PROBLEMS IN THE SET THEORY OF BOOLEAN ALGEBRAS

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In this survey article we state some open problems and give appropriate background in the set-theoretic aspect of Boolean algebras. These are a selection of problems from a forthcoming book of the author which is a greatly expanded version of Monk [90]. The problems mentioned are among the most interesting in that book, in the author's opinion. Some of them may be quite easy, but others are known to be difficult. We assume only a modest familiarity with the notions of Boolean algebras, and hope that even non-experts in this field can follow the exposition. We give the basic definitions to establish notation, and to recall notions that the reader may have forgotten. But we do not give elementary examples to go along with the basic notions, in order to be able to concentrate on the problems.

A Boolean algebra (BA) is an abstract structure of the form $\langle A, +, \cdot, -, 0, 1\rangle$ subject to some simple equational axioms. It is not useful to write these down, since one of the first things one learns about BA's is that each BA is isomorphic to a field of sets, i.e., a structure $\langle A, \cup, \cap, \setminus, \emptyset, U\rangle$ where A is a collection of subsets of U closed under the indicated operations. The abstract equational definition is useful mainly in that, because of it, one can make common algebraic operations on BA's like products, homomorphisms, free products (co-products, categorically), etc. and stay in the class of BA's. The Handbook of Boolean Algebras listed in the references is what we recommend in order to learn about BA's, or recall facts; volume I is the most essential thing for this. In any BA one can define a partial order $\leq$ by setting $x \leq y$ iff $x + y = y$. The generalizations of $+$ and $\cdot$ to finite sets are denoted by $\sum$ and $\prod$ respectively. Under $\leq$, $x+y$ and $x \cdot y$ are the least upper and greatest lower bounds of x and y respectively. So one can discuss infinitary $\sum$ and $\prod$ also; $\sum X$, when it exists, is by definition the least upper bound of X, and $\prod X$ the greatest lower bound. If both of these always exist, the BA is called *complete.* Given a BA A and two subsets X and Y of A, we say that X is *dense* in Y if for every nonzero $y \in Y$ there is a nonzero $x \in X$ such that $x \leq y$. Every BA A has a *completion*, which is a complete BA which includes A, in which A is dense. If L is a linear ordering with smallest element 0_L, we can form a natural BA, called the *interval algebra* of L, by taking all subsets of L of the form

$$[a_0, b_0) \cup [a_1, b_1) \cup \ldots \cup [a_{m-1}, b_{m-1}),$$

where $a_0 < b_0 < a_1 < b_1 < \cdots < a_{m-1} < b_{m-1} \leq \infty$.

An *ultrafilter* on a BA A is a subset F of A with the following properties:
(i) $1 \in F$.
(ii) $0 \notin F$.

(iii) If $x, y \in F$, then $x \cdot y \in F$.
(iv) If $x \in F$ and $x \leq y$, then $y \in F$.
(v) For any element $a \in A$, either $a \in F$ or $-a \in F$.

Ultrafilters play a central role in the study of BA's. In particular, the set Ult$A$ of all ultrafilters on A can be given a topology making this set into a compact space with special properties; this is the *Stone space* of A, and there is a categorical duality between BA's and the corresponding class of spaces.

An *atom* in a BA A is a nonzero element $a \in A$ such that there are no nonzero elements $< a$. A BA A is *atomic* if for every nonzero element $b \in A$ there is an atom $a \leq b$. If a is an atom, then $\{x \in A : a \leq x\}$ is an ultrafilter of a rather trivial sort; it is called a *principal* ultrafilter. Other ultrafilters, which always exist in infinite BA's by Zorn's lemma, are called *non-principal.*

There is an interesting special class of atomic BA's. A BA A is *superatomic* if not only A but also every homomorphic image and subalgebra of A is atomic. Surprisingly, there are many such algebras. The simplest of them are the finite-cofinite algebras on infinite sets. There is a more complicated characterization of them which is useful, and brings out another way in which they are specially simple. We construct a transfinite sequence of ideals of A as follows. Let $I_0 = \{0\}$. Having constructed I_α (α an ordinal number), let $I_{\alpha+1}$ be the ideal of A generated by all $a \in A$ such that either $a/I_\alpha = 0$ or a/I_α is an atom of A/I_α. For λ a limit ordinal, let $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$. Now it is not hard to prove that A is superatomic iff the first α such that $I_\alpha = I_{\alpha+1}$ is a successor ordinal $\beta + 1$, and A/I_β is finite. Here is another fact which shows how simple superatomic BA's are, on the face of it. If F is a principal ultrafilter on A/I_α, generated as above by an atom a/I_α, then $\{x \in A : a/I_\alpha \leq x/I_\alpha\}$ is an ultrafilter on A. And every ultrafilter on A is obtained from a principal ultrafilter on some A/I_α in this fashion.

Free caliber. A collection X of elements of a BA is *independent* if for all disjoint finite subsets Y and Z of X we have $\prod_{x\in Y} x \cdot \prod_{x\in Z} -x \neq 0$. This is clearly related to the notion of independence in measure theory. If X is independent and generates A, then A is a free BA on X, in the usual categorical sense. Naturally, the notion of a free BA and this related notion of independence have been intensively studied. For example, it is known that if A is an infinite complete BA, then A has an independent subset of size $|A|$; but A is never itself free. Now let κ be an infinite cardinal. A BA A is said to have *free caliber* κ provided that $\kappa \leq |A|$ and for every subset X of size κ there is an independent subset Y of X of size κ. We give some background on this notion from Monk [83] and state some problems. It is easy to see that if $\lambda^+ \leq |\prod_{i\in I} A_i|$ and $\lambda^{|I|} = \lambda$, then $\prod_{i\in I} A_i$ has free caliber λ^+ iff every A_i of size at least λ^+ has free caliber λ^+. If $\kappa \leq 2^{|I|}$ and there is a linear order of power $2^{|I|}$ with a dense subset of size $|I|$, then $\prod_{i\in I} A_i$ does not have free caliber κ (all A_i non-trivial). If A is the product of the finite-cofinite algebra on $\beth_\omega$ and the free algebra on $(\beth_\omega)^+$ free generators, then A has caliber $(\beth_\omega)^+$ but ${}^\omega A$ does not. Here is a related simple open problem.

Problem 1. *For each $n \in \omega$ let A_n be the free BA on $\beth_n$ free generators. Does $\prod_{n\in\omega} A_n$ have free caliber $(\beth_\omega)^+$?*

Although a complete BA always has an independent subset of the size of the BA, the free caliber is a stronger requirement. We give two problems concerning this.

Problem 2. *Let A be a free BA on $\beth_{\omega+1}$ free generators, and let B be the completion of A. Does B have free caliber $\beth_{\omega+1}$?*

Problem 3. *Is there for every infinite cardinal κ a complete BA A of size 2^κ such that for every infinite cardinal λ, A does not have free caliber λ?*

In this connection we mention that if L is a linear order of size 2^κ with a dense subset of size κ, and A is the completion of the interval algebra on L, then $|A| = 2^\kappa$ and A satisfies the conclusion of Problem 3.

Problems 1–3 have not been worked on much, and may be easy.

Chain conditions. A BA A satisfies the *κ-chain condition* if every set of pairwise disjoint elements of A is of size less than κ. There is a related cardinal function, cellularity, on BA's. The *cellularity* of A, denoted by cA, is the supremum of the cardinalities of pairwise disjoint subsets of A. Chain conditions and cellularity have been studied a lot. There are several open problems. One which is probably a little harder than the problems above is as follows.

Problem 4. *Is it consistent with ZFC that there is a BA A of size $\aleph_2$ with the following properties?*

(i) c$B = \aleph_2$ *for every homomorphic image B of A of size $\aleph_2$.*
(ii) A has no countable homomorphic image.
(iii) A has a homomorphic image B of size $\aleph_1$ with cellularity $\aleph_1$.
(iv) A has a homomorphic image B of size $\aleph_1$ with cellularity $\aleph_0$.

The most relevant result to this problem is in Koppelberg [77]: if MA (Martin's axiom) and $2^\omega > \omega_2$, then every BA of size ω_2 has a countable homomorphic image. That is why Problem 4 asks for consistency, not provability in ZFC.

Depth and subalgebras. The *depth* of a BA A, Depth$A$, is

$$\sup\{|X| : X \text{ is a subset of } A \text{ well-ordered by } \leq\}.$$

We state a problem about this function which is motivated by a fact about cellularity. Shelah [80] proved that if κ and λ are infinite regular cardinals and $\forall\mu < \lambda(\mu^{<\kappa} < \lambda)$, and if A is a BA of size at least λ satisfying the κ-chain condition, then A has free caliber λ. The corollary of this of most interest to us is that if κ is infinite, $(2^\kappa)^+ \leq |A|$, and A satisfies the κ^+-chain condition, then A has an independent subset of size $(2^\kappa)^+$. Now any free BA has cellularity ω. It follows that if A has size $(2^\kappa)^+$ and cellularity κ, then A has a subalgebra of size $(2^\kappa)^+$ with cellularity ω. We wonder if the analogous result for depth holds:

Problem 5. *Is there an infinite cardinal κ and a BA A of size $(2^\kappa)^+$ such that* Depth$A = \kappa$*, while A has no subalgebra of size $(2^\kappa)^+$ with depth ω?*

We feel that this problem is not very hard.

A density problem. There are three cardinal functions connected with density. The *algebraic density* of a BA A is the smallest cardinality of a subset of A which is dense in A; this cardinal is denoted by πA (because it comes from a topological notion traditionally denoted by π). If X is dense in A, then it is easy to see that every element of A is a sum of elements of X; so $|A| \leq 2^{|X|}$. Thus $|A| \leq 2^{\pi A}$ for any BA A. For any infinite cardinal κ and any cardinal λ with $\kappa \leq \lambda \leq 2^\kappa$ there is a BA A such that $\pi A = \kappa$ and $|A| = \lambda$. Another density notion is obtained by modifying this one as follows. Let F be an ultrafilter on a BA A. The *π-character* of F, denoted by $\pi\chi F$, is the smallest cardinality of a subset D of A which is dense in F. Note that it is not required that $D \subseteq F$; adding this requirement leads to the notion of the *character* of the ultrafilter, which we discuss later. Now the *π-character* of A itself is $\sup\{\pi\chi F : F \text{ is an ultrafilter on } A\}$; we denote it by $\pi\chi A$. This notion may seem to be rather special, but it occurs a lot in the theory of BA's, and is important in set-theoretic topology. It is actually quite different from π. For example, for any infinite cardinal κ, the finite-cofinite algebra on κ has π-character ω and algebraic density κ.

It is a rather deep result of Bozeman [91] that, under GCH, if A is complete then $\pi A = \pi\chi A$. We do not know whether this holds in ZFC:

Problem 6. *Is it true in ZFC that $\pi A = \pi\chi A$ for every complete BA A?*

This problem may be hard. One can reformulate it in terms of another density notion of a topological character. Namely, the *topological density* of A is the smallest cardinality of a dense subspace of Ult$A$; it is denoted by dA. An easy result connecting these three density notions is that $\mathrm{d}A \cdot \pi\chi A = \pi A$. Thus an equivalent formulation of Problem 6 is whether one can prove in ZFC that $\mathrm{d}A \leq \pi\chi A$ for every complete BA A.

Tightness and superatomic algebras. The *tightness* of a BA A, denoted by tA, is

$$\sup\{\mathrm{Depth}B : B \text{ is a homomorphic image of } A\}.$$

There are several equivalent definitions of this notion. For example, also

$$\mathrm{t}A = \sup\{\pi\chi B : B \text{ is a homomorphic image of } A\}.$$

And there is a purely topological definition (actually the standard one, from which the name is derived), and one involving "free sequences". We mention two problems about tightness and superatomic BA's. One concerns how large the gap between Depth$A$ and tA can be for A superatomic. A theorem of Dow, Monk [94] is that if $\kappa \to (\kappa)^{<\omega}_2$, then every superatomic Boolean algebra with tightness at least κ^+ has depth at least κ. The hypothesis $\kappa \to (\kappa)^{<\omega}_2$ is one of the partition relations that implies that κ is an inaccessible cardinal. This result shows that the gap between Depth$A$ and tA cannot be arbitrarily large. A somewhat small gap can be obtained like this. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be a sequence of infinite subsets of ω such that $\alpha < \beta < \omega_1$ implies that $a_\alpha \backslash a_\beta$ is finite and $a_\beta \backslash a_\alpha$ is infinite; it is an exercise in some set theory books to show that such a sequence exists. Let A be the BA of subsets of ω generated by the singletons and all of these sets a_α. Then $\mathrm{Depth}A = \omega$ while $\mathrm{t}A = \omega_1$. The following problem seems to be open.

Problem 7. *Is there a superatomic BA A such that $\mathrm{t}A = (2^\omega)^+$ while $\mathrm{Depth}A = \omega$?*

Our opinion here is that this problem is not trivial, but not extremely hard either.

The other problem we want to mention about tightness involves the number of automorphisms of a superatomic BA. It is easy to see that any infinite superatomic BA has at least 2^ω automorphisms. One way to see this is to look at an atom a/I_1. One can permute arbitrarily all of the atoms below a and fix all other atoms; this map can be extended to an automorphism of the algebra. So the question arises to find a superatomic BA with a relatively small group of automorphisms. Unlike BA's in general, where there are even rigid algebras (ones with no non-trivial automorphisms), we see that here there are in general many of them. The first natural question is to find a superatomic BA with fewer automorphisms than elements. This was done by M. Rubin in 1992/1993, using diamond; the algebra has size 2^{ω_1} and only ω_1 automorphisms. Note also that there are at least as many automorphisms as atoms, since a finite permutation of atoms can be extended to an automorphism. There are superatomic BA's with tightness greater than the number of atoms—see, for example, the algebra above with depth less than tightness. So here is the next obvious problem along these lines:

Problem 8. *Is there a superatomic BA A such that the number of automorphisms of A is less than* $\mathrm{t}A$*?*

This problem has not been worked on much.

Irredundance. A subset X of a BA A is *irredundant* provided that for all $x \in X$, x is not in the subalgebra of A generated by $X\backslash\{x\}$. The *irredundance* of a BA A is

$$\mathrm{Irr}A = \sup\{|X| : X \text{ is an irredundant subset of } A\}.$$

This is a universal algebraic notion. Most commonly considered BA's have irredundance $|A|$. In fact, the only BA's A known which have $\mathrm{Irr}A < |A|$ have been constructed using additional axioms of set theory, or using forcing; see Monk [90] for a survey. Todorčević also showed that it is consistent that every uncountable BA has uncountable irredundance. The following problem is probably difficult:

Problem 9. *Can one construct in ZFC a BA A such that* $\mathrm{Irr}A < |A|$*?*

A very simple problem about irredundance which is probably easy is as follows:

Problem 10. *Is* $\mathrm{Irr}(A \times B) = \max\{\mathrm{Irr}A, \mathrm{Irr}B\}$*?*

Here $A \times B$ is the cartesian product of A and B: the collection of all ordered pairs (a, b) with $a \in A$ and $b \in B$, with coordinatewise operations. Since $\mathrm{Irr}A$ is "usually" the same as $|A|$, a counterexample for the statement of Problem 10 might be hard to construct. In algebras A of Kunen and of Todorčević constructed in Monk [90] with $\mathrm{Irr}A < |A|$, one has $\mathrm{Irr}(A \times A) = \mathrm{Irr}A\ (= \omega)$.

***S* and *L* problems.** Topologists have extensively studied the problem of existence of two related special kinds of spaces, called S-spaces and L-spaces. An *S-space* is a regular Hausdorff space which is hereditarily separable (each subspace separable) but not hereditarily Lindelöf. An *L-space* is a regular Hausdorff space which is hereditarily Lindelöf by not hereditarily separable. There are five cardinal functions on BA's related to these notions: d, defined above, and the following new functions:

Character. For any ultrafilter F on a BA A, let χF, the character of F, be the smallest cardinality of a subset of F which generates it. The character of the BA itself is

$$\chi A = \sup\{\chi F : F \text{ is an ultrafilter on } A\}.$$

Hereditary Lindelöf degree. The topological definition of this function is clear, but algebraically an equivalent definition is as follows:

$$\mathrm{hL}A = \sup\{\kappa : \text{ there is an ideal not generated by } < \kappa \text{ elements}\}.$$

Hereditary density. Again, you can imagine what the topological form of this cardinal function is. An algebraic definition is:

$$\mathrm{hd}A = \sup\{\pi B : B \text{ is a homomorphic image of } A\}.$$

Spread. An equivalent definition of this function is

$$\mathrm{s}A = \sup\{|X| : X \text{ is an ideal-independent subset of } A\},$$

where a subset X of A is *ideal independent* if for all distinct $x, y_0, \ldots, y_{m-1} \in X$ we have $x \not\leq y_0 + \ldots y_{m-1}$.

We mention some results, and four problems, concerning these functions; they are related to the work on S and L spaces mentioned at the outset. Under CH there is a locally compact topology on $\mathbb{R}$ such that the compactification of $\mathbb{R}$ under this topology gives a Boolean space whose BA A of clopen subsets, called the *Kunen line BA*, is such that $\mathrm{s}A = \omega$ and $\chi A = \omega_1$. It is unknown whether an example with these functions different can be found in ZFC:

Problem 11. *Can one find in ZFC a BA A such that* $\mathrm{s}A < \chi A$?

This is equivalent to the problem of finding in ZFC a BA A such that $\mathrm{s}A < \mathrm{hL}A$. The Kunen line BA is also such that $\mathrm{hd}A < \chi A$. It is also unknown whether this can be done in ZFC:

Problem 12. *Can one find in ZFC a BA A such that* $\mathrm{hd}A < \chi A$?

This is equivalent to finding in ZFC a BA A such that $\mathrm{hd}A < \mathrm{hL}A$ (the generalized S-space problem).

The interval algebra A on a Suslin line is a BA such that $\mathrm{hL}A < \mathrm{d}A$ and $\mathrm{s}A < \mathrm{hd}A$. This gives rise to the following two questions:

Problem 13. *Can one find in ZFC a BA A such that* $\mathrm{hL}A < \mathrm{d}A$?

This is equivalent to the problem of finding in ZFC a BA A such that $\mathrm{hL}A < \mathrm{hd}A$ (the generalized L-space problem).

Problem 14. *Can one find in ZFC a BA A such that* $\mathrm{s}A < \mathrm{hd}A$?

This is equivalent to finding in ZFC a BA A such that $\mathrm{s}A < \mathrm{d}A$.

We also do not know whether some two of these four problems are equivalent.

Our guess is that all four problems are difficult.

More problems on irredundance. A simple theorem of Heindorf says that $\mathrm{Irr}A \leq \mathrm{s}(A \oplus A)$, where $\oplus$ is free product. It may be that equality actually holds here:

Problem 15. *Is* $\mathrm{Irr}A = \mathrm{s}(A \oplus A)$ *for every BA A?*

Here are two problems concerning irredundance in superatomic algebras:

Problem 16. *Is there a superatomic BA A such that* $\mathrm{s}A < \mathrm{Irr}A$?

Problem 17. *Is there a superatomic BA A such that* $\mathrm{Irr}A < \mathrm{Inc}A$?

Here $\mathrm{Inc}A$ is the supremum of cardinalities of subsets X of A such that members of X are pairwise incomparable under the Boolean ordering.

Problems 16–17 may be easy on the basis of what is known about superatomic Boolean algebras.

REFERENCES

1. Bozeman, K. [91], *On the relationship between density and weak density in Boolean algebras*, Proc. Amer. Math. Soc. 112, no. 4, 1137–1141.
2. Dow, A.; Monk, J. D. [94], *Depth, π-character, and tightness for superatomic Boolean algebras*, Preprint.
3. Koppelberg, S. [77], *Boolean algebras as unions of chains of subalgebras*, Alg. Univ. **7**, 195–203.
4. Koppelberg, S. [89], *Handbook of Boolean algebras, Part I*, North-Holland 1989, 312pp.
5. Monk, J. D. [83], *Independence in Boolean algebras*, Per. Math. Hungar. 14, 269–308.
6. Monk, J. D. [90], *Cardinal functions on Boolean algebras*, Birkhäuser Verlag, 152pp.
7. Shelah, S. [80], *Remarks on Boolean algebras*, Alg. Univ. 11, 77–89.