = \mathrm{RO}(\mathrm{Fn}(E(\kappa), 2, \kappa))$. Define

$$B_E = \Big\{f \in \prod_{\kappa\in I} A_{E\kappa} : \text{for every infinite regular } \lambda,\\ |\{\kappa \in \lambda \cap I : f(\kappa) \ne 1\}| < \lambda\\ \text{or } |\{\kappa \in \lambda \cap I : f(\kappa) \ne 0\}| < \lambda\Big\}.$$

Clearly, B_E is a finitely closed subalgebra of $\prod_{\kappa\in I} A_{E\kappa}$. Let $C_E = \mathrm{RO}(B_E^\star)$.

For any cardinal λ, let

$$J_\lambda^- = \{\kappa \in I : \kappa \le \lambda\}, \qquad B_{E\lambda}^- = \{f\restriction J_\lambda^- : f \in B_E\},$$
$$J_\lambda^+ = \{\kappa \in I : \lambda < \kappa\}, \qquad B_{E\lambda}^+ = \{f\restriction J_\lambda^+ : f \in B_E\}.$$

Then $B_E \cong B_{E\lambda}^- \times B_{E\lambda}^+$ via $f \mapsto (f\restriction J_\lambda^-, f\restriction J_\lambda^+)$. So by Corollary 4.4 we have

$$\mathrm{RO}(B_E^\star) \cong \overline{\mathrm{RO}((B_{E\lambda}^-)^\star) \oplus \mathrm{RO}((B_{E\lambda}^+)^\star))}.$$

Next, there is an isomorphism of $\mathbb{P}(E)$ (defined in Kunen [80]) onto a dense subset of $\mathrm{RO}(B_E^\star)$. In fact, for each $p \in \mathbb{P}(E)$ define $f(p) \in \prod_{\kappa\in I} A_{E\kappa}$ by setting $f(p)_\kappa = \mathscr{O}_{p(\kappa)}$. Clearly, $f(p)_\kappa \in A_{E\kappa}$. Note that $1_{A_{E\kappa}} = \mathscr{O}_0$. Now for any $\kappa \in \lambda \cap I$ we have $f(p)_\kappa \ne 1$ iff $p(k) \ne 0$. It follows that $f(p) \in B_E^\star$. Now

$$\begin{aligned} p \le q \quad &\text{iff} \quad \forall \kappa \in I\ (p(\kappa) \le q(\kappa)) \\ &\text{iff} \quad \forall \kappa \in I\ (\mathscr{O}_{p(\kappa)} \subseteq \mathscr{O}_{q(\kappa)}) \\ &\text{iff} \quad f(p) \le f(q). \end{aligned}$$

Finally, $\mathrm{rng}(f)$ is dense, since if $b \in B_E^\star$, then we can choose $p(\kappa) \in \mathrm{Fn}(E(\kappa), 2, \kappa)$ such that $\mathscr{O}_{p(\kappa)} \subseteq b_k$ for every $\kappa \in I$. Clearly, $p \in \mathbb{P}(E)$, and $f(p) \subseteq \mathscr{O}_b$.

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