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The spectrum of partitions of a Boolean algebra

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Abstract. The main notion dealt with in this article is

$$\mathrm{PT}(A) = \{|\mathscr{P}| : \mathscr{P} \text{ is a partition of 1 in } A\},$$

where A is a Boolean algebra. A *partition of* 1 is a family of nonzero pairwise disjoint elements with sum 1. One of the main reasons for interest in this notion is from investigations about maximal almost disjoint families of subsets of sets X, especially $X = \omega$. We begin the paper with a few results about this set-theoretical notion.

Some of the main results of the paper are:

• (1) If there is a maximal family of size $\lambda \geq \kappa$ of pairwise almost disjoint subsets of κ each of size κ, then there is a maximal family of size λ of pairwise almost disjoint subsets of κ^+ each of size κ.

• (2) A characterization of the class of all cardinalities of partitions of 1 in a product in terms of such classes for the factors; and a similar characterization for weak products.

• (3) A cardinal number characterization of sets of cardinals with a largest element which are for some BA the set of all cardinalities of partitions of 1 of that BA.

• (4) A computation of the set of cardinalities of partitions of 1 in a free product of finite-cofinite algebras.

The set-theoretical background

We introduce some notation which will make it easier to state our results. Suppose that κ, ν, μ are infinite cardinals such that $\nu \leq \mu \leq \kappa$.

Sets A and B are *ν-almost disjoint* (*ν-ad*) if $|A \cap B| < \nu$. A family $\mathscr{A}$ of sets is *ν-almost disjoint* (*ν-ad*) if any two members of $\mathscr{A}$ are ν-ad. Now let $\mathscr{F}$ be a family of sets each of size at least ν. We say that $\mathscr{A}$ is *$\mathscr{F}$-maximal ν-almost disjoint* (*$\mathscr{F}$, ν-mad*) if $\mathscr{A} \subseteq \mathscr{F}$, it is ν-ad, and it is maximal among subsets of $\mathscr{F}$ which are ν-ad. Equivalently, $\mathscr{A}$ is $\mathscr{F}$-maximal ν-almost disjoint if $\mathscr{A} \subseteq \mathscr{F}$, it is ν-ad, and for each $X \in \mathscr{F}$ there is a $Y \in \mathscr{A}$ such that $|X \cap Y| \geq \nu$. Instead of $[\kappa]^\kappa$, *κ-mad* we say *κ-mad* or *κ-maximal almost disjoint*. Now we define

$$\mathrm{MAD}(\kappa, \nu, \mu) = \{|\mathscr{A}| : \mathscr{A} \text{ is} [\kappa]^\mu, \nu\text{-}mad\};$$

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$$\begin{aligned}
\mathrm{MAD}(\kappa) &= \mathrm{MAD}(\kappa,\kappa,\kappa);\\
\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu) &= \{|\mathscr{A}| : \text{ there is a partition } \mathscr{D} \text{ of } \kappa \text{ into } \lambda \text{ sets of size } \mu\\
&\qquad \text{such that } \mathscr{D}\cap\mathscr{A}=0 \text{ and } \mathscr{D}\cup\mathscr{A} \text{ is } [\kappa]^\mu,\ \nu\text{-mad}\};\\
\mathrm{MAD}_1(\kappa) &= \mathrm{MAD}_1(\kappa,\kappa,\kappa,\kappa);\\
\mathfrak{a}_{\kappa\nu\mu} &= \min(\mathrm{MAD}(\kappa,\nu,\mu)\cap[\mathrm{cf}\kappa,\infty));\\
\mathfrak{a}_\kappa &= \mathfrak{a}_{\kappa\kappa\kappa};\\
\mathfrak{a}_{\kappa\lambda\mu\nu 1} &= \min(\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu));\\
\mathfrak{a}_{\kappa 1} &= \mathfrak{a}_{\kappa\kappa\kappa\kappa 1};
\end{aligned}$$

Obviously $\mathrm{MAD}(\kappa,\lambda,\mu)\cap[\kappa,\infty)\neq 0$, so $\mathfrak{a}_{\kappa\lambda\mu}$ is well-defined. $\mathfrak{a}_{\kappa\lambda\mu\nu 1}$, however, does not always exist; see Proposition 1. The MAD_1 notion is generalized from a notion in van Douwen [84]. Having in mind as we do only the cardinality of maximal almost disjoint families, the main references are given in the list of references at the end of the paper. Baumgartner [76] has a treatment of simple relationships between the parameters, although he really considers the sizes of almost disjoint families which are not necessarily maximal. Mad families themselves are treated in most of the other references. Two important consistency results should be mentioned. Theorem 6.1 of Baumgartner [76] implies, with mild restrictions, that for any κ,λ,μ,ρ in the ground model one can define a forcing extension preserving cardinals in which there are members of $\mathrm{MAD}(\kappa,\lambda,\mu)$ of size at least ρ. Blass [91] allows one to even specify to a great extent the class of all members of $\mathrm{MAD}(\omega)$ in some model of set theory. His construction easily generalizes to give similar specifications for $\mathrm{MAD}(\kappa)$ for any infinite regular κ and for $\mathrm{MAD}(\kappa,\mu,\mu)$ with $\omega\leq\mu\leq\kappa$, both regular.

We begin our the paper with a simple proposition about the MAD_1 notion:

Proposition 1. *(i) If* $\kappa>\mu$ *and* $\kappa>\lambda$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu)=0$.

(ii) If $\lambda<\mathrm{cf}\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\nu)=\{0\}$.

(iii) If $\mathrm{cf}\kappa\leq\lambda<\kappa$ *and* $\nu<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\nu)=\{0\}$.

(iv) If $\mathrm{cf}\kappa\leq\lambda<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)\subseteq[(\mathrm{cf}\kappa)^+,\infty)$.

(v) If $\mathrm{cf}\kappa\leq\lambda<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)\cap[\kappa,\infty)\neq 0$.

(vi) If $\mu<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\kappa,\mu,\nu)\subseteq[\kappa,\infty)$.

(vii) If $\nu<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\kappa,\kappa,\nu)\subseteq[\kappa,\infty)$.

(viii) If κ *is regular*, *then* $\mathrm{MAD}_1(\kappa)\subseteq[\kappa^+,\infty)$.

Proof. Under the assumptions of *(i)*, there is no partition of κ into λ sets, each of power μ. For *(ii)*, if $\mathscr{D}$ is a partition of κ into λ sets each of power κ, then $\mathscr{D}$ is $[\kappa]^\kappa$, ν-mad. The same is true under the assumptions of *(iii)*. In fact, suppose that $\Gamma\in[\kappa]^\kappa$ and $|\Gamma\cap X|<\nu$ for all $X\in\mathscr{D}$. Then $\Gamma=\bigcup_{X\in\mathscr{D}}(\Gamma\cap X)$, which has size at most $\lambda\cdot\nu<\kappa$, contradiction.

We turn to *(iv)*. Assume that $\mathrm{cf}\kappa\leq\lambda<\kappa$ and $\rho\in\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)$. Accordingly, let $\mathscr{D}$ be a partition of κ into λ sets, each of size κ; say that $\mathscr{D}=\{D_\alpha : \alpha<\lambda\}$, without repetitions, and let $|\mathscr{A}|=\rho$, $\mathscr{D}\cap\mathscr{A}=0$, $\mathscr{D}\cup\mathscr{A}$ is κ-mad. Say $\mathscr{A}=\{A_\xi : \xi<\rho\}$ without repetitions.

First suppose that $\rho < \mathrm{cf}\kappa$. Now $|A \cap D_\alpha| < \kappa$ for all $A \in \mathscr{A}$ and $\alpha < \lambda$, so $\left|D_\alpha \cap \bigcup_{A\in\mathscr{A}} A\right| < \kappa$ for all $\alpha < \lambda$. Hence it is easy to construct a set $B \in [\kappa]^\kappa$ which is disjoint from each member of $\mathscr{A}$ and has intersection of size less than κ with each member of $\mathscr{D}$, contradiction.

Second, suppose that $\rho = \mathrm{cf}\kappa$. Hence Say $\kappa_\xi \uparrow \kappa$ for $\xi < \mathrm{cf}\kappa$. We claim
(1) $\forall \xi, \eta, \theta < \mathrm{cf}\kappa \exists \alpha \in \mathrm{cf}\kappa\backslash\{\xi+1, \eta+1, \theta+1\}[|A_\xi \cap D_\alpha| \geq \kappa_\eta]$.
For, assume otherwise; choose $\xi, \eta, \theta < \mathrm{cf}\kappa$ such that for all $\alpha \in \mathrm{cf}\kappa\backslash\{\xi+1, \eta+1, \theta+1\}$ we have $|A_\xi \cap D_\alpha| < \kappa_\eta$. Hence

$$\left|\bigcup\{A_\xi \cap D_\alpha : \alpha \in \mathrm{cf}\kappa\backslash\{\xi+1, \eta+1, \theta+1\}\right| \leq \mathrm{cf}\kappa \cdot \kappa_\eta < \kappa.$$

So

$$\left|\bigcup\{A_\xi \cap D_\alpha : \alpha < \max(\xi+1, \eta+1, \theta+1)\}\right| = \kappa.$$

But $|A_\xi \cap D_\alpha| < \kappa$ for all $\alpha < \max(\xi+1, \eta+1, \theta+1)$, and $\max(\xi+1, \eta+1, \theta+1) < \mathrm{cf}\kappa$, contradiction. So (1) holds.

Now we define $B_\xi \subseteq \kappa$ and $\alpha_\xi < \mathrm{cf}\kappa$ for $\xi < \mathrm{cf}\kappa$ so that always $|B_\xi| = \kappa_\xi$. Suppose defined for all $\eta < \xi$. Then

$$D \stackrel{\mathrm{def}}{=} \bigcup_{\eta<\xi}(A_\xi \cap A_\eta) \cup \bigcup_{\eta<\xi} B_\eta \cup \kappa_\xi^+$$

has size less than κ; say that its size is less than κ_τ, where $\xi < \tau < \mathrm{cf}\kappa$. By (1), choose $\alpha_\xi \in \mathrm{cf}\kappa\backslash\{\sup\{\alpha_\eta : \eta < \xi\}+1\}$ so that $|A_\xi \cap D_{\alpha_\xi}| \geq \kappa_\tau$. Choose $B_\xi \subseteq A_\xi \cap D_{\alpha_\xi}$ so that $|B_\xi| = \kappa_\xi$ and $B_\xi \cap D = 0$.

Let $B = \bigcup_{\xi<\mathrm{cf}\kappa} B_\xi$. Then $|B| = \kappa$, $|B \cap A_\xi| < \kappa$ for all $\xi < \mathrm{cf}\kappa$, $B \cap D_{\alpha_\xi} = B_\xi$, which has size less than κ, for each $\xi < \mathrm{cf}\kappa$, and $B \cap D_\beta = 0$ for all $\beta \in \lambda\backslash\{\alpha_\xi : \xi < \mathrm{cf}\kappa\}$. Thus $\mathscr{D} \cup \mathscr{A}$ is not κ-mad, contradiction. So *(iv)* holds.

Next, we take *(v)*; assume that $\mathrm{cf}\kappa \leq \lambda < \kappa$. Let $\omega \leq \kappa_\alpha \uparrow \kappa$ for $\alpha < \mathrm{cf}\kappa$. Let $\mathscr{D}$ be a partition of κ into λ sets each of size κ; say $\mathscr{D} = \{D_\alpha : \alpha < \lambda\}$. For each $\alpha < \mathrm{cf}\kappa$, let $\langle E_{\alpha\beta} : \beta < \kappa\rangle$ be a partition of D_α into sets of size κ. For all $\beta < \kappa$ let A_β be defined by requiring that $A_\beta \cap D_\alpha$ is a subset of $E_{\alpha\beta}$ of size κ_α for each $\alpha < \mathrm{cf}\kappa$, while $A_\beta \cap D_\alpha = 0$ if $\mathrm{cf}\kappa \leq \alpha < \lambda$. Thus $|A_\beta| = \kappa$, the A_β's are pairwise disjoint, and $|A_\beta \cap D_\alpha| < \kappa$ for all $\beta < \kappa$ and $\alpha < \mathrm{cf}\lambda$, as desired.

For *(vi)*, assume that $\mathscr{D}$ is a partition of κ into κ sets each of size $\mu < \kappa$, $\mathscr{A} \subseteq [\kappa]^\mu$, $\mathscr{D} \cap \mathscr{A} = 0$, $\mathscr{D} \cup \mathscr{A}$ is ν-ad, and $|\mathscr{A}| < \kappa$. For each $X \in \mathscr{A}$ let $M_X = \{Y \in \mathscr{D} : X \cap Y \neq 0\}$. Clearly $|M_X| \leq \mu$ for each $X \in \mathscr{A}$. Hence $\bigcup_{X\in\mathscr{A}} M_X$ has fewer than κ elements. So there is a subset Z of $\kappa\backslash\bigcup\bigcup_{X\in\mathscr{A}} M_X$ of size μ which has at most one element in common with each member of $\mathscr{D}$. Thus $\mathscr{D} \cup \mathscr{A}$ is not $[\kappa]^\mu$, ν-mad.

For *(vii)*, assume that $\mathscr{D}$ is a partition of κ into κ sets each of size κ, $\mathscr{A} \subseteq [\kappa]^\kappa$, $\mathscr{D} \cap \mathscr{A} = 0$, $\mathscr{D} \cup \mathscr{A}$ is ν-ad, and $|\mathscr{A}| < \kappa$. For each $D \in \mathscr{D}$ the set $\{D \cap A : A \in \mathscr{A}\}$ has size less than κ, and each set $D \cap A$ has size less than ν, so $\bigcup_{A\in\mathscr{A}} D \cap A$ has fewer than κ elements; so choose $a_D \in D\backslash\bigcup_{A\in\mathscr{A}} D \cap A$. Let $E = \{a_D : D \in \mathscr{D}\}$. Now $|\mathscr{D}| = \kappa$, so $|E| = \kappa$. Since $|E \cap X| < \nu$ for all $X \in \mathscr{D} \cup \mathscr{A}$, this shows that $\mathscr{D} \cup \mathscr{A}$ is not $[\kappa]^\kappa$, ν-mad.

Finally, *(viii)* is well-known. $\square$

Proposition 2. $\mathrm{MAD}_1(\kappa,\kappa,\mu,\nu) \subseteq \mathrm{MAD}(\kappa,\mu,\nu)$ *if* $\mu<\kappa$ *or* $\nu<\kappa$ *or* κ *is regular.*

Proof. By Proposition 1*(vi)–(viii)*. □

Proposition 3. *If* κ *is regular, then* $\mathrm{MAD}(\kappa)\cap[\kappa,\infty)=\mathrm{MAD}_1(\kappa)$.

Proof. By Proposition 1*(viii)* and Proposition 2, it remains only to prove $\subseteq$. Suppose that $\mathscr{A}$ is $[\kappa]^\kappa,\kappa$-mad and $|\mathscr{A}|\geq\kappa$. Let $X\in{}^\kappa\mathscr{A}$ be one-one. We define $\langle\alpha_\zeta:\zeta<\kappa\rangle$. Let $\zeta<\kappa$ be given. For every $\xi<\zeta$ choose $\beta_\xi<\kappa$ such that $X_\xi\cap X_\zeta\subseteq\beta_\xi$. Let $\alpha_\zeta=\left(\sup_{\xi<\zeta}\beta_\xi\right)\cup(\zeta+1)$. Thus
(*) For all $\xi,\zeta<\kappa$, if $\xi<\zeta$, then $X_\xi\cap X_\zeta\subseteq\alpha_\zeta$; moreover, $\zeta<\alpha_\zeta$.
Now define, for any $\zeta<\kappa$,

$$Y_\zeta=\begin{cases}(X_\zeta\backslash\alpha_\zeta)\cup\{\zeta\} & \text{if } \zeta\notin\bigcup_{\xi<\zeta}X_\xi,\\ X_\zeta\backslash\alpha_\zeta & \text{otherwise.}\end{cases}$$

If $\xi<\zeta<\kappa$, there are two possibilities. If $\zeta\in\bigcup_{\lambda<\zeta}X_\lambda$, then

$$Y_\xi\cap Y_\zeta\subseteq((X_\xi\backslash\alpha_\xi)\cup\{\xi\})\cap(X_\zeta\backslash\alpha_\zeta)=0.$$

If $\zeta\notin\bigcup_{\lambda<\zeta}X_\lambda$, then

$$Y_\xi\cap Y_\zeta\subseteq((X_\xi\backslash\alpha_\xi)\cup\{\xi\})\cap((X_\zeta\backslash\alpha_\zeta)\cup\{\zeta\})=0.$$

It follows that $\langle Y_\zeta:\zeta<\kappa\rangle$ is a partition. Let $\mathscr{A}'=\mathscr{A}\backslash\{X_\zeta:\zeta<\kappa\}$. Clearly $\mathscr{A}'\cup\{Y_\zeta:\zeta<\kappa\}$ is $[\kappa]^\kappa,\kappa$-mad. □

Proposition 4. $\mathrm{MAD}(\kappa,\mu,\mu)=\mathrm{MAD}_1(\kappa,\kappa,\mu,\mu)$ *if* κ *is regular and* $\mu<\kappa$.

Proof. Again, we only need to prove $\subseteq$. Let $\mathscr{A}$ be $[\kappa]^\mu,\mu$-mad.
(1) We may assume that $\forall\alpha<\kappa\ \exists X\in\mathscr{A}(|X\cap\alpha|<\mu)$.
To prove this, we consider two cases.

Case 1. $\forall\alpha<\kappa\ \exists\beta>\alpha\ \exists X\in\mathscr{A}(|X\cap\beta|=\mu$ and $|X\backslash\beta|=\mu)$. Now we define $X_\zeta\in\mathscr{A}$ and $\beta_\zeta<\kappa$ by induction, for each $\zeta<\kappa$. Suppose defined for all $\xi<\zeta$. Choose

$$\beta_\zeta>\bigcup_{\xi<\zeta}\beta_\xi\cup\bigcup_{\xi<\zeta}\sup X_\xi$$

and $X_\zeta\in\mathscr{A}$ such that $|X_\zeta\cap\beta_\zeta|=\mu$ and $|X_\zeta\backslash\beta_\zeta|=\mu$. The second part of the definition of β_ζ assures that the X_ζ's are distinct from each other. In $\mathscr{A}$, replace each X_ζ by $X_\zeta\cap\beta_\zeta$ and $X_\zeta\backslash\beta_\zeta$. The resulting set $\mathscr{A}'$ is still $[\kappa]^\mu,\mu$-mad, and (1) now holds.

Case 2. $\exists\alpha<\kappa\ \forall\beta>\alpha\ \forall X\in\mathscr{A}(|X\cap\beta|<\mu$ or $|X\backslash\beta|<\mu)$. We show that (1) holds for $\mathscr{A}$ itself. Let $\gamma<\kappa$ be given. Choose $\beta>\gamma,\alpha$, and let $Y\in[\kappa\backslash\beta]^\mu$. Choose $X\in\mathscr{A}$ such that $|X\cap Y|\geq\mu$. Hence $|X\backslash\beta|\geq\mu$. It follows that $|X\cap\beta|<\mu$, and hence $|X\cap\gamma|<\mu$, as desired.

Thus (1) holds, and we make the indicated assumption.

Now we define $X_\xi \in \mathscr{A}$ for all $\xi < \kappa$. Suppose that X_ξ has been defined for all $\xi < \zeta$. Let $Y_\zeta = \bigcup_{\xi<\zeta} X_\xi \cup \zeta$. Note that $\sup(Y_\zeta) < \kappa$, since $\mu < \kappa$. Choose $X_\zeta \in \mathscr{A}$ such that $|X_\zeta \cap \sup(Y_\zeta)| < \mu$. Next, define

$$X'_\zeta = (X_\zeta \cup \{\zeta\}) \backslash Y_\zeta.$$

Then
(2) If $\xi < \zeta < \kappa$, then $X'_\xi \cap X'_\zeta = 0$.
For, suppose that $\alpha \in X'_\xi \cap X'_\zeta$. Then $\alpha \in X_\xi \cup \{\xi\}$ because $\alpha \in X'_\xi$, and this contradicts $\alpha \in X'_\zeta$. So (2) holds.

Now
(3) $\zeta \in \bigcup_{\xi\leq\zeta} X'_\xi$ for all $\zeta < \kappa$.
In fact, suppose that $\zeta \notin \bigcup_{\xi\leq\zeta} X'_\xi$. Then $\zeta \in Y_\zeta$, so there is a $\xi < \zeta$ such that $\zeta \in X_\xi$; take the least such ξ. Then $\zeta \notin Y_\xi$, so $\zeta \in X'_\xi$, contradiction.

Now let $\mathscr{A}' = (\mathscr{A}\backslash\{X_\zeta : \zeta < \kappa\}) \cup \{X'_\zeta : \zeta < \kappa\}$. Clearly $\mathscr{A}'$ is still μ-ad. Suppose that $Y \in [\kappa]^\mu$. Choose $Z \in \mathscr{A}$ such that $|Y \cap Z| = \mu$. If $Z \notin \{X_\zeta : \zeta < \kappa\}$, this is fine. Suppose that $Z = X_\zeta$ with $\zeta < \kappa$, while $|Y \cap X'_\zeta| < \mu$. Then $|Y \cap X_\zeta \backslash X'_\zeta| \geq \mu$. But

$$Y \cap X_\zeta \backslash X'_\zeta \subseteq (X_\zeta \cap \sup Y_\zeta) \cup \{\zeta\},$$

and this has size less than μ, contradiction. So $\mathscr{A}'$ is $[\kappa]^\mu$, μ-mad, and it includes a partition $\mathscr{C}$ of κ into κ sets of size μ, as desired (see also Proposition 1*(vi)*). $\square$

The main result of the first part of this paper is:

Theorem 5. *If* κ *is regular*, *then* $\mathrm{MAD}(\kappa) \cap [\kappa, \infty) \subseteq \mathrm{MAD}(\kappa^+, \kappa, \kappa)$.

Proof. Let $\lambda \in \mathrm{MAD}(\kappa) \cap [\kappa, \infty)$. It is well-known that $\kappa < \lambda$. Now we construct $\mathscr{A}_\alpha$ for $\kappa \leq \alpha < \kappa^+$ so that the following conditions hold:
(1) $\mathscr{A}_\alpha$ is $[\alpha]^\kappa$, κ-mad.
(2) If $\kappa \leq \beta < \alpha$, then $\mathscr{A}_\beta \subseteq \mathscr{A}_\alpha$.
(3) $|\mathscr{A}_\alpha| = \lambda$.

To start with, let $\mathscr{A}_\kappa$ be obtained by the definition of $\mathrm{MAD}(\kappa)$ so that (1) and (3) hold; (2) does not apply yet. Now suppose that $\kappa < \alpha < \kappa^+$ and $\mathscr{A}_\beta$ has been defined for all $\beta \in [\kappa, \alpha)$.

Case 1. α is a successor ordinal $\beta + 1$. Let $\mathscr{A}_\alpha = \mathscr{A}_\beta$. Clearly (1)–(3) continue to hold.

Case 2. α is a limit ordinal, and $\mathrm{cf}\alpha < \kappa$. Let $\mathscr{A}_\alpha = \bigcup_{\kappa\leq\beta<\alpha} \mathscr{A}_\beta$. By (2) it is clear that $\mathscr{A}_\alpha$ is κ-ad. Suppose now that $\Gamma \in [\alpha]^\kappa$. Let $\langle \beta_\xi : \xi < \mathrm{cf}\alpha \rangle$ be a strictly increasing sequence of ordinals with supremum α, $\beta_0 \geq \kappa$. Then there is a $\xi < \mathrm{cf}\alpha$ such that $|\Gamma \cap \beta_\xi| = \kappa$. It follows that there is a $X \in \mathscr{A}_{\beta_\xi}$ such that $|\Gamma \cap X| = \kappa$. Thus $\mathscr{A}_\alpha$ is $[\alpha]^\kappa$, κ-mad. So (1) holds. Clearly (2) holds, as does (3).

Case 3. α is a limit ordinal, $\mathrm{cf}\alpha = \kappa$, and $\exists\beta < \alpha\forall\gamma \in (\beta, \alpha)[\mathrm{cf}\gamma < \kappa]$. Then α must have the form $\gamma + \kappa$ for some γ. Note that $\mathscr{A}_\delta = \mathscr{A}_\gamma$ for all $\delta \in \alpha\backslash\gamma$. Let

$\mathscr{A}$ be $[\alpha\backslash\gamma]^\kappa$, κ-mad and of size λ. Then we set $\mathscr{A}_\alpha = \mathscr{A}_\gamma \cup \mathscr{A}$. Clearly (1)–(3) hold.

Case 4. α is a limit ordinal, $\mathrm{cf}\alpha = \kappa$, and $\forall\beta < \alpha\exists\gamma \in (\beta, \alpha)[\mathrm{cf}\gamma = \kappa]$. Then there is a continuous strictly increasing sequence $\langle\beta_\xi : \xi < \kappa\rangle$ of ordinals with supremum α, with $\kappa = \beta_0$, and with $|\beta_{\xi+1}\backslash\beta_\xi| = \kappa$ for every $\xi < \kappa$. By Proposition 3, $\lambda \in \mathrm{MAD}_1(\kappa)$. Hence let $\mathscr{A}$ and $\mathscr{D}$ be as in the definition of $\mathrm{MAD}_1(\kappa)$, with $|\mathscr{D}| = \kappa$ and $|\mathscr{A}| = \lambda$. Let $\langle D_\xi : \xi < \kappa\rangle$ be a one-one enumeration of $\mathscr{D}$. Let f be a one-one function mapping κ onto α such that $f[D_0] = \kappa$ and $f[D_{1+\xi}] = \beta_{\xi+1}\backslash\beta_\xi$ for every $\xi < \kappa$. Then we define

$$\mathscr{A}_\alpha = \bigcup_{\kappa\leq\gamma<\alpha} \mathscr{A}_\gamma \cup \{f[a] : a \in \mathscr{A}\}.$$

(5) $\mathscr{A}_\alpha$ is κ-ad.
For, suppose that x and y are distinct elements of $\mathscr{A}_\alpha$. If both are in $\bigcup_{\kappa\leq\gamma<\alpha}\mathscr{A}_\gamma$, then they are both in $\mathscr{A}_\gamma$ for some $\gamma \in [\kappa, \alpha)$, and so $|x \cap y| < \kappa$. Suppose that $x \in \mathscr{A}_\gamma$ with $\gamma \in [\kappa, \alpha)$, and $y = f[a]$ with $a \in \mathscr{A}$. Choose $\xi < \kappa$ so that $\gamma < \beta_\xi$. Then

$$\begin{aligned} x \cap y &\subseteq \bigcup_{\eta<\xi}(\beta_{\eta+1}\backslash\beta_\eta) \cap y \\ &\subseteq f\left[\left(\bigcup_{\eta<\xi} D_\eta\right) \cap a\right], \end{aligned}$$

which has size less than κ. If $x = f[a]$ and $y = f[b]$ with $a, b \in \mathscr{A}$, clearly $|x \cap y| < \kappa$. By symmetry, these are all possibilities.
(6) $\mathscr{A}_\alpha$ is $[\alpha]^\kappa$, κ-mad.
For, let $x \in [\alpha]^\kappa$.

Case 1. $|x \cap (\beta_{\xi+1}\backslash\beta_\xi)| = \kappa$ for some $\xi < \kappa$. Choose $y \in \mathscr{A}_{\beta_{\xi+1}}$ such that $|x \cap y| = \kappa$.

Case 2. $x \cap (\beta_{\xi+1}\backslash\beta_\xi)$ has size less than κ for all $\xi < \kappa$. So, $|f^{-1}[x] \cap D_\xi| < \kappa$ for all $\xi < \kappa$. Choose $a \in \mathscr{A}$ such that $|f^{-1}[x] \cap a| = \kappa$. So $|x \cap f[a]| = \kappa$.

The construction is completed. Clearly $\bigcup_{\alpha<\kappa^+}\mathscr{A}_\alpha$ is $[\kappa^+]^\kappa$, κ-mad, as desired. □

Corollary 6. *If κ is regular, then $\mathfrak{a}_\kappa = \mathfrak{a}_{\kappa^+\kappa\kappa}$.* □

Proof. By Theorem 5 we have $\mathfrak{a}_\kappa \geq \mathfrak{a}_{\kappa^+\kappa\kappa}$.

For the converse, suppose that $\mathscr{A}$ is $[\kappa^+]^\kappa$, κ-mad and $|\mathscr{A}| = \mathfrak{a}_{\kappa^+\kappa\kappa}$. Then
(1) $\forall\alpha < \kappa^+\exists a \in \mathscr{A}[|a \cap (\kappa^+\backslash\alpha)| = \kappa]$.
For, otherwise we obtain $\alpha < \kappa^+$ such that for all $a \in \mathscr{A}$ we have $|a\cap(\kappa^+\backslash\alpha)| < \kappa$. Choose any $b \in [\kappa^+\backslash\alpha]^\kappa$. Then $\forall a \in \mathscr{A}[|a\cap b| < \kappa]$, contradicting the maximality of $\mathscr{A}$. So (1) holds.

Now we define $\langle\alpha_\xi : \alpha < \kappa\rangle$ and $\langle a_\xi : \xi < \kappa\rangle$ by recursion. If they are defined for all $\eta < \xi$, let

$$\alpha_\xi = \max\{\sup_{\eta<\xi} \alpha_\eta + 1, \sup_{\eta<\xi}(\sup a_\eta) + 1\}.$$

By (1), choose $a_\xi \in \mathscr{A}$ such that $|a_\xi \cap (\kappa^+ \backslash \alpha_\xi)| = \kappa$.

Let $\beta = \bigcup_{\xi<\kappa} \alpha_\xi$ and $\mathscr{B} = \{a \cap \beta : a \in \mathscr{A}, |a \cap \beta| = \kappa\}$. Then $\mathscr{B}$ is clearly κ-ad, and $a_\xi \in \mathscr{B}$ for each $\xi < \kappa$, so $|\mathscr{B}| \geq \kappa$. If $b \in [\beta]^\kappa$, choose $a \in \mathscr{A}$ such that $|a \cap b| = \kappa$ (by the maximality of $\mathscr{A}$). Then $a \cap \beta \in \mathscr{B}$ and $|a \cap \beta \cap b| = |a \cap b| = \kappa$. So $\mathscr{B}$ is $[\beta]^\kappa$, κ-mad. $\square$

The partition spectrum of a Boolean algebra

Suppose that A is a BA, and $J \subseteq I$ are two ideals in A. Then we set

$$\begin{aligned} \mathrm{PT}(A, I, J) &= \{|\mathscr{P}| : \mathscr{P} \text{ is a maximal pairwise disjoint subset of } I \backslash J\}; \\ \mathrm{PT}(A, I) &= \mathrm{PT}(A, I, \{0\}); \\ \mathrm{PT}(A) &= \mathrm{PT}(A, A, \{0\}). \end{aligned}$$

Proposition 7. *Suppose that $\omega \leq \lambda \leq \mu \leq \kappa$. Let $A = \mathscr{P}\kappa/[\kappa]^{<\lambda}$, $I = \{[X] : |X| \leq \mu\}$, and $J = \{[X] : |X| < \mu\}$. Then* $\mathrm{PT}(A, I, J) = \mathrm{MAD}(\kappa, \lambda, \mu)$.

Proof. Let $\mathscr{P}$ be a maximal pairwise disjoint subset of $I \backslash J$. Write $\mathscr{P} = \{[X] : X \in \mathscr{Q}\}$, with $[X] \neq [Y]$ for distinct $X, Y \in \mathscr{Q}$, and $\mathscr{Q} \subseteq [\kappa]^\mu$. Clearly $\mathscr{Q}$ is λ-ad. Suppose that $Y \in [\kappa]^\mu$. Then $[Y] \in I \backslash J$, so there is an $X \in \mathscr{Q}$ such that $[X] \cdot [Y] \neq 0$. Thus $|X \cap Y| \geq \lambda$. Thus $\mathscr{Q}$ is $[\kappa]^\mu$, λ-mad. This proves $\subseteq$.

The proof of $\supseteq$ is equally obvious. $\square$

Proposition 8. *If I is a dense ideal of A, then*

$$\mathrm{PT}(A, I) = \{|\mathscr{P}| : \mathscr{P} \text{ is a partition of } 1 \text{ and } \mathscr{P} \subseteq I\}.$$

Proof. Suppose that $\mathscr{P}$ is a maximal pairwise disjoint subset of $I \backslash \{0\}$. Suppose that $0 \neq y \in A$. Choose $x \in I \backslash \{0\}$ such that $x \leq y$. Then there is a $z \in \mathscr{P}$ such that $x \cdot z \neq 0$. So also $y \cdot z \neq 0$. Hence $\mathscr{P}$ is a partition of 1. This proves $\subseteq$.

$\supseteq$ is obvious. $\square$

Corollary 9. *Suppose that $\omega \leq \mu \leq \kappa$. Let $A = \mathscr{P}\kappa/[\kappa]^{<\mu}$, and let $I = \{[X] : |X| \leq \mu\}$. Then* $\mathrm{PT}(A, I) = \mathrm{MAD}(\kappa, \mu, \mu)$. $\square$

Corollary 10. *Suppose that $\omega \leq \kappa$. Let $A = \mathscr{P}\kappa/[\kappa]^{<\kappa}$. Then* $\mathrm{PT}(A) = \mathrm{MAD}(\kappa)$. $\square$

There is a slightly different way of seeing that PT generalizes MAD, given in the following easy results.

Proposition 11. *Suppose that $\omega \leq \lambda \leq \mu \leq \kappa$. Let $B = [\kappa]^{\leq\mu} \cup \{X \subseteq \kappa : \kappa \backslash X \in [\kappa]^{\leq\mu}\}$, and let $A = B/[\kappa]^{<\lambda}$. Further, let $I = \{[X] : X \in B,\ |X| \leq \mu\}$, and $J = \{[X] : X \in B,\ |X| < \mu\}$. Then* $\mathrm{PT}(A, I, J) = \mathrm{MAD}(\kappa, \lambda, \mu)$. $\square$

Corollary 12. *Suppose that $\omega \leq \mu \leq \kappa$. Let $B = [\kappa]^{\leq\mu} \cup \{X \subseteq \kappa : \kappa \backslash X \in [\kappa]^{\leq\mu}\}$, and let $A = B/[\kappa]^{<\mu}$. Further, let $I = \{[X] : X \in B,\ |X| \leq \mu\}$. Then* $\mathrm{PT}(A, I) = \mathrm{MAD}(\kappa, \mu, \mu)$. *Furthermore, I is a maximal ideal of B.* $\square$

For the rest of the paper we concentrate on the most natural algebraic notion, PT itself. First we mention some facts which are easy to prove. Recall that the *cellularity* of a BA A, denoted by $\mathrm{c}A$, is the supremum of cardinalities of disjoint subsets of A.

(1) If A is infinite, then $[1,\omega) \subseteq \mathrm{PT}(A) \subseteq [1, \mathrm{c}A]$.

(2) Let A be infinite. Then $\sup(\mathrm{PT}(A)) = \mathrm{c}A$, and $\mathrm{c}A$ is attained iff $\mathrm{c}A \in \mathrm{PT}(A)$.

(3) Suppose that A is complete. Then:

(i) If $\mathrm{c}A$ is attained, then $\mathrm{PT}(A) = [1, \mathrm{c}A]$.

(ii) If $\mathrm{c}A$ is not attained, then $\mathrm{PT}(A) = [1, \mathrm{c}A)$.

(4) $\mathrm{PT}(\mathrm{Finco}\kappa) = [1,\omega) \cup \{\kappa\}$.

(5) If A is a regular subalgebra of B, then $\mathrm{PT}(A) \subseteq \mathrm{PT}(B)$. Recall that A is a regular subalgebra of B if whenever $X \subseteq A$ and $\sum^A X$ exists, then also $\sum^B X$ exists and equals $\sum^A X$.

(6) For every infinite BA A there is a BA B such that A is a subalgebra of B and $\mathrm{PT}(A) \cap \mathrm{PT}(B) = [1,\omega)$.

In fact, let κ be a regular cardinal greater than $|A|$, and embed A into a κ-saturated BA B; this is as desired.

(7) For every infinite BA B and every infinite disjoint $P \subseteq B$ there is a subalgebra A of B such that $\mathrm{PT}(A) = [1,\omega) \cup \{|P|\}$.

In fact, let A be generated by P.

(8) Since there are BAs A and B with $\mathrm{c}(A \oplus B) > \max\{\mathrm{c}A, \mathrm{c}B\}$, it is not possible to compute $\mathrm{PT}(A \oplus B)$ solely in terms of $\mathrm{PT}(A)$ and $\mathrm{PT}(B)$. But we do have $\mathrm{PT}(A) \cup \mathrm{PT}(B) \subseteq \mathrm{PT}(A \oplus B)$.

(9) It is natural to try to characterize the set $\mathrm{PT}(A)$ of cardinals in cardinal-theoretic terms. Clearly any such set has an infinite member (assuming that A is infinite), and either has a largest element, or if not, its supremum is weakly inaccessible, by the Erdös, Tarski theorem. Two easy examples also shed light on what to expect for $\mathrm{PT}(A)$.

In these examples and later on in the text, intervals like $[1,\kappa)$ *or* $[\kappa,\lambda]$ *are understood as the set of all* **cardinals** *in those intervals.*

First, let κ be any uncountable weakly inaccessible cardinal, and let f be a function mapping κ^+ onto κ. For each $\alpha < \kappa^+$ let $A_\alpha = \mathrm{Finco}(|f(\alpha)|)$. Then $\mathrm{PT}\left(\prod^{\mathrm{w}}_{\alpha<\kappa^+} A_\alpha\right) = [1,\kappa) \cup \{\kappa^+\}$. Second, let κ be a singular cardinal, and define for each $\alpha < \kappa^+$

$$A_\alpha = \begin{cases} \mathrm{Finco}(\mathrm{cf}\alpha) & \text{if } \alpha \text{ is a limit ordinal,} \\ \mathrm{Finco}\omega & \text{otherwise.} \end{cases}$$

Then $\mathrm{PT}\left(\prod^{\mathrm{w}}_{\alpha<\kappa^+} A_\alpha\right) = [1,\kappa) \cup \{\kappa^+\}$. This example shows that in general the analog of the Erdös, Tarski theorem for partitions of unity does not hold.

Thus we can formulate the characterization problem as follows.

Problem 1. *Suppose that C is a nonempty collection of infinite cardinals such that either C has a largest element or else* $\sup C$ *is weakly inaccessible. Is there a BA A such that* $\mathrm{PT}(A) = [1,\omega) \cup C$*?*

We prove two results giving affirmative answers to special cases of this question.

First we characterize what happens to PT under products.

Proposition 13. *If A or B is infinite, then* $\mathrm{PT}(A \times B) = \mathrm{PT}(A) \cup \mathrm{PT}(B)$.

Proof. Clearly $\omega\backslash\{0\}$ is a subset of both sides. Now suppose that $\kappa \in \mathrm{PT}(A \times B)$, $\kappa \geq \omega$. Say $\kappa = |X|$, X maximal disjoint in $A \times B$. Then either $\kappa = |\mathrm{pr}_0[X]\backslash\{0\}|$ or $\kappa = |\mathrm{pr}_1[X]\backslash\{0\}|$, and each of the indicated sets is maximal disjoint in A resp. B. So $\kappa \in \mathrm{PT}(A) \cup \mathrm{PT}(B)$. Conversely, if for example $\omega \leq \kappa \in \mathrm{PT}(A)$, say $\kappa = |X|$, with X maximal disjoint in A. Then $(X \times \{0\}) \cup \{(0, 1)\}$ is maximal disjoint in $A \times B$, and it has size κ. □

For infinite products the situation is more complicated.

Theorem 14. *If I is infinite and each A_i is nontrivial, then*

$$\mathrm{PT}\left(\prod_{i\in I} A_i\right) = [1, |I|] \cup \bigcup_{i\in I} \mathrm{PT}(A_i)$$
$$\cup \Big\{\kappa : \kappa \text{ is singular and there exist a sequence } \langle \mu_\alpha : \alpha < \mathrm{cf}\kappa\rangle \text{ of cardinals and a one-one sequence } \langle i_\alpha : \alpha < \mathrm{cf}\kappa\rangle \text{ of members of } I \text{ such that } \mu_\alpha \in \mathrm{PT}(A_{i_\alpha}) \text{ for all } \alpha < \mathrm{cf}\kappa \text{ and } \sup_{\alpha<\mathrm{cf}\kappa} \mu_\alpha = \kappa\Big\}.$$

Proof. $\subseteq$: Suppose that $\kappa \in \mathrm{PT}\left(\prod_{i\in I} A_i\right)$. Suppose that $|I| < \kappa$ and $\kappa \notin \bigcup_{i\in I} \mathrm{PT}(A_i)$. Say $\kappa = |X|$, where X is a maximal disjoint subset of $\prod_{i\in I} A_i$. First note that for any $i \in I$, the set $\{x_i : x \in X,\ x_i \neq 0\}$ is a partition of unity in A_i, and hence has size $< \kappa$. Now $|\{x_i : x \in X,\ x_i \neq 0\}| = |\{x \in X : x_i \neq 0\}|$, and $X = \bigcup_{i\in I}\{x \in X : x_i \neq 0\}$, and $|I| < \kappa$, so κ is singular and desired conclusion is clear.

$\supseteq$: First suppose that $\kappa \in [1, |I|]$. Let $\mathscr{A}$ be a partition of I into κ sets. For each $A \in \mathscr{A}$, define $\chi_A \in \prod_{i\in I} A_i$ by setting, for any $i \in I$,

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\chi_A : A \in \mathscr{A}\}$ is maximal disjoint and of size κ.

Second, suppose that $\kappa > |I|$ and $\kappa \in \mathrm{PT}(A_i)$ for some $i \in I$. Choose a maximal disjoint subset X of A_i such that $|X| = \kappa$. Note that X is infinite. Fix $x_0 \in X$. Let $g(i) = x_0$, and $g(j) = 1$ for all $j \in I\backslash\{i\}$. For each $x \in X\backslash\{x_0\}$ let $f^{(x)} \in \prod_{j\in I} A_j$ be such that $f^{(x)}(i) = x$ and $f^{(x)}(j) = 0$ for all $j \in I\backslash\{i\}$. Then

$$\{f^{(x)} : x \in X\backslash\{x_0\}\} \cup \{g\}$$

is maximal disjoint in $\prod_{j\in I} A_j$ and has size κ.

Finally, suppose that κ is a singular cardinal $> |I|$ with $\kappa \notin \bigcup_{i\in I} \mathrm{PT}(A_i)$, $\langle \mu_\alpha : \alpha < \mathrm{cf}\kappa\rangle$ is a sequence of cardinals, $\langle i_\alpha : \alpha < \mathrm{cf}\kappa\rangle$ is a one-one sequence of members of I, $\mu_\alpha \in \mathrm{PT}(A_{i_\alpha})$ for all $\alpha < \mathrm{cf}\kappa$ and $\sup_{\alpha<\mathrm{cf}\kappa} \mu_\alpha = \kappa$. Say X_α is a partition of unity in A_{i_α} with $|X_\alpha| = \mu_\alpha$, for every $\alpha < \mathrm{cf}\kappa$. For each $\alpha < \mathrm{cf}\kappa$

and each $x \in X_\alpha$ define $f^{\alpha x} \in \prod_{j\in I} A_j$ by setting $f^{\alpha x}(i_\alpha) = x$ and $f^{\alpha x}(j) = 0$ otherwise. Then

$$\{f^{\alpha x} : \alpha < \mathrm{cf}\kappa,\ x \in X_\alpha\} \cup \{\chi_{\{i\}} : i \in I \backslash \mathrm{rng}(i)\}$$

is a partition of unity in $\prod_{j\in J} A_j$ of size κ, as desired. □

A modified theorem and proof work for weak products. Recall that the weak product $\prod^{\mathrm{w}}_{i\in I} A_i$ consists of all functions f in the full direct product such that $\{i \in I : f(i) \neq 0\}$ is finite or $\{i \in I : f(i) \neq 1\}$ is finite.

Theorem 15. *If I is infinite and each A_i is nontrivial, then*

$$\begin{aligned}\mathrm{PT}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) &= [1,\omega) \cup \{|I|\} \cup \bigcup_{i\in I} \mathrm{PT}(A_i)\\ &\cup \Big\{\kappa : \kappa \text{ is singular and there exist a sequence } \langle \mu_\alpha : \alpha < \mathrm{cf}\kappa\rangle\\ &\quad \text{of cardinals and a one-one sequence } \langle i_\alpha : \alpha < \mathrm{cf}\kappa\rangle \text{ of members}\\ &\quad \text{of } I \text{ such that } \mu_\alpha \in \mathrm{PT}(A_{i_\alpha}) \text{ for all } \alpha < \mathrm{cf}\kappa \text{ and } \sup_{\alpha<\mathrm{cf}\kappa} \mu_\alpha = \kappa\Big\}.\end{aligned}$$

Proof. $\subseteq$: Suppose that $\kappa \in \mathrm{PT}\big(\prod^{\mathrm{w}}_{i\in I} A_i\big)$. Suppose that κ is infinite, $|I| \neq \kappa$, and $\kappa \notin \bigcup_{i\in I} \mathrm{PT}(A_i)$. Say $\kappa = |X|$, X disjoint, $X \subseteq \prod^{\mathrm{w}}_{i\in I} A_i$.

Case 1. $\kappa < |I|$. Say $|X| = \kappa$ with X maximal disjoint in $\prod^{\mathrm{w}}_{i\in I} A_i$. *Subcase 1.1.* There is an $x \in X$ such that $\Gamma \overset{\text{def}}{=} \{i \in I : x_i \neq 1\}$ is finite. Then the proof of Proposition 13 shows that $|X| \in \mathrm{PT}(A_i)$ for some $i \in \Gamma$, contradiction. *Subcase 1.2.* $\{i \in I : x_i \neq 0\}$ is finite for all $x \in X$. Since $\kappa < |I|$, this is clearly impossible.

Case 2. $|I| < \kappa$. Then the argument in the proof of Theorem 14 works.

$\supseteq$: see the proof of Theorem 14. □

A partial solution of Problem 1 above is as follows.

Proposition 16. *Suppose that Γ is a set of infinite cardinals with a largest element. Then there is a BA A such that* $\mathrm{PT}(A) = [1,\omega) \cup \Gamma$.

Proof. Note that $|\Gamma| \leq \max\Gamma$. Let κ be a function from $\max\Gamma$ onto Γ. For each $i \in \max\Gamma$ let $A_i = \mathrm{Finco}\kappa_i$. Then by Theorem 16, $\prod^{\mathrm{w}}_{i\in\max\Gamma} A_i$ is as desired. □

Although we do not have any general results about PT for free products (see (8)), we can prove some special results.

Proposition 17. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least 4 elements and I is infinite, then $\omega \in \mathrm{PT}(\bigoplus_{i\in I} A_i)$.*

Proof. Let f be a one-one function from ω into I. For each $i < \omega$ choose $a_{f(i)} \in A_{f(i)}$ such that $0 < a_{f(i)} < 1$. Now we define $b_i = a_{f(i)} \cdot \prod_{j<i} -a_{f(j)}$ for each $i < \omega$. Then $\langle b_i : i < \omega\rangle$ is a partition of 1. □

Proposition 18. *Suppose that C is a set of infinite cardinals, and λ is a singular cardinal not in C, but $C \cap \lambda$ is unbounded in λ. Then* $\lambda \in \mathrm{PT}(\oplus_{\nu \in C}\mathrm{Finco}\nu)$.

Proof. We have

$$\oplus_{\nu \in C}\mathrm{Finco}\nu \cong \left(\oplus_{\substack{\nu\in C,\\ \nu<\lambda}} \mathrm{Finco}\nu\right) \oplus \left(\oplus_{\substack{\nu\in C,\\ \nu>\lambda}} \mathrm{Finco}\nu\right),$$

and $\lambda \in \mathrm{PT}\left(\oplus_{\substack{\nu\in C,\\ \nu<\lambda}} \mathrm{Finco}\nu\right)$ by the Erdös, Tarski theorem, so also $\lambda \in \mathrm{PT}\left(\oplus_{\nu\in C}\mathrm{Finco}\nu\right)$. (see (8)) $\square$

Theorem 19. *Suppose that C is a set of infinite cardinals with $\omega \in C$, and such that for every singular cardinal λ, if $\lambda \cap C$ is unbounded in λ, then $\lambda \in C$. Then* $\mathrm{PT}(\oplus_{\nu\in C}\mathrm{Finco}\nu) = [1, \omega) \cup C$.

Proof. If $\nu \in C$, then $\langle\{\alpha\} : \alpha < \nu\rangle$ is a partition of 1 in $\oplus_{\nu\in C}\mathrm{Finco}\nu$. (Each $\{\alpha\}$ is considered to be in the free factor $\mathrm{Finco}\nu$.) Thus $\supseteq$ in the conclusion of the theorem holds.

Now suppose that λ is an infinite cardinal not in C. Then λ is uncountable, since $\omega \in C$. Moreover, if λ is a limit of members of C, then λ is weakly inaccessible. It follows that there is a regular cardinal $\rho \le \lambda$ such that $\nu < \rho$ for any member ν of $C \cap \lambda$. Suppose that $\langle a_\alpha : \alpha < \lambda\rangle$ is a partition of 1 in $\oplus_{\nu\in C}\mathrm{Finco}\nu$. Without loss of generality, for each $\alpha < \lambda$, $a_\alpha = \prod_{\nu\in A_\alpha} x_{\nu\alpha}$, where each A_α is a finite subset of C and each $x_{\nu\alpha}$ is in $\mathrm{Finco}\nu$. Without loss of generality, if $x_{\nu\alpha}$ is finite then $x_{\nu\alpha} = \{\beta_{\nu\alpha}\}$ and if $\nu < \lambda$ then $x_{\nu\alpha} = \{\beta_{\nu\alpha}\}$. For all $\nu \in C\backslash\lambda$ choose

$$\gamma_\nu \in \nu\backslash\{\beta_{\nu\alpha} : \alpha < \lambda, x_{\nu\alpha} \text{ finite}\}.$$

Let $\langle A_\alpha : \alpha \in \Gamma\rangle$ be a Δ-system, say with kernel F, where $|\Gamma| = \rho$. Now

$$\Gamma = \bigcup_{\delta\in\prod_{\nu\in F\cap\lambda}\nu} \{\alpha \in \Gamma : \forall \nu \in F \cap \lambda(\beta_{\nu\alpha} = \delta_\nu)\}$$

and $\left|\prod_{\nu\in F\cap\lambda}\nu\right| < \rho$ and ρ is regular, so there is a $\Gamma_0 \in [\Gamma]^\rho$ and a $\delta \in \prod_{\nu\in F\cap\lambda}\nu$ such that for all $\nu \in F \cap \lambda$ and all $\alpha \in \Gamma_0$ we have $\beta_{\nu\alpha} = \delta_\nu$.

Let $m = |F\backslash\lambda|$. We now construct Γ_i for all $i \le m$ and $\varepsilon_i, \nu_i, \alpha_i$ for all $i < m$. Suppose that for a certain $i < m$ we have already constructed Γ_j for all $j \le i$ and ε_j, ν_j, and α_j for all $j < i$ so that each Γ_j is a subset of Γ_0 of size ρ^+, $\nu_0, \ldots, \nu_{i-1}$ are distinct members of $F\backslash\lambda$, each $\varepsilon_j < \nu_j$, if $j < i$ then for all $\beta \in \Gamma_{j+1}$, $x_{\nu_j\beta} = \{\varepsilon_j\}$, and each $\alpha_j < \lambda$. Note that this has been done trivially for $i = 0$. (Note also the possibility that $m = 0$.) Choose $\alpha_i < \lambda$ such that

$$\prod_{\nu\in F\cap\lambda}\{\delta_\nu\} \cdot \prod_{j<i}\{\varepsilon_j\} \cdot \prod_{\nu\in F\backslash(\{\nu_0,\ldots,\nu_{i-1}\}\cup\lambda)}\{\gamma_\nu\} \cdot a_{\alpha_i} \ne 0.$$

Let $\Gamma'_{i+1} = \Gamma_i\backslash\{\alpha_i\}$. (Note, however, that perhaps $\alpha_i \notin \Gamma_i$.) Now

$$\Gamma'_{i+1} = \bigcup_{\nu\in A_{\alpha_i}} \{\beta \in \Gamma'_{i+1} : \nu \in A_\beta \text{ and } x_{\nu\alpha_i} \cap x_{\nu\beta} = 0\},$$

so there exists $\nu_i \in A_{\alpha_i}$ such that

$$\Gamma''_{i+1} \stackrel{\text{def}}{=} \{\beta \in \Gamma'_{i+1} : \nu_i \in A_\beta \text{ and } x_{\nu_i\alpha_i} \cap x_{\nu_i\beta} = 0\}$$

has size ρ. We claim that
(*) $\nu_i \in F \backslash (\{\nu_0, \ldots, \nu_{i-1}\} \cup \lambda)$.
In fact, first choose β, γ distinct ordinals in Γ''_{i+1}. Then $\nu_i \in A_\beta \cap A_\gamma = F$. If $\nu_i \in \lambda$, then $x_{\nu_i\beta} = \{\delta_{\nu_i}\}$ and so $x_{\nu_i\beta} \cap x_{\nu_i\alpha_i} \neq 0$, contradiction. If $\nu_i = \nu_j$ for some $j < i$, then $x_{\nu_i\beta} = \{\varepsilon_j\}$, and again we get a contradiction. So (*) holds.

From (*) it follows that $\gamma_{\nu_i} \in x_{\nu_i\alpha_i}$. Hence $x_{\nu_i\alpha_i}$ is cofinite. Hence for any $\beta \in \Gamma''_{i+1}$, $x_{\nu_i\beta}$ is contained in the finite set $\nu_i \backslash x_{\nu_i\alpha_i}$. So there exist an $\varepsilon_i < \nu_i$ and a subset Γ_{i+1} of Γ''_{i+1} of size ρ such that for all $\beta \in \Gamma_{i+1}$, $x_{\nu_i\beta} = \{\varepsilon_i\}$. This finishes the construction.

For any two distinct $\beta, \gamma \in \Gamma_m$ we have $x_{\nu\beta} = x_{\nu\gamma}$ for all $\nu \in F$. It follows that $a_\beta \cdot a_\gamma \neq 0$, contradiction. $\square$

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