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Depth, π-character, and tightness in superatomic Boolean algebras

Alan Dow *, J. Donald Monk

Department of Mathematics, York University, North York, Ontario M3J 1P3, Canada

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Abstract

Theorem A. *There is a superatomic Boolean algebra B with size and π-character equal to ω_1 and countable depth.* **Theorem B.** *If B is a superatomic Boolean algebra with π-character greater than ω_1, then the π-character and depth of B are the same.* **Theorem C.** *If $\kappa \to (\kappa)_2^{<\omega}$, then every superatomic Boolean algebra with tightness at least κ^+ has depth at least κ.*

Keywords: Tightness; π-character; Superatomic Boolean algebra

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Introduction

We abbreviate "Boolean algebra" by "BA". A BA B is superatomic if every homomorphic image of B is atomic. The *depth* of a BA B is the supremum of all the cardinals κ such that there is a sequence $\langle b_\alpha\colon \alpha < \kappa\rangle$ of elements of B with $b_\alpha < b_\beta$ for all $\alpha < \beta < \kappa$. If F is an ultrafilter on a Boolean algebra B, then the *π-character* of F, denoted by $\pi\chi F$, is the smallest cardinal κ such that there is a subset D of B^+ (not necessarily of F) of size κ such that D is dense in F. Here $B^+ = B \setminus \{0\}$, and D dense in F means that for all $a \in F$ there is a $b \in D$ such that $b \leqslant a$. The *π-character* of B itself, denoted by $\pi\chi B$, is the supremum of $\pi\chi F$ for F an ultrafilter on B. The *tightness* of B is the supremum of the cardinals κ such that B has a free sequence of length κ, where a sequence $\langle b_\alpha\colon \alpha < \kappa\rangle$ is a free sequence provided that if Γ and Δ are finite subsets of κ such that $\alpha < \beta$ for all $\alpha \in \Gamma$ and $\beta \in \Delta$, then

$$\prod_{\alpha\in\Gamma} -b_\alpha \cdot \prod_{\beta\in\Delta} b_\beta \neq 0.$$

* Corresponding author. E-mail: adow@yorku.ca.

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The following relations hold between these cardinal functions in general:

$$\text{depth}(B) \leqslant \text{tightness}(B) \quad \text{and} \quad \pi\chi B \leqslant \text{tightness}(B);$$

the gaps in the inequalities can be arbitrarily large, and there is in general no inequality between depth and π-character. Moreover, tightness(B) is the supremum of depth(A) for A a homomorphic image of B and is also the supremum of $\pi\chi A$ for A a homomorphic image of B.

As most readers will be aware, all results about superatomic Boolean algebras are dual to results about compact scattered spaces. The cardinal invariants of tightness and π-character are well-known topologically and the Boolean algebraic versions correspond exactly to the usual topological notions. The depth of a Boolean algebra B obviously is equal to the supremum of those cardinals κ such that the Stone space of B maps continuously onto the ordinal space $\kappa + 1$. We are not aware of a naming convention for this topological cardinal invariant, nor do we propose to introduce one. If $\{x_\alpha\colon \alpha < \kappa\}$ is a free sequence in a compact space X, then the depth of the closure of this free sequence is κ. For the sake of consistency it is best to choose to work either completely algebraically or topologically. Clearly some proofs may benefit from one approach or the other but on balance the results in this paper are best worked algebraically.

In a version of Monk [5], the following two problems were stated.

Problem 1. *Is there a superatomic BA B such that* depth$(B) < \pi\chi(B)$?

No example, under any set-theoretic assumptions, was known; Theorems A and B answer this question fairly completely.

Problem 2. *Can the difference between* depth(B) *and* tightness(B) *be arbitrarily large*?

Theorem C answers this question, but there remains the question of how large the gap can be. In this connection recall that there is a system $\langle b_\alpha\colon \alpha < \omega_1\rangle$ of infinite subsets of ω such that $b_\alpha \setminus b_\beta$ is finite and $b_\beta \setminus b_\alpha$ infinite whenever $\alpha < \beta < \omega_1$. Letting B be the algebra of subsets of ω generated by the b_α's and the singletons, we have a superatomic BA with tightness ω_1 and depth ω. Also, Hechler [1] generalized this by showing that under Martin's axiom there is a system $\langle b_\alpha\colon \alpha < 2^\omega\rangle$ of infinite subsets of ω such that $b_\alpha \setminus b_\beta$ is finite and $b_\beta \setminus b_\alpha$ infinite whenever $\alpha < \beta < 2^\omega$. This gives a superatomic BA B with countable depth and tightness 2^ω. These results form a background for Theorem C.

Notation. We use standard set-theoretic notation, and for BAs we follow the notation of [3]. We now set up some notation for superatomic BAs. For any BA A we define the standard sequence of ideals I_α^A on A as follows:

$$I_0^A = \{0\}, \qquad I_{\alpha+1}^A = \langle\{x : /I_\alpha^A \text{ is an atom}\}\rangle^{\text{Id}}, \qquad I_\lambda^A = \bigcup_{\alpha<\lambda} I_\alpha^A \text{ for } \lambda \text{ limit}.$$

We usually omit the superscript ^{A}. We let π_α^A denote the natural homomorphism from A onto A/I_α.

Recall [3, 17.8] that A is superatomic iff $I_\alpha = A$ for some α. It is easy to see that A is superatomic iff A/I_α is finite for some α, and that if α is minimum such that A/I_α is finite, then $|A/I_\alpha| > 1$ (provided, of course, that $|A| > 1$ to start with). This least α is denoted by λ_A; it is called the *first invariant* of A. Let $I_A = I_{\lambda_A}$. We also let λ_A^2 be the number of atoms of A/I_A; this is the *second invariant* of A. Usually we will arrange things such that this second invariant is 1, so that I_A is a maximal ideal. For any $a \in A$ we let $\rho_A a$ be the least α such that $a \in I_{\alpha+1}$. Thus $\rho_A 1 = \lambda_A$, if A is nontrivial. Let $A' = \{a \in A\colon a/I_{\rho a} \text{ is an atom}\}$. Note that if $a \in A'$, then the set $F_a \stackrel{\text{def}}{=} \{x \in A\colon \rho(a \cdot x) = \rho a\}$ is an ultrafilter. Conversely, if F is an ultrafilter, then $F \cap A' \neq 0$, and if we choose a member $a \in F \cap A'$ of smallest rank, then $F = \{x \in A\colon \rho(a \cdot x) = \rho a\}$.

For any BA A, we let $\mathrm{At}A$ denote the collection of all atoms of A.

1. Preliminaries

We now give some elementary facts about superatomic BAs, most of which are needed later. For some of these results see [2, pp. 363ff].

Lemma 1.1. *Suppose that A and B are superatomic and A is a subalgebra of B. Then $A \cap I_\alpha^B \subseteq I_\alpha^A$ for any α.*

Proof. We proceed by induction on α. The cases $\alpha = 0$ and α a limit ordinal are easy. For the successor case we note

($*$) If $a_0/I_\alpha^A, \ldots, a_{m-1}/I_\alpha^A$ are atoms, then there exist $a'_0, \ldots, a'_{m-1} \in A$ such that $a_i/I_\alpha^A = a'_i/I_\alpha^A$ for all $i < m$ and $a'_0/I_\alpha^B, \ldots, a'_{m-1}/I_\alpha^B$ are nonzero pairwise disjoint elements. In fact, simply choose $a'_0, \ldots, a'_{m-1}$ to be disjoint elements of A such that $a_i/I_\alpha^A = a'_i/I_\alpha^A$ for all $i < m$; the desired conclusion is clear by the induction hypothesis.

Now suppose that $a \in A \cap I_{\alpha+1}^B$. Then a/I_α^B is the sum of a finite number, say m, of atoms of B/I_α^B. Now if $x \in A$ and $x/I_\alpha^A \leqslant a/I_\alpha^A$, then by the induction hypothesis, $x/I_\alpha^B \leqslant a/I_\alpha^B$. Hence by ($*$) it follows that a/I_α^A is the sum of at most m atoms, completing the inductive proof. □

Corollary 1.2. *If A and B are superatomic and A is a subalgebra of B, then $\lambda_A \leqslant \lambda_B$. If $\lambda_A = \lambda_B$, then $\lambda^2 A \leqslant \lambda^2 B$.*

Proof. The first part is direct from Lemma 1.1. For the second part, use ($*$) in the proof of Lemma 1.1. □

We leave the proof of the following simple but useful lemma to the reader:

Lemma 1.3. *If A and B are BA's and f is a homomorphism from A onto B, then any atom of A is either mapped to 0 or to an atom under f.*

Lemma 1.4. *Suppose that B is a superatomic BA and J is an ideal in B. Then for any α and any $a \in I_\alpha^B$ we have $(a/J) \in I_\alpha^{B/J}$. Also, there is a homomorphism g from B/I_α^B onto $(B/J)/I_\alpha^{B/J}$ such that $g(a/I_\alpha^B) = (a/J)/I_\alpha^{B/J}$ for any $a \in B$.*

Proof. The second assertion follows from the first for any α. We prove the first assertion by induction on α. Again, the cases $\alpha = 0$ and α limit are clear. Suppose that $a \in I_{\alpha+1}^B$. Say that $b_0/I_\alpha^B, \ldots, b_{m-1}^B/I_\alpha^B$ are atoms and $a/I_\alpha^B = b_0/I_\alpha^B + \cdots + b_{m-1}/I_\alpha^B$. Then $a\triangle(b_0 + \cdots + b_{m-1}) \in I_\alpha^B$, and so by the inductive hypothesis,

$$\big((a/J)\triangle(b_0/J + \cdots + b_{m-1}/J)\big) \in I_\alpha^{B/J}.$$

Also, Lemma 1.3 says that each $(b_i/J)/I_\alpha^{B/J}$ is either 0 or an atom. So $(a/J) \in I_{\alpha+1}^{B/J}$, as desired. □

Corollary 1.5. *Suppose that B is a superatomic BA and J is an ideal in B. Then $\lambda_{B/J} \leqslant \lambda_B$, and if $\lambda_{B/J} = \lambda_B$, then $\lambda_{B/J}^2 \leqslant \lambda_B^2$.* □

The following lemma is well known, and can be easily proved by induction on α:

Lemma 1.6. *Let A be any BA, $a \in A$, and let α be any ordinal. Then the following conditions hold:*

(i) $I_\alpha^{A\restriction a} = I_\alpha^A \cap (A \restriction a)$;

(ii) $(\pi_\alpha^{A\restriction a})^{-1}[\mathrm{At}((A \restriction a)/I_\alpha)] = (\pi_\alpha^A)^{-1}[\mathrm{At}(A/I_\alpha)] \cap (A \restriction a)$.

(iii) *There is an isomorphism g from A/I_α onto $(A \restriction a)/I_\alpha \times (A \restriction -a)/I_\alpha$ such that for any $x \in A$,*

$$g(\pi_\alpha^A x) = \big(\pi_\alpha^{A\restriction a}(x \cdot a), \pi_\alpha^{A\restriction -a}(x \cdot -a)\big).$$

Note that from this lemma it follows that $\rho_A a = \lambda_{A\restriction a}$.

Corollary 1.7. *Let A and B be superatomic BAs.*

(i) *If $\lambda_A < \lambda_B$, then $\lambda_{A\times B} = \lambda_B$ and $\lambda_{A\times B}^2 = \lambda_B^2$.*

(ii) *If $\lambda_A = \lambda_B$, then $\lambda_{A\times B} = \lambda_A$ and $\lambda_{A\times B}^2 = \lambda_A^2 + \lambda_B^2$.*

Corollary 1.8. *If $a \leqslant b$, then $\rho_A a \leqslant \rho_A b$.*

It is also necessary to discuss the situation with weak products. Here we give a more complete proof, and we do things in somewhat more generality than is needed below.

Lemma 1.9. *Let $B = \prod_{i\in I}^{\mathrm{w}} A_i$. Choose α minimum such that $\{i \in I\colon I_\alpha^{A_i} \neq A_i\}$ is finite. Assume that $\beta < \alpha$. Then*

(i) $I_\beta^B = \{b \in B\colon \forall i \in I(b_i \in I_\beta^{A_i})\}$.

(ii) *If $b \in I_\beta^B$, then $\{i \in I\colon b_i \neq 0\}$ is finite.*

(iii) $B/I_\beta^B \cong \prod_{i\in I}^{\mathrm{w}} A_i/I_\beta^{A_i}$ *via*

$$b/I_\beta^B \mapsto \langle b_i/I_\beta^{A_i}\colon i \in I\rangle.$$

Proof. We prove all three statements simultaneously by induction on β. They are all obvious if $\beta = 0$. Suppose now that they hold for β; we prove them for $\beta+1$, where $\beta+1 < \alpha$. First we show (i) for $\beta+1$. Suppose that $b \in I^B_{\beta+1}$. Say

$$b/I^B_\beta \leqslant c^0/I^B_\beta + \cdots + c^{m-1}/I^B_\beta$$

with each c^k/I^B_β an atom. Fix $k < m$. By (iii) for β, $\langle c^k_i/I^{A_i}_\beta\colon i \in I\rangle$ is an atom of $\prod^{\mathrm{w}}_{i\in I} A_i/I^{A_i}_\beta$. Hence there is an $i(k) \in I$ such that $c^k_{i(k)}/I^{A_{i(k)}}_\beta$ is an atom of $A_{i(k)}/I^{A_{i(k)}}_\beta$, while $c^k_j/I^{A_j}_\beta = 0$ for all $j \neq i(k)$. Hence $c^k_j \in I^{A_j}_{\beta+1}$ for all $j \in I$. Now $b \cdot -c^0 \cdot \cdots \cdot -c^{m-1} \in I^B_\beta$, and so by the induction hypothesis

$$\left(b \cdot -c^0 \cdot \cdots \cdot -c^{m-1}\right)_j \in I^{A_j}_\beta \quad \text{for all } j \in I.$$

It follows easily that $b_j \in I^{A_j}_{\beta+1}$ for all $j \in I$.

Conversely, suppose that $b_i \in I^{A_i}_{\beta+1}$ for all $i \in I$. Now if $b_i = 1$, then $I^{A_i}_{\beta+1} = A_i$. Hence, since $\beta+1 < \alpha$, we have that $F \stackrel{\text{def}}{=} \{i \in I\colon b_i \neq 0\}$ is finite. For each $i \in F$ write $b_i/I^{A_i}_\beta = \sum_{c\in G_i} c/I^{A_i}_\beta$, each $c/I^{A_i}_\beta$ an atom, although perhaps $G_i = 0$. Fix $c \in G_i$. Let $d_c i = c$, $d_c j = 0$ for $j \neq i$. Then $\langle d_c/I^{A_j}_\beta\colon j \in I\rangle$ is an atom of $\prod^{\mathrm{w}}_{j\in J} A_j/I^{A_j}_\beta$, and so by (iii), d_c/I^B_β is an atom of B_β. Now

$$\begin{aligned} b_i/I^{A_i}_\beta &= \sum_{c\in G_i} c/I^{A_i}_\beta = \sum_{c\in G_i} (d_c)_i/I^{A_i}_\beta = \Bigl(\sum_{c\in G_i} d_c\Bigr)_i/I^{A_i}_\beta \\ &\leqslant \Bigl(\sum_{j\in F}\sum_{c\in G_j} d_c\Bigr)_i/I^{A_i}_\beta. \end{aligned}$$

Hence by (iii),

$$b/I^B_\beta \leqslant \sum_{j\in F}\sum_{c\in G_j} d_c/I^B_\beta,$$

and so $b \in I^B_{\beta+1}$. This proves (i) for $\beta+1$.

To prove (iii), note that the given mapping is well-defined and one-one by (i); it is clearly onto and preserves the operations. Condition (ii) follows from (i).

The case of β limit, but still less than α, is even easier. □

Lemma 1.10. *Let $B = \prod^{\mathrm{w}}_{i\in I} A_i$. Choose α minimum such that $H \stackrel{\text{def}}{=} \{i \in I\colon I^{A_i}_\alpha \neq A_i\}$ is finite. Then*

(i) $I^B_\alpha = \{b \in B\colon \forall i \in I(b_i \in I^{A_i}_\alpha)$ *and* $\{i \in I\colon b_i \neq 0\}$ *is finite*$\}$.

(ii) $B/I_\alpha \cong 2 \times \prod_{i\in H} A_i/I_\alpha$ *via the map* $b/I^B_\alpha \mapsto (u_b, \langle b_i/I^{A_i}_\alpha\colon i \in I\rangle)$, *where*

$$u_b = \begin{cases} 1 & \textit{if } \{i \in I\colon b_i \neq 1\} \textit{ is finite}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. To prove (i), we take two cases.

Case 1: *α is a successor ordinal $\beta+1$.* Here we continue the first part of the proof of Lemma 4, which gives that $b_i \in I_\alpha^{A_i}$ for all $i \in I$. Since $\{i \in I\colon I_\beta^{A_i} \neq A_i\}$ is infinite, $\{j \in I\colon c_j^k \neq 0\}$ is finite for all $k < m$. Also, by Lemma 4(ii) for the element $b \cdot -c^0 \cdot \cdots \cdot -c^{m-1}$ of I_β^B, the set $\{i \in I\colon (b \cdot -c^0 \cdot \cdots \cdot -c^{m-1})_i \neq 0\}$ is finite. It follows that $\{i \in I\colon b_i \neq 0\}$ is also finite. The second part of the proof of Lemma 4 gives the converse inclusion.

Case 2: *α is a limit ordinal.* If $b \in I_\alpha^B$, then $b \in I_\beta^B$ for some $\beta < \alpha$, and so by Lemma 4(i),(ii), b satisfies the desired condition. The converse is similar. This proves (i).

For (ii), first we check that the given mapping is well-defined. Suppose that $b/I_\alpha^B = d/I_\alpha^B$. Thus $b \triangle d \in I_\alpha^B$, so by (i), all entries on the right side are the same. Condition (i) also yields that the mapping is one-one. For ontoness, suppose that $(\varepsilon, \langle d_i/I_\alpha^{A_i}\colon i \in H\rangle)$ is given. Let $b_i = d_i$ for all $i \in H$, and $b_i = \varepsilon$ for all $i \notin H$; this is the required preimage. $\square$

Lemma 1.11. *Suppose that A is a subalgebra of B, both superatomic, and for all $a \in I_A$ and all ordinals β, a/I_β^A is an atom iff a/I_β^B is an atom. Then*

(i) *For all $a \in I_A$ and all ordinals β, $a \in I_\beta^A$ iff $a \in I_\beta^B$.*

(ii) *$\rho_A a = \rho_B a$ for all $a \in I_A$.*

(iii) *$\lambda_A \leqslant \lambda_B$.*

Proof. We prove (i) by induction on β. The case $\beta = 0$ is trivial. Assume that it is true for β. Suppose that $a \in I_A$. First suppose that $a \in I_{\beta+1}^A$. We may assume that $a \notin I_\beta^A$, and hence $\rho_A a = \beta < \lambda_A$. Say

$$a/I_\beta^A \leqslant c_0/I_\beta^A + \cdots + c_{m-1}/I_\beta^A,$$

with each c_k/I_β^A an atom. Now $\rho_A c_k = \beta$, so $c_k \in I_A$. Hence by assumption, c_k/I_β^B is an atom. Also, $a \cdot -c_0 \cdot \cdots \cdot c_{m-1} \in I_\beta^A$, so by the inductive hypothesis, $a \cdot -c_0 \cdot \cdots \cdot c_{m-1} \in I_\beta^B$. Hence $a \in I_{\beta+1}^B$.

Conversely, suppose that $a \notin I_{\beta+1}^A$. Thus $\beta + 1 \leqslant \rho_A a$. Let $\langle c_k/I_\beta^A\colon k < \omega\rangle$ be a system of distinct atoms $\leqslant a/I_\beta^A$. Since $\beta < \lambda_A$, each c_k is in I_A. So by assumption, c_k/I_β^B is an atom for each $k < \omega$. If $k, l < \omega$ and $k \neq l$, then $c_k \triangle c_l \in I_\beta^A$, and hence by the inductive hypothesis $c_k \triangle c_l \in I_\beta^B$. For each $k < \omega$, $c_k/I_\beta^A \leqslant a/I_\beta^A$, and hence $c_k \cdot -a \in I_\beta^A$; the inductive hypothesis implies that $c_k \cdot -a \in I_\beta^B$. All of this shows that $a \notin I_{\beta+1}^B$.

The case of limit β is easy, so (i) has been proved.

(ii) follows easily: Let $a \in I_A$, say $\gamma = \rho_A a$. Thus $a \in I_{\gamma+1}^A \setminus I_\gamma^A$. So by (i), $a \in I_{\gamma+1}^B \setminus I_\gamma^B$. Thus $\rho_A a = \rho_B a$.

For (iii), suppose that $\lambda_B < \lambda_A$. Let $\langle c_k/I_{\lambda_B}^A\colon k < \omega\rangle$ be a system of distinct atoms. Then by assumption, each $c_k/I_{\lambda_B}^B$ is an atom. For distinct $k, l < \omega$ we have $c_k \triangle c_l \in I_{\lambda_B}^A$, and so by (i), also $c_k \triangle c_l \in I_{\lambda_B}^B$. This is impossible. $\square$

Lemma 1.12. *Suppose that $\langle B_\alpha\colon \alpha < \lambda\rangle$ is a strictly increasing continuous sequence of infinite superatomic BAs, λ a limit ordinal. Assume that $\lambda_{B_\alpha}^2 = 1$ for all $\alpha < \lambda$, and:*

(*) *For all* β, *all* $\gamma < \delta < \lambda$, *and all* $a \in I_{B_\gamma}$, ($a/I_\beta^{B_\gamma}$ *is an atom iff* $a/I_\beta^{B_\delta}$ *is an atom*). *Then the following conditions hold, where* $C = \bigcup_{\gamma<\lambda} B_\gamma$:

(i) *For each* $\beta < \sup_{\gamma<\lambda} \lambda_{B_\gamma}$ *we have* $I_\beta^C = \bigcup\{I_\beta^{B_\gamma}\colon \beta < \lambda_{B_\gamma}\}$.

(ii) *For all* β, *all* $\gamma < \lambda$, *and all* $a \in I_{B_\gamma}$, $a/I_\beta^{B_\gamma}$ *is an atom iff* a/I_β^C *is an atom.*

(iii) $\lambda_C = \sup_{\gamma<\lambda} \lambda_{B_\gamma}$.

(iv) $I_C = \bigcup_{\gamma<\lambda} I_{B_\gamma}$.

(v) C *is superatomic.*

(vi) $\lambda_C^2 = 1$.

Proof. We prove (i) and (ii) simultaneously by induction on β. First suppose that $\beta = 0$. Then (i) is obvious. For (ii), suppose that $a \in I_{B_\gamma}$, and first suppose that a is an atom of B_γ. Then $a \neq 0$. Suppose that $0 < b < a$ in C. Say $b \in B_\delta$, where $\gamma < \delta < \lambda$. But by (*), a is an atom of B_δ, contradiction. Conversely, if a is an atom of C, it is obviously an atom of B_γ.

Now we assume (i) and (ii) for β and prove them for $\beta + 1$. First we take (i). Suppose that $\beta + 1 < \sup_{\gamma<\lambda} \lambda_{B_\gamma}$. Suppose that $a \in I_{\beta+1}^C$. Say

$$a/I_\beta^C \leqslant b_0/I_\beta^C + \cdots + b_{m-1}/I_\beta^C,$$

where $b_0/I_\beta^C, \ldots, b_{m-1}/I_\beta^C$ are atoms. Say $a, b_0, \ldots, b_{m-1} \in B_\gamma$, $\gamma < \lambda$, and $\beta + 1 < \lambda_{B_\gamma}$. Then by (ii) for β, each $b_i/I_\beta^{B_\gamma}$ is an atom. Now $a \cdot -b_0 \cdot \cdots \cdot -b_{m-1} \in I_\beta^C$, so by (i) for β, (*), and Lemma 1.11, we may assume that $a \cdot -b_0 \cdot \cdots \cdot -b_{m-1} \in I_\beta^{B_\gamma}$. This shows that $a \in I_{\beta+1}^{B_\gamma}$. The converse part of (i) is proved similarly.

For (ii), suppose that $a \in I_{B_\gamma}$; and first suppose that $a/I_{\beta+1}^{B_\gamma}$ is an atom. Thus $\rho_{B_\gamma} a = \beta + 1 < \lambda_{B_\gamma}$. If $a \in I_{\beta+1}^C$, then (i) for $\beta + 1$ plus (*) gives a contradiction. So $a \notin I_{\beta+1}^C$. Suppose that $b/I_{\beta+1}^C \leqslant a/I_{\beta+1}^C$. Thus by (i) for $\beta + 1$ we have $b \cdot -a \in I_{\beta+1}^{B_\delta}$ for some δ, and we may assume that $\gamma \leqslant \delta$ and $a, b \in B_\delta$. By (*), $a/I_{\beta+1}^{B_\delta}$ is an atom, so we have two cases: (1) $b \in I_{\beta+1}^{B_\delta}$; then by (i) for $\beta + 1$, $b \in I_{\beta+1}^C$; (2) $a \cdot -b \in I_{\beta+1}^{B_\delta}$; then $a \cdot -b \in I_{\beta+1}^C$. So, $a/I_{\beta+1}^C$ is an atom. For the converse, suppose that $a/I_{\beta+1}^C$ is an atom. If $a \in I_{\beta+1}^{B_\gamma}$, then $a \in I_{\beta+1}^C$ by (i) for $\beta + 1$, contradiction. Suppose that $b/I_{\beta+1}^{B_\gamma} \leqslant a/I_{\beta+1}^{B_\gamma}$. Thus $b \cdot -a \in I_{\beta+1}^{B_\gamma}$, so $b \cdot -a \in I_{\beta+1}^C$. If $b \in I_{\beta+1}^C$, then $b \in I_{\beta+1}^{B_\delta}$ for some δ. Then $b \in I_{\beta+1}^{B_\gamma}$ by Lemma 1.11. The rest of the proof goes similarly.

The case of limit β is treated similarly. So (i) and (ii) hold.

Next, let $\alpha = \sup_{\gamma<\lambda} \lambda_{B_\gamma}$. We show that $I_{B_\gamma} \subseteq I_\alpha^C$ for all $\gamma < \lambda$. Let $a \in I_{B_\gamma}$. Say $\rho_{B_\gamma} a = \beta < \lambda_{B_\gamma}$. Write $a/I_\gamma^{B_\gamma} = c_0/I_\beta^{B_\gamma} + \cdots + c_{m-1}/I_\beta^{B_\gamma}$ with each $c_k/I_\beta^{B_\gamma}$ an atom. Then (ii), each c_k/I_β^C is an atom. Moreover, $a \triangle (c_0 + \cdots + c_{m-1}) \in I_\beta^{B_\gamma}$, so by (i), this element is in I_β^C too. This shows that $a \in I_{\beta+1}^C$; hence $a \in I_\alpha^C$, as desired.

Now to prove (iii), note that $\lambda_{B_\gamma} \leqslant \lambda_C$ for all $\gamma < \lambda$, by (ii) and Lemma 1.6. Thus $\alpha \leqslant \lambda_C$. Suppose that $\alpha < \lambda_C$. Let a/I_α^C be an atom. Say $a \in B_\gamma$. By the preceding paragraph we have $a \notin I_{B\gamma}$. So $-a \in I_{\beta_\gamma}$, and hence by the previous paragraph again, $\rho_C(-a) < \alpha$. It follows then from the product lemma that $\rho_C 1 = \alpha$, contradiction. So (iii) holds.

For (iv), we have already shown $\supseteq$. Now suppose that $a \in I_C$. Thus $\rho_C a < \lambda_C$, so we can choose $\gamma < \lambda$ such that $a \in B_\gamma$ and $\rho_C a < \lambda_{B_\gamma}$. If $-a \in I_{B_\gamma}$, then by (ii) and Lemma 1.11 we would get $\rho_C(-a) = \rho_{B_\gamma}(-a) < \lambda_{B_\gamma}$, hence $\rho_C 1 < \lambda_{B_\gamma}$ by the product lemma, contradiction. Thus $a \in I_{B_\gamma}$. This proves (iv).

By (iv), I_C is a maximal ideal. So (v) and (vi) follow. □

Lemma 1.13. *Suppose that A and B are BAs, and I and J are maximal ideals of A and B respectively. Suppose that $f : I \to J$, and for any $a, b \in I$, $f(a \cdot b) = fa \cdot fb$, $f(a + b) = fa + fb$, and $f(a \cdot -b) = fa \cdot -fb$. Furthermore, suppose that $fa = 0$ only if $a = 0$. Then f can be extended to an isomorphism from A into B.*

Proof. Define, for any $a \in A$,

$$f^+a = \begin{cases} fa & \text{if } a \in I, \\ -f(-a) & \text{if } a \notin I. \end{cases}$$

We check that f preserves $\cdot$: suppose that $a, b \in A$.

Case 1: $a, b \in I$. Then $f^+(a \cdot b) = f(a \cdot b) = fa \cdot fb = f^+a \cdot f^+b$.

Case 2: $a \in I$, $b \notin I$. Then

$$\begin{aligned} f^+(a \cdot b) &= f(a \cdot b) = f(a \cdot -(-b)) \\ &= fa \cdot -f(-b) = f^+a \cdot f^+b. \end{aligned}$$

Case 3: $a \notin I$, $b \in I$. Symmetric to Case 2.

Case 4: $a, b \notin I$. Then also $a \cdot b \notin I$. So

$$\begin{aligned} f^+(a \cdot b) &= -f\big(-(a \cdot b)\big) = -f((-a) + (-b)) \\ &= -\big(f(-a) + f(-b)\big) = -f(-a) \cdot -f(-b) = f^+a \cdot f^+b. \end{aligned}$$

Next, if $a \in I$, then $f^+(-a) = -f(-(-a)) = -fa = -f^+a$; and if $a \notin I$, then $f^+(-a) = f(-a) = -(-f(-a)) = -f^+a$. So f^+ is a homomorphism from A into B.

Suppose that $f^+a = 0$. If $a \in I$, then $f^+a = fa$ and hence $a = 0$ by hypothesis. If $a \notin I$, then $f^+a = -f(-a)$, hence $f(-a) = 1$, contradiction. □

The following result is not needed in what follows, but it may help the intuition on these problems.

Proposition 1.14. *If B is a superatomic BA, then* tightness$(B) \leqslant \lambda_B$.

Proof. Since λ does not go up in homomorphic images (Corollary 1.5), it suffices to show that depth$(B) \leqslant \lambda_B$. But then since λ does not go up when passing to a subalgebra (Corollary 1.2), it suffices to note that the interval algebra A on a cardinal κ is such that $\lambda_B = \kappa$. □

We conclude this section with examples, given in the following proposition.

Proposition 1.15. *For each infinite cardinal κ there is a superatomic BA B such that $\lambda_B = |B| = \kappa$ and* tightness$(B) = \omega$.

Proof. Recall that the tightness of a weak product is the supremum of the tightnesses of all the factors (see [4]). So the following simple construction is what is desired:

$A_0 =$ finite-cofinite algebra on ω;

$A_{\alpha+1} =$ weak product of ω copies of A_α;

$A_\lambda =$ weak product of all A_α, $\alpha < \lambda$, for λ limit. $\square$

2. Depth and π-character

Lemma 2.1. *Suppose that B is a countable superatomic BA with λ_B infinite and $\lambda_B^2 = 1$. Assume that $0 \neq A \subseteq B$ and*

$$(\forall b \in I_B)(\forall n \in \omega)\ \big[\{a \in A\colon\ \rho(a \cdot -b) < n\} \textit{ is finite}\big].$$

Then B is a subalgebra of a countable superatomic BA C with the following properties:

(i) *If $b \in I_B$ then b/I_ξ^B is an atom iff b/I_ξ^C is an atom, for all ξ.*

(ii) *There is a $c \in C$ such that $\rho(b \cdot -c) < \rho_B b$ for all $b \in I_B$.*

(iii) *For all $b \in I_C$ and all $n \in \omega$, the set $\{a \in A\colon\ \rho_C(a \cdot -b) < n\}$ is finite.*

(iv) *The function $b \mapsto b \cdot c$ is an isomorphism from B onto $C \restriction c$.*

(v) *$\lambda_C = \lambda_B + 1$ and $\lambda_C^2 = 1$.*

Proof. Let $\{b_n\colon\ 0 < n < \omega\}$ enumerate I_B, and $\{a_n\colon\ n < \omega\}$ enumerate A. Suppose we have defined $b'_k \leqslant b_k$ for each $k < n$. Now $\sum_{k \leqslant n} b_k \in I_B$, so by the hypothesis, the set

$$A_n \stackrel{\text{def}}{=} \left\{a \in A\colon\ \rho\Big(a \cdot -\sum_{k \leqslant n} b_k\Big) < n\right\} \cup \{a_0, \dots, a_n\}$$

is finite; say $A_n = \{c_0, \dots, c_m\}$. For each $k \leqslant m$, if $\rho(c_k \cdot b_n \cdot -\sum_{l<n} b_l) \geqslant \omega$, choose $b_n^k \leqslant c_k \cdot b_n \cdot -\sum_{l<n} b_l$ so that $\omega > \rho b_n^k \geqslant n$; otherwise simply let $b_n^k = c_k \cdot b_n \cdot -\sum_{l<n} b_l$. Finally, let $b'_n = \sum_{k \leqslant m} b_n^k$. Thus the following conditions hold:

(1) $b'_n \leqslant b_n \cdot -\sum_{k<n} b_k$.

(2) $\rho b'_n < \omega$.

(3) For each $a \in A_n$, $\rho(a \cdot b'_n) \geqslant \min\{n, \rho(a \cdot b_n \cdot -\sum_{k<n} b_k)\}$.

Let $B_0 = B$ and, for $n > 0$ let $B_n = B/I_1$. Let $C = \prod_{n<\omega}^{\mathrm{w}} B_n$. Let $c = \langle 1, 0, 0, \dots\rangle$. Thus C is a countable superatomic BA. Note that $\lambda_{B_n} = \lambda_B$ and $\lambda_{B_n}^2 = \lambda_B^2 = 1$ for all $n \in \omega$. Hence from Lemmas 1.9 and 1.10 it follows that

(4) $\lambda_C = \lambda_B + 1$, $\lambda_C^2 = 1$, and $I_C = \{x \in C\colon\ \{n < \omega\colon\ x_n \neq 0\}$ is finite$\}$.

In particular, (v) of the lemma holds. Now for any $b \in I_{\lambda_B}$ define $fb \in \prod_{n<\omega} B_n$ by setting, for any $n \in \omega$,

$$(fb)_n = \begin{cases} b & \text{if } n = 0, \\ (b \cdot b'_n)/I_1 & \text{if } n > 0. \end{cases} \tag{1}$$

Now f maps into I_C, since for any $b \in I_{\lambda_B}$, if $b = b_m$, then $(fb)_n = 0$ if $n > m$. Clearly f satisfies the conditions of Lemma 1.13. So f extends to an isomorphism from B into C as in the proof of that lemma. We want to show that this embedding satisfies the conditions (i)–(iv) of our lemma. Clearly (iv) holds (in the form that $b \mapsto fb \cdot c$ defines an isomorphism from B onto $C \restriction c$).

We now prove three conditions (5)–(7) for any ordinal ξ. The condition (i) follows from (5).

(5) b/I^B_ξ is an atom iff fb/I^C_ξ is an atom, for any $b \in I_B$.

(6) $b \in I^B_\xi$ iff $fb \in I^C_\xi$, for any $b \in I_B$.

(7) The mapping $b/I^B_\xi \mapsto fb/I^C_\xi$ is a well-defined isomorphism from B/I^B_ξ into C/I^C_ξ.

We prove these statments by induction on ξ. First suppose that $\xi = 0$. Then (6) and (7) are trivial, as is the direction $\Leftarrow$ in (5). For $\Rightarrow$ in (5), note that if b is an atom, then $(b \cdot b'_n)/I_1 = 0$ for all $n > 0$, so that fb is an atom of C.

Now assume the conditions for ξ; we prove them for $\xi + 1$. First we take (6). Let $b \in I_B$. Suppose that $b \in I^B_{\xi+1}$. Say

$$b/I^B_\xi \leqslant a_0/I^B_\xi + \cdots + a_{m-1}/I^B_\xi,$$

where each a_k/I^B_ξ is an atom. Then $b \cdot -a_0 \cdot \cdots \cdot -a_{m-1} \in I^B_\xi$. So by the inductive hypothesis,

$$f(b \cdot -a_0 \cdot \cdots \cdot -a_{m-1}) \in I^C_\xi,$$

and each fa_k/I^C_ξ is an atom. Hence $fb \in I^C_{\xi+1}$. Conversely, if $b \notin I^B_{\xi+1}$, then there are infinitely many atoms $\leqslant b/I^B_\xi$, and so by (7) for ξ, there are infinitely many atoms $\leqslant fb/I^C_\xi$. So (6) holds for $\xi + 1$.

Condition (7) for $\xi+1$ follows easily from (6), using Lemma 1.13 again. The direction $\Leftarrow$ of (5) then follows from (7). For the direction $\Rightarrow$ of (5), suppose that $b/I^B_{\xi+1}$ is an atom. By the above lemmas, we need to see that $((b \cdot b'_n)/I_1)/I^{B_n}_{\xi+1} = 0$ for all $n > 0$. For $\xi < \omega$ this is true since $B_n/I^{B_n}_{\xi+1}$ is naturally isomorphic to $B/I^B_{\xi+2}$ via $(d/I_1)/I_{\xi+1} \mapsto d/I_{\xi+2}$. For $\xi \geqslant \omega$ it is true since each $b'_n \in I_\xi$.

For ξ limit, the arguments are similar but simpler. So (5)–(7) hold.

Next we look at condition (ii). If $b \in I_B$, then

$$fb \cdot -c = \langle 0, b \cdot b'_1/I_1, b \cdot b'_2/I_1, \ldots \rangle.$$

Now for each positive integer i, $b \cdot b'_i$ has finite rank (since b'_i does), and its rank is at most that of b. Hence $(b \cdot b'_i)/I_1$ has rank strictly less than that of b. Also recall from the above that fb is 0 except for finitely many places. So (ii) follows from the lemma on rank in products.

Thus it remains only to take care of (iii). Fix $x \in I_C$ and $n < \omega$. Without loss of generality, $x_m = 0$ for all $m > n$. Now $\sum_{k\leqslant n} b_k \in I_B$, so by the hypothesis of the lemma, there is an $M > n$ such that $\rho(a_m \cdot -\sum_{k\leqslant n} b_k) > n$ for all $m \geqslant M$. Now take any $m > M$. We claim that $\rho(fa_m \cdot -x) \geqslant n$. Now

$$fa_m \cdot -x \geqslant \langle 0, 0, \ldots, 0, a_m \cdot b'_{n+1}/I_1, a_m \cdot b'_{n+2}/I_1, \ldots \rangle.$$

Hence it suffices to show

(8) There is a $k > n$ such that $\rho(a_m \cdot b'_k) \geqslant n$.

Assume that there is no $k \in (n, m)$ such that $\rho(a_m \cdot b'_k) \geqslant n$. Then

(9) $\forall k \in [n, m) \left[\rho\left(a_m \cdot -\sum_{l\leqslant k} b_l\right) > n\right]$.

We prove this by induction on k. It is given for $k = n$. Assume it for k, where $n + 1 \leqslant k + 1 < m$. Suppose that $\rho(a_m \cdot -\sum_{l\leqslant k+1} b_l) \leqslant n$. So $a_m \in A_{k+1}$. Hence

$$\rho\left(a_m \cdot b_{k+1} \cdot -\sum_{l<k+1} b_l\right) \leqslant \rho(a_m \cdot b'_{k+1}) < n < \rho\left(a_m \cdot -\sum_{l<k+1} b_l\right),$$

and hence

$$\rho\left(a_m \cdot -\sum_{l\leqslant k+1} b_l\right) = \rho\left(a_m \cdot -\sum_{l<k+1} b_l\right) > n,$$

contradiction. So, (9) holds.

In particular, $\rho\left(a_m \cdot -\sum_{l<m} b_l\right) > n$. Let k be minimum such that

$$\rho\left(a_m \cdot -\sum_{l<m} b_l \cdot \sum_{l\leqslant k} b_l\right) > n.$$

Obviously $k \geqslant m$. Set $c_l = b_l \cdot -\sum_{s<l} b_s$ for all $l \leqslant k$. Then $\sum_{l<u} b_l = \sum_{l<u} c_l$ for all $u \leqslant k + 1$. By the minimality of k it follows that $\rho\left(a_m \cdot -\sum_{l<m} b_l \cdot c_k\right) > n$. Thus $\rho\left(a_m \cdot -\sum_{l<k} b_l \cdot b_k\right) > n$. Now $a_m \in A_k$, so

$$\rho(a_m \cdot b'_k) \geqslant \min\left\{\rho\left(a_m \cdot -\sum_{l<k} b_l \cdot b_k\right), k\right\} > n,$$

as desired. □

The construction for Theorem A. We construct $\langle B_\alpha\colon\ \alpha \leqslant \omega_1\rangle$ and $\langle b_\alpha\colon\ \alpha \leqslant \omega_1\rangle$ by induction so that the following conditions hold, with $A_\alpha = \{b_\gamma : \gamma < \alpha\} \setminus \{0\}$:

(A_α) B_α is an infinite superatomic BA, and $\lambda^2_{B_\alpha} = 1$; if $\alpha < \omega_1$, then B_α is countable.

(B_α) For all $\beta < \alpha$, all $a \in I_{B_\beta}$, and all γ, $a/I^{B_\beta}_\gamma$ is an atom iff $a/I^{B_\alpha}_\gamma$ is an atom.

(C_α) For all $b \in I_{B_\alpha}$ and all $n \in \omega$, the set $\{a \in A_\alpha\colon\ \rho_{B_\alpha}(a \cdot -b) < n\}$ is finite.

(D_α) Either α is not a successor and $b_\alpha = 0$, or $\alpha = \beta + 1$ for some β, $b_\alpha \in B_\alpha$, and $\rho_{B_\alpha} b_\alpha = \lambda_{B_\beta}$.

(E_α) $\lambda_{B_\alpha} = \omega + \alpha$.

(F_α) If $\alpha = \beta + 1$, then $\rho_{B_\alpha}(b \cdot -b_\alpha) < \rho_{B_\alpha} b$ for all $b \in I_{B_\beta}$.

Let B_0 be a countable superatomic BA with $\lambda_{B_0} = \omega$ and $\lambda^2_{B_0} = 1$, and set $b_0 = 0$. Clearly (A_0)–(F_0) hold.

Now suppose that α is a limit ordinal $\leqslant \omega_1$, and things have been defined for all $\beta < \alpha$ so that (A_β)–(F_β) hold. We define $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and $b_\alpha = 0$. Then (A_α), (B_α), and (E_α) hold by Lemma 1.12. (D_α) and (F_α) are trivial. For (C_α), suppose that $b \in I_{B_\alpha}$ and $n \in \omega$. Say $\rho_{B_\alpha} b = \beta < \lambda_{B_\alpha}$. Thus $b \in I^{B_\alpha}_{\beta+1}$ and $\beta + 1 < \lambda_{B_\alpha}$. So by Lemma 1.12, choose $\gamma < \alpha$ such that $\beta + 1 < \lambda_{B_\gamma}$ and $b \in I^{B_\gamma}_{\beta+1}$. Hence by Lemma 1.11, $\rho_{B_\gamma} b = \rho_{B_\alpha} b = \beta$. Now by (C_γ), $\{a \in A_\gamma\colon \rho_{B_\gamma}(a \cdot -b) < n\}$ is finite. If $a \in A_\gamma$ and $\rho_{B_\alpha}(a \cdot -b) < n$, then $a \cdot -b \in I^{B_\alpha}_n$. Hence $a \cdot -b \in I^{B_\gamma}_n$ by Lemmas 1.11 and 1.12, so $\rho_{B_\alpha}(a \cdot -b) = \rho_{B_\gamma}(a \cdot -b)$. So $\{a \in A_\gamma\colon \rho_{B_\alpha}(a \cdot -b) < n\}$ is finite. Suppose that $\gamma \leqslant \delta < \alpha$ and $b_\delta \neq 0$. By (D_δ), write $\delta = \zeta + 1$ with $\rho_{B_\delta} b_\delta = \lambda_{B_\zeta} = \omega + \zeta$. Now

$$\rho_{B_\alpha}(b \cdot b_\delta) = \rho_{B_\delta}(b \cdot b_\delta) \leqslant \rho_{B_\delta} b = \rho_{B_\gamma} b = \beta < \beta + 1 < \lambda_{B_\gamma} = \omega + \gamma \leqslant \omega + \delta,$$

so $\rho_{B_\alpha}(b \cdot b_\delta) < \omega + \zeta = \rho_{B_\delta} b_\delta = \rho_{B_\alpha} b_\delta$. Hence $\rho(b_\delta \cdot -b) = \rho_{B_\alpha} b_\delta \geqslant \omega$. Thus (C_α) holds.

Now suppose that B_β and b_β have been constructed for all $\beta \leqslant \alpha$ so that (A_β)–(F_β) hold, with $\alpha < \omega_1$. Let $B_{\alpha+1}$ be obtained by Lemma 2.1 from A_α and B_α, with $b_{\alpha+1}$ equal to the "c" there. The conditions (A_α)–(F_α) are all clear.

Lemma 2.2. *Suppose that $b \in B'_\alpha \cap I_{B_\alpha}$ with $\alpha < \beta + 1 < \omega_1$. Then $b/I^{B_{\beta+1}}_{\rho b} \leqslant b_{\beta+1}/I^{B_{\beta+1}}_{\rho b}$.*

Proof. By $(B_{\beta+1})$ and Lemma 1.11 we have $b \in B'_\beta \cap I_{B_\beta}$. So by Lemma 2.1(iv), $(b \cdot b_{\beta+1})/I^{B_{\beta+1}}_{\rho b}$ is an atom. Thus the desired conclusion follows. □

Lemma 2.3. *Suppose that $\beta + 1 \leqslant \alpha \leqslant \omega_1$ and $a \in B_\alpha$. Then there is a $b_a \in B_{\beta+1}$ such that $b_a \leqslant b_{\beta+1}$ and for any $b \in B'_{\beta+1}$ the following conditions are equivalent:*

(i) *$\rho_{B_{\beta+1}}(b \cdot b_{\beta+1}) = \rho_{B_{\beta+1}} b$ and $\rho_{B_\alpha}(b \cdot a) = \rho_{B_\alpha} b$.*

(ii) *$\rho_{B_{\beta+1}}(b \cdot b_a) = \rho_{B_{\beta+1}} b$.*

Proof. We proceed by induction on α. For $\alpha = \beta + 1$, let $b_a = a \cdot b_{\beta+1}$; the desired conclusion is clear. Now assume the statement true for $\alpha \geqslant \beta + 1$, and suppose that $a \in B_{\alpha+1}$. By Lemma 2.1(iv), choose $c \in B_\alpha$ such that $c \cdot b_{\alpha+1} = a \cdot b_{\alpha+1}$. Now we apply the inductive hypothesis to c to obtain an element b_c with the indicated properties. We want to show that b_c works for a too. Suppose that $b \in B'_{\beta+1}$. First note:

(1) If $\rho_{B_{\beta+1}}(b \cdot b_{\beta+1}) = \rho_{B_{\beta+1}} b$, then $b \in I_{B_{\beta+1}}$ and $b/I^{B_{\alpha+1}}_{\rho b} \leqslant b_{\alpha+1}/I^{B_{\alpha+1}}_{\rho b}$.

In fact, $\rho_{B_{\beta+1}} b = \rho_{B_{\beta+1}}(b \cdot b_{\beta+1}) \leqslant \rho_{B_{\beta+1}} b_{\beta+1} < \lambda_{B_{\beta+1}}$, so $b \in I_{B_{\beta+1}}$. By Lemma 2.2, $b/I^{B_{\alpha+1}}_{\rho b} \leqslant b_{\alpha+1}/I^{B_{\alpha+1}}_{\rho b}$. So, (1) holds.

Now suppose that $\rho_{B_{\beta+1}}(b \cdot b_{\beta+1}) = \rho_{B_{\beta+1}} b$ and $\rho_{B_{\alpha+1}}(b \cdot a) = \rho_{B_{\alpha+1}} b$. Then by (1),

$$b/I^{B_{\alpha+1}}_{\rho b} \leqslant (a \cdot b_{\alpha+1})/I^{B_{\alpha+1}}_{\rho b} \leqslant c/I^{B_{\alpha+1}}_{\rho b}.$$

Hence $b \cdot -c \in I_{\rho b}^{B_{\alpha+1}}$, so $b \cdot -c \in I_{\rho b}^{B_\alpha}$ by Lemma 1.11. So $\rho_{B_\alpha}(b \cdot c) = \rho_{B_\alpha} b$. Hence $\rho_{B_{\beta+1}}(b \cdot b_c) = \rho_{B_{\beta+1}} b$ by the choice of c. The converse is similar.

For α limit, suppose that $a \in B_\alpha$. Say $a \in B_\gamma$ with $\beta + 1 \leqslant \gamma < \alpha$. Applying the inductive hypothesis easily gives the desired conclusion. □

Let $B = B_{\omega_1}$.

Lemma 2.4. $\pi\chi(-I_B) = \omega_1$.

Proof. Suppose that $S \subseteq B^+$, S a countable set dense in $-I_B$. Without loss of generality each member of S is an atom. Choose $\lambda < \omega_1$ such that $S \subseteq B_\lambda$. By $(F_{\lambda+1})$ we have $x \leqslant b_{\lambda+1}$ for all $x \in S$. Now $b_{\lambda+1} \in I_B$, so $-b_{\lambda+1} \in -I_B$, and hence there is no $x \in S$ such that $x \leqslant -b_{\lambda+1}$, contradiction. □

The proof of Theorem A is completed by the following lemma.

Lemma 2.5. *There is no uncountable well-ordered chain in B.*

Proof. Suppose that $\langle u_\alpha\colon \alpha < \omega_1\rangle$ is a strictly increasing chain in B. If $\rho u_\alpha = \omega_1$, then $\langle u_\beta \cdot -u_\alpha\colon \alpha \leqslant \beta < \omega_1\rangle$ is strictly increasing and $\rho(u_\beta \cdot -u_\alpha) < \omega_1$ for all $\beta \in [\alpha, \omega_1)$. So without loss of generality we may assume that $\rho u_\alpha < \omega_1$ for all $\alpha < \omega_1$.

(1) There is no $\tau < \omega_1$ such that $\rho u_\alpha < \tau$ for all $\alpha < \omega_1$.

For, otherwise choose τ minimum such that there exists a strictly increasing sequence of elements all of rank less than τ. Then there is a $\beta < \omega_1$ such that $\rho u_\alpha = \rho u_\beta$ for all $\alpha > \beta$. Then $\rho(u_\beta \cdot -u_\alpha) < \rho u_\alpha$ for all $\alpha > \beta$. Thus the sequence $\langle u_\beta \cdot -u_\alpha : \alpha < \beta < \omega_1\rangle$ contradicts the choice of τ.

By (1) we may assume that $\rho u_\alpha < \rho u_\beta$ for $\alpha < \beta < \omega_1$.

Let $\{d_\alpha\colon \alpha < \omega_1\}$ enumerate I_B.

(2) Suppose that C is a subset of ω_1 such that for all $\lambda \in C$ the following conditions hold:

(a) For every $\alpha < \lambda$ there is a $\beta < \lambda$ such that $u_\beta \cdot -d_\alpha \neq 0$.

(b) $I_{B_\alpha} \subseteq \{d_\beta\colon \beta < \lambda\}$ for all $\alpha < \lambda$.

(c) λ is a limit ordinal.

(d) If $\alpha < \lambda$, then $u_\alpha \in B_\lambda$.

Then $\rho(b_{\lambda+1} \cdot u_\lambda) = \rho b_{\lambda+1}$ for each $\lambda \in C$.

To prove this, choose α so that $u_\lambda \in I_{B_\alpha}$ and $\lambda+1 \leqslant \alpha$. By Lemma 2.3 we get $c \in B_{\lambda+1}$ such that $c \leqslant b_{\lambda+1}$ and for any $b \in B'_{\lambda+1}$ the following two conditions are equivalent:

(e) $\rho_{B_{\lambda+1}}(b \cdot b_{\lambda+1}) = \rho_{B_{\lambda+1}} b$ and $\rho_{B_\alpha}(b \cdot u_\lambda) = \rho_{B_\alpha} b$.

(f) $\rho_{B_{\lambda+1}}(b \cdot c) = \rho_{B_{\lambda+1}} b$.

Now we claim

(3) For every $b \in I_{B_\lambda}$ there is an $\alpha < \lambda$ such that $u_\alpha \cdot -b \neq 0$.

For, say $b = d_\beta$, $\beta < \lambda$, by (b). Then by (a) choose $\alpha < \lambda$ such that $u_\alpha \cdot -b \neq 0$.

Write $c = y \cdot b_{\lambda+1}$ with $y \in B_\lambda$. Suppose that $\rho_{B_{\lambda+1}}(b_{\lambda+1} \cdot c) \neq \rho_{B_{\lambda+1}} b_{\lambda+1}$. Since $c \leqslant b_{\lambda+1}$, it follows that $\rho_{B_{\lambda+1}} y = \rho_{B_{\lambda+1}} c < \rho_{B_{\lambda+1}} b_{\lambda+1} = \lambda_{B_\lambda}$, so $y \in I_{B_\lambda}$.

By (3) we get $\alpha < \lambda$ such that $u_\alpha \cdot -y \neq 0$. Now $u_\alpha \cdot -y \in B_\lambda$ by (d), so let b be an atom of B_λ which is $\leqslant u_\alpha \cdot -y$. Then $b \leqslant b_{\lambda+1}$ by Lemma 2.2, and $b \leqslant u_\alpha \leqslant u_\lambda$. So by the choice of c, $\rho_{B_{\lambda+1}}(b \cdot c) = \rho_{B_{\lambda+1}} b$. But $b \cdot c = 0$, contradiction. Hence $\rho_{B_{\lambda+1}}(b_{\lambda+1} \cdot c) = \rho_{B_{\lambda+1}} b_{\lambda+1}$. Let $d \in B'_{\lambda+1}$ be such that $d/I_{\rho b_{\lambda+1}}$ is an atom $\leqslant (b_{\lambda+1} \cdot c)/I_{\rho b_{\lambda+1}}$. Thus $\rho_{B_{\lambda+1}}(d \cdot c) = \rho_{B_{\lambda+1}} d$. By the equivalence of (e) and (f) we then get $\rho_{B_\alpha} b_{\lambda+1} = \rho_{B_\alpha} d = \rho_{B_\alpha}(d \cdot u_\lambda) \leqslant \rho_{B_\alpha} u_\lambda$. So (2) holds.

Now let $J = \langle \{u_\alpha \colon \alpha < \omega_1\} \rangle^{\mathrm{Id}}$.

(4) There is a $b \in I_B$ such that $\{\alpha < \omega_1 \colon b_\alpha \cdot -b \notin J\}$ is countable.

To prove (4), suppose that there is no such b. We define an increasing sequence $\langle \lambda_\alpha \colon \alpha < \omega_1 \rangle$ of ordinals less than ω_1. If λ_β has been defined for all $\beta < \alpha$, then $\{\mu \colon b_\mu \cdot -d_\alpha \notin J\}$ is not countable, so $\{\mu \colon b_\mu \cdot -d_\alpha \notin J\} \cap (\sup_{\beta<\alpha} \lambda_\beta, \omega_1) \neq 0$, and hence there is a $\lambda_\alpha > \sup_{\beta<\alpha} \lambda_\beta$ such that $b_{\lambda_\alpha} \cdot -d_\alpha \notin J$.

Now for each $\alpha < \omega_1$ the set

$$\{b_{\lambda_\alpha} \cdot -d_\alpha\} \cup \{-u_\beta \colon \beta < \omega_1\}$$

has the fip, and so is included in an ultrafilter F_α. Say F_α is determined by $c_\alpha \in B'$: $F_\alpha = \{x \in B \colon \rho_B(c_\alpha \cdot x) = \rho_B c_\alpha\}$. Without loss of generality, $c_\alpha \leqslant b_{\lambda_\alpha} \cdot -d_\alpha$ for all $\alpha < \omega_1$. Now let C be the set of all $\lambda < \omega_1$ such that (2)(a)–(d) hold, along with

(g) $c_\alpha \in B_\lambda$ for all $\alpha < \lambda$.

Note that $C \neq 0$ (a club argument). Fix $\lambda \in C$. Say $u_\lambda \in B_\delta$ with $\lambda + 1 \leqslant \delta < \omega_1$. By Lemma 2.3 choose $c \in B_{\lambda+1}$ such that $c \leqslant b_{\lambda+1}$ and for all $b \in B'_{\lambda+1}$ the following are equivalent:

(i) $\rho_{B_{\lambda+1}}(b \cdot b_{\lambda+1}) = \rho_{B_{\lambda+1}} b$ and $\rho_{B_\delta}(b \cdot u_\lambda) = \rho_{B_\delta} b$.

(ii) $\rho_{B_{\lambda+1}}(b \cdot c) = \rho_{B_{\lambda+1}} b$.

By (2) we have $\rho(b_{\lambda+1} \cdot u_\lambda) = \rho b_{\lambda+1}$. Choose $b \in B'_{\lambda+1}$ such that $b/I_{\rho b_{\lambda+1}}$ is an atom $\leqslant (b_{\lambda+1} \cdot u_\lambda)/I_{\rho b_{\lambda+1}}$. By the equivalence of (i) and (ii) we get $b/I_{\rho b_{\lambda+1}} \leqslant c/I_{\rho b_{\lambda+1}}$. Thus $\rho_{B_{\lambda+1}} c = \rho_{B_{\lambda+1}} b_{\lambda+1}$. Write $b_{\lambda+1} \cdot -c = d \cdot b_{\lambda+1}$ with $d \in I_{B_\lambda}$. Then, we claim,

(5) $(c_\alpha/I_{\rho c_\alpha}) \cdot ((b_{\lambda+1} \cdot -d)/I_{\rho c_\alpha}) = 0$ for all $\alpha < \lambda$.

For, suppose not. Now $c_\alpha, b_{\lambda+1}, d \in B_{\lambda+1}$, so choose $e \in B'_{\lambda+1}$ so that $e/I_{\rho c_\alpha}$ is an atom $\leqslant (c_\alpha/I_{\rho c_\alpha}) \cdot ((b_{\lambda+1} \cdot -d)/I_{\rho c_\alpha})$. Now $b_{\lambda+1} \cdot -d = b_{\lambda+1} \cdot c$, so $e/I_{\rho c_\alpha} \leqslant c/I_{\rho c_\alpha}$. Hence by the equivalence of (i) and (ii) we have $e/I_{\rho c_\alpha} \leqslant u_\lambda/I_{\rho c_\alpha}$. But also $e/I_{\rho c_\alpha} \leqslant c_\alpha/I_{\rho c_\alpha} \leqslant -u_\lambda/I_{\rho c_\alpha}$, contradiction. So (5) holds.

By (5) it follows that $c_\alpha/I_{\rho c_\alpha} \cdot -d/I_{\rho c_\alpha} = 0$ for all $\alpha < \lambda$. Write $d = d_\beta$ with $\beta < \lambda$. Thus $c_\beta \cdot -d_\beta \in I_{\rho c_\beta}$. But $c_\beta \leqslant -d_\beta$, contradiction. So (4) holds.

Fix b as in (4). Now $b_{\alpha+1} \nleqslant b$ if $\omega + \alpha > \rho b$, since $\rho b_{\alpha+1} = \omega + \alpha$. So we can choose $\lambda < \omega_1$ such that $S \overset{\mathrm{def}}{=} \{\alpha < \lambda \colon 0 \neq b_\alpha \cdot -b \in J\}$ is infinite. For each $\alpha \in S$ there is a $\beta_\alpha < \omega_1$ such that $b_\alpha \cdot -b \leqslant u_{\beta_\alpha}$. Let $\mu < \omega_1$ be greater than β_α for each $\alpha \in S$. Then $\{\beta < \lambda \colon 0 < b_\beta \text{ and } \rho(b_\alpha \cdot -(b + u_\mu)) = 0\}$ is infinite. This contradicts (C_γ), where γ is chosen so that $b + u_\mu \in I_{B_\gamma}$ and $\lambda \leqslant \gamma$. $\square$

Now we turn to Theorem B. It follows easily from the following result.

Theorem 2.6. *Suppose that κ is a regular cardinal greater than ω_1, and B is a superatomic BA with $\pi\chi B = \kappa$ attained (i.e., there is an ultrafilter F on B such that $\pi\chi F = \kappa$). Then B has a chain of order type κ.*

Proof. Let $a \in B'$ have smallest rank such that $\pi\chi F_a = \kappa$. Thus without loss of generality $\lambda_B^2 = 1$ and if $b \in B'$ and $\rho b < \lambda_B$, then $\pi\chi F_b < \kappa$. Thus $F \stackrel{\text{def}}{=} \{b \in B\colon \rho b = \kappa\}$ is the only ultrafilter with $\pi\chi F = \kappa$. Now we construct $\langle b_\alpha\colon \alpha < \kappa\rangle$ by induction, all members of F. Suppose that $\langle b_\alpha\colon \alpha < \beta\rangle$ has been constructed. Then $\langle\{b_\alpha\colon \alpha < \beta\}\rangle$ is not dense in F, so there is a $b_\beta \in F$ such that there is no nonzero element of $\langle\{b_\alpha\colon \alpha < \beta\}\rangle$ below b_β. Let $C = \langle\{b_\alpha\colon \alpha < \kappa\}\rangle$. Then clearly $\pi\chi(C \cap F) = \kappa$. Thus we may assume that $|B| = \kappa$.

Now we choose a big θ, and work within $H(\theta)$, taking elementary substructures, where $H(\theta)$ is supplied with various additional relations for the arguments below, in the usual fashion. Let $\langle M_\alpha\colon \alpha \leqslant \kappa\rangle$ be an increasing continuous sequence of elementary submodels of $H(\theta)$ with the following properties:

(1) $B, \kappa \in M_0$.

(2) $M_\alpha \in M_{\alpha+1}$.

(3) $|M_\alpha| < \kappa$.

(4) $B \cap M_\alpha$ is a subalgebra of B, and $B = \bigcup_{\alpha<\kappa}(B \cap M_\alpha)$.

(5) If $J \in M_\alpha$ and $|J| < \kappa$, then $J \subseteq M_{\alpha+1}$.

Note that (5) is possible because $|\bigcup\{J\colon J \in M_\alpha,\ |J| < \kappa\}| < \kappa$.

We claim

(6) For all $\alpha < \kappa$ there is a $b \in B$ such that $\rho b < \kappa$ and $\rho(c \cdot -b) < \rho c$ for all $c \in I_B^+ \cap M_\alpha$. For, $I_B^+ \cap M_{\alpha+1}$ is not dense in $\{b \in B\colon \rho b = \lambda_B\}$, so there is a $b \in B$ such that $\rho b = \lambda_B$ and no member of $I_B^+ \cap M_{\alpha+1}$ is $\leqslant b$. Suppose that $c \in I_B^+ \cap M_\alpha$ and $\rho(c \cdot b) = \rho c$. Then there is a $d \in B'$ such that $\rho d = \rho c$ and $d/I_{\rho c} \leqslant c/I_{\rho c}$. By elementarity we may assume that $d \in M_\alpha$. Now $\pi\chi F_d < \kappa$, so there is a set J of atoms of B such that $|J| < \kappa$ and J is dense in F_d. By elementarity we may assume that $J \in M_\alpha$. So by (5), $J \subseteq M_{\alpha+1}$. Now $\rho(d \cdot b) = \rho d$, so $b \in F_d$ and hence there is a $j \in J$ such that $j \leqslant b$. This is a contradiction. Thus (6) holds (with $-b$ in place of b).

(7) For every $\alpha < \kappa$ there is a $b_\alpha \in B \cap M_{\alpha+1}$ such that $\rho b_\alpha < \kappa$ and $\rho(c \cdot -b_\alpha) < \rho c$ for all $c \in I_B^+ \cap M_\alpha$.

This follows by elementarity from (6).

Let $T = \{\alpha < \kappa\colon \operatorname{cf}\alpha = \omega_1\}$. Fix $\alpha \in T$ and $\beta < \alpha$. For each $n \in \omega$ let $c_n = \sum_{k<n} b_{\alpha+k}$. The ranks of the elements $b_\beta \cdot -c_n$ are decreasing for increasing n, so there is an n_β such that $b_\beta < c_{n_\beta}$. Now $\alpha = \bigcup_{n<\omega}\{\beta < \alpha\colon n_\beta = n\}$, so there is an n such that $S_n \stackrel{\text{def}}{=} \{\beta < \alpha\colon n_\beta = n\}$ is cofinal in α.

(8) $b < c_n$ for all $b \in M_\alpha$.

In fact, fix $\beta < \alpha$ such that $b \in B_\beta$. Then there are infinitely many members of S_n above β. As in the argument with the c_n's, it follows that there is a finite $F \subseteq S_n$ such that $b < \sum_{\gamma\in F} b_\gamma$. Since $b_\gamma < c_n$ for each $\gamma \in F$, (8) follows.

Let $T = \{\alpha_\xi\colon \xi < \kappa\}$, listed in increasing order. By (8), for each $\xi < \kappa$ there is a $d_\xi \in M_{\alpha_{\xi+1}}$ such that $b < d_\xi$ for all $b \in M_{\alpha_\xi}$. Hence $\langle d_\xi\colon \xi < \kappa\rangle$ is strictly increasing, as desired. □

Corollary 2.7. *If B is a superatomic BA and $\pi\chi B = \lambda^+$ with $\lambda > \omega$, then B has a chain of order type λ^+.*

Lemma 2.8. *If B is a superatomic BA, $\pi\chi B$ is a limit cardinal, and $\mu < \pi\chi B$, then B has a subalgebra C such that $\pi\chi C = \mu^+$.*

Proof. Let F be an ultrafilter of B such that $\pi\chi F \geqslant \mu^+$. We define the sequence $\langle b_\alpha\colon \alpha < \pi\chi F\rangle$ as in the first part of the proof of the theorem. Let $C = \langle\{b_\alpha\colon \alpha < \mu^+\}\rangle$. Clearly C is as desired. □

Theorem B now follows. Note, however, that if $\pi\chi B$ is a limit cardinal, the proof does not show that depth B, which is the same as $\pi\chi B$, is attained.

3. Tightness and depth

We prove a result slightly stronger than Theorem C of the abstract: if $\kappa \to (\kappa)_2^{<\omega}$ and B is a BA which has a free sequence of length κ, then B has depth κ. Recall that κ is a limit cardinal. We may assume that B has tightness exactly κ, that the tightness of $B \restriction b$ is less than κ for all $b \in I_B$, and that $\lambda_B^2 = 1$. Let $\langle b_\alpha\colon \alpha < \kappa\rangle$ be a free sequence. If b_β has rank λ_B, then the sequence $\langle b_\alpha \cdot -b_\beta\colon \kappa > \alpha > \beta\rangle$ is still a free sequence, and all elements have rank less than λ_B; thus we may assume that each b_α has rank less than λ_B. For each nonzero $m \in \omega$ we partition $[\kappa]^{2m+1}$ into two parts, as follows:

$$\begin{aligned}\Gamma_m = \bigl\{\{\alpha, \beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{m-1}\}\colon\ & \alpha < \beta_0 < \cdots < \beta_{m-1} < \gamma_0 < \cdots \\ & < \gamma_{m-1} \text{ and } b_\alpha \cdot -b_{\beta_0} \cdot \cdots \cdot -b_{\beta_{m-1}} \cdot b_{\gamma_0} \cdot \cdots \cdot b_{\gamma_{m-1}} = 0\bigr\}; \\ \Delta_m = \bigl\{\Theta \in [\kappa]^{2m+1}\colon\ & \Theta \notin \Gamma_m\bigr\}.\end{aligned}$$

By the partition relation $\kappa \to (\kappa)_2^{<\omega}$ we may assume that κ is homogeneous.

Now for each $\alpha < \kappa$ we have tightness$(B \restriction b_\alpha) < \kappa$. We apply this to the sequence $\langle b_\alpha \cdot b_\beta\colon \beta$ a limit ordinal greater than $\alpha\rangle$; this yields finite sets Γ and Δ such that $\alpha < \beta < \gamma$ whenever $\beta \in \Gamma$ and $\gamma \in \Delta$, with $b_\alpha \cdot \prod_{\beta\in\Gamma} -b_\beta \cdot \prod_{\gamma\in\Delta} b_\gamma = 0$. Filling in beyond Γ or Δ if necessary, we may assume that Γ and Δ have the same size (but they no longer have to consist exclusively of limit ordinals). By the homogeneity we thus have this equality for any α, Γ, Δ in the indicated order, with Γ and Δ of the same size, say n. From this we show that for any $\lambda < \kappa$ there is a chain of order type λ. In fact, select a disjoint system $\langle F_\alpha\colon \alpha < \lambda\rangle$ of members of $[\lambda]^n$ such that $\max F_\alpha < \min F_\beta$ if $\alpha < \beta < \lambda$. Define

$$c_\alpha = \sum_{\beta\in F_\alpha} b_\beta \cdot \prod_{i<n} b_{\lambda+i}.$$

If $\alpha < \beta$ and $\xi \in F_\alpha$, then $b_\xi \cdot \prod_{i<n} b_{\lambda+i} \leqslant c_\beta$. Hence $c_\alpha \leqslant c_\beta$. Actually $c_\alpha < c_\beta$. For, suppose that they are equal. Then

$$\sum_{\gamma \in F_\beta} b_\gamma \cdot \prod_{i<n} b_{\lambda+i} \cdot \prod_{\beta \in F_\alpha} - b_\beta = 0,$$

contradicting the free sequence property. □

Note that the proof of Theorem C does not show that depth $B = \kappa$ is attained.

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Erratum

Erratum to "Depth, π-character, and tightness in superatomic Boolean algebras" Topology and its Applications 75 (1997) 183–199 ☆

A. Dow *, J. Donald Monk

Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3

Theorem B in the abstract should read as follows:

> *If B is a superatomic Boolean algebra with π-character greater than ω_1, then the π-character is less than or equal to the depth of B.*

Also, on page 198, delete the sentence after the sentence "*Theorem B now follows*."

The authors are indebted to Juan Carlos Martinez for pointing out this error.

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* Corresponding author. Email: dowa@mathstat.yorku.ca.

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