TREE ALGEBRAS AND CHAINS[1)]

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For any tree T, one may define the tree algebra B_T as the field of subsets of T generated by the collection of sets $S_t = \{s \in T : t \leq s\}$ for $t \in T$. The notion was introduced and used to construct rigid Boolean algebras and to solve a question about weakly homogeneous Boolean algebras in Brenner [B]. Here we relate this notion to linear orderings.

We concern ourselves with two main questions. First, which chains occur in tree algebras? This is the topic of section 2. Our main results are that if C is a chain in a tree algebra and the cardinality of C is regular uncountable, then C contains a well-ordered or inversely well-ordered subset of the same power; and for κ regular uncountable, T has a branch of power $\geq\kappa$ iff B_T contains a well-ordered chain of power $\geq\kappa$.

In section 3 we consider the relationship between the classes of tree algebras and of interval algebras. We show that every tree algebra is isomorphic to a subalgebra of an interval algebra and use results from section 2 to show that some interval algebras embed in no tree algebra.

Section 1 contains definitions and lemmas used in the later sections, some of independent interest, as well as a discussion of several other methods of generating Boolean algebras from trees.

1. BASIC DEFINITIONS AND FACTS. We use "BA" to abbreviate "Boolean algebra." For any tree T and any $t \in T$, let $S_t^T = S_t = \{s \in T : t \leq s\}$. The <u>tree algebra</u> B_T

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over T is the field of subsets of T generated by $\{S_t : t \in T\}$. For any $t \in T$, A_t is the set of immediate successors to t. A tree T is splitting if for any $t \in T$, $|A_t| \neq 1$; it is infinitely splitting if for all $t \in T$, A_t is infinite. The level of $t \in T$ is the order type of $\{s : s < t\}$; for any ordinal α, level $\alpha = \{t \in T :$ level of t is $\alpha\}$. The height of T is $\sup\{(\text{level of } t) + 1 : t \in T\}$. If $t \in T$, then S_t may be considered as a tree in its own right. An initial chain in T is a chain C in T such that $x \leq y \in C \rightarrow x \in C$. T is limit-normal if for every initial chain C in T without last element, there is at most one $t \in T$ such that $C = \{s \in T : s < t\}$. A maximal initial chain in T is called a branch.

If A is any BA and $a \in A$, we set $a^1 = a$, $a^0 = -a$. If $(L,\leq)$ is a linear ordering, the interval algebra on L is the BA of subsets of L generated by $\{[a,b) : a < b \text{ in } L\} \cup \{[a,\infty) : a \in L\}$. For any BA A, depth $A = \sup\{|X| : X \subseteq A, X \text{ well-ordered}\}$, and length $A = \sup\{|X| : X \subseteq A, X \text{ a chain}\}$. A partition of A is a subset P of A which is pairwise disjoint, with $0 \notin P$, and with $\sum P = 1$. For any cardinal κ, κ^+ is the least cardinal greater than κ, $\kappa^{+0} = \kappa$ and $\kappa^{+(n+1)} = (\kappa^{+n})^+$.

Other ways of associating Boolean algebras with trees are known, and for background, we mention several.

In Horn, Tarski [HT] the following notion is discussed. A ramification set in a BA A is a subset X of A satisfying the following conditions:

(1) For all $x \in X$, $x \neq 0$.

(2) For all $x,y \in X$, $x \leq y$ or $y \leq x$ or $x \cdot y = 0$.

(3) For all $x \in X$, $\{y : x < y\}$ is well-ordered by $\geq$.

PROPOSITION 1.1. B is isomorphic to a tree algebra iff B has a ramification set as a generating set.

PROOF. ($\rightarrow$) obvious. ($\leftarrow$): Let X be a ramification set in B. Then under $\geq$, X forms a tree. If X has finitely many roots, we may assume that $\sum \{t : t \text{ a root of } X\} = 1$. We now define a homomorphism f of B_X onto B. For any $x \in X$, let $fS_x = x$. Then f extends to a homomorphism by the Sikorski extension criterion. Since B is a homomorphic image of B_X, B is isomorphic to a tree algebra by Theorem 5.2 of the first author's dissertation.

Note that the mapping f in the proof of 1.1 is not in general an isomorphism. For example, if the ramification set has an element t with A_t finite, and $t = \sum A_t$, then f is not one-one.

Notice the similarity of 1.1 to the well-known fact that a BA B is isomorphic to an interval algebra iff B contains an ordered basis (i.e., a linearly ordered set which is a generating set).

One method of associating a BA with a tree is common in forcing arguments. Make T into a topological space by letting $\{S_t : t \in T\}$ be a base for the topology. Then RO(T) is the BA of regular open sets of T. It is a complete BA, so for T infinite, it is never isomorphic to B_T by Theorem 3.1. However, there is a close connection with B_T, given by our next result.

PROPOSITION 1.2. If T is infinitely splitting, then B_T is isomorphic to the subalgebra of RO(T) generated by $\{S_t : t \in T\}$.

PROOF. By the Sikorski extension criterion.

Note the following about RO(T) and the operations in it.

$$\text{int cl } S_t = \{s : \text{For all } r \geqq s,\ r \text{ and } t \text{ are comparable}\}.$$

Thus for T splitting, int cl $S_t = S_t$.

$$S_t \cdot S_s = S_t \cap S_s$$

$$S_{t_1} + \ldots + S_{t_m} = \{s: \text{For all } r \geqq s,\ r \text{ is comparable with one of } t_1, \ldots, t_m\}.$$

Hence for T infinitely splitting,

$$S_{t_1} + \ldots + S_{t_m} = S_{t_1} \cup \ldots \cup S_{t_m}$$

$$-S_t = \{s : s \text{ and } t \text{ are incomparable}\}.$$

Thus $-S_t \neq {\sim}S_t$ unless t is the unique root of T.

For T not splitting, Proposition 1.2 can fail. For example, if T is a well-ordered chain, then RO(T) = 2, while B_T is isomorphic to the interval algebra on T.

Shelah in [S] makes use of the following construction. If T is a tree, let F_T be a BA freely generated by $\{x_t : t \in T\}$ subject to $x_t \leqq x_s$ for $s \leqq t$. Of course B_T

is a homomorphic image of F_T, but in general F_T is not isomorphic to a tree algebra. For example, if T consists just of uncountably many roots, then F_T is an uncountable free BA, which by Theorem 3.1 cannot be embedded in a tree algebra. Shelah also considers F'_T, which is freely generated by $\{x_t : t \in T\}$ subject to $x_t \leq x_s$ for $s \leq t$ and $x_t \cdot x_s = 0$ for s and t incomparable.

PROPOSITION 1.3. If T has infinitely many roots, then $B_T \cong F'_T$. If T has finitely many roots, then $a = \Pi\{-x_r : r \text{ is a root}\}$ is an atom of F'_T and $B_T \cong F'_T \restriction -a$.

PROOF. Note that $S_{t_1} \cdot \ldots \cdot S_{t_m} \cdot -S_{s_1} \cdot \ldots \cdot -S_{s_n} = 0$ iff one of the following conditions holds:

(1) t_i and t_j are incomparable for some $i,j \in \{1,\ldots,m\}$,

(2) $s_i \leq t_j$ for some $i \in \{1,\ldots,n\}$, some $j \in \{1,\ldots,m\}$,

(3) R is a subset of $\{s_1,\ldots,s_n\}$.

On the other hand, $x_{t_1} \cdot \ldots \cdot x_{t_m} \cdot -x_{s_1} \cdot \ldots \cdot -x_{s_n} = 0$ iff (1) or (2) holds. Hence the proposition follows.

The final method of associating a BA with a tree has been used implicitly in several constructions, and was made explicit in a letter from Judy Roitman to the second author. Ch(T) is the BA of subsets of T generated by {C : C is an initial chain of T}. It is easy to see that Ch(T) is always hereditarily atomic, and hence there are tree algebras not embeddable in any algebra Ch(T) (see, e.g., Theorem 2.9)

We now turn to several simple facts about tree algebras which will be used later. First we have a normal form lemma.

LEMMA 1.4. Let $b \in B_T$. Then b can be expressed in the form $\sum_{i \leq n} f_i$ where $n \in \omega$ and

(i) For all $i,j \leq n$, $i \neq j \to f_i \cdot f_j = 0$;

(ii) If $b \leq \sum_{t \in J} S_t$ for some $J \in [T]^{<\omega}$, then $f_n = 0$ and $J_n = 0$; otherwise $f_n = -\sum_{t \in J_n} S_t$ with $J_n \in [T]^{<\omega}$;

(iii) For all $i < n$, $f_i = S_{t_i} - \sum_{s \in J_i} S_s$ with J_i a finite set of successors to t_i;

(iv) for all $i \leqq n$, J_i is pairwise incomparable;

(v) for all $i < n$, for all $j \leqq n$, for all $s \in J_j$, $t_i \neq s$.

PROOF. Since $\{S_t : t \in T\}$ generates B_T, we can write

$$b = \sum_{i \in p} f_i,$$

each $f_i \neq 0$, $f_i \cdot f_j = 0$ for $i \neq j$, and

$$f_i = \prod_{j < m_i} S_{t_{i,j}}^{e(i,j)}$$

each $e(i,j) \in \{0,1\}$. Since $S_t \cdot S_s = 0$ if s and t are incomparable and $S_t \cdot S_s = S_t$ if $s \leqq t$, we may assume that each f_i has the form

(1) $f_i = S_{t_i} - \sum_{s \in J_i} S_s$, J_i a pairwise incomparable set of successors to t_i, or

(2) $f_i = -\sum_{s \in J_i} S_s$, J_i a pairwise incomparable set.

If $f_i \leqq \sum_{u \in K} S_u$ for some $K \in [T]^{<\omega}$, then $f_i = \sum_{u \in K} (f_i \cdot S_u)$; hence we may assume that (2) holds only if $f_i \not\leqq \sum_{u \in K} S_u$ for all $K \in [T]^{<\omega}$. Clearly (2) occurs for some i iff $b \not\leqq \sum_{u \in K} S_u$ for all $K \in [T]^{<\omega}$. This is possible only if T has infinitely many roots. Now (2) can hold for at most one $i < p$; for if it holds for $i,j < p$ with $i \neq j$, then choose a root v of T such that $v \not\in \sum_{u \in J_i \cup J_j} S_u$. Then $v \in f_i \cdot f_j$, a contradiction. Finally, note that if f_i and f_j have the form (1) and $t_j \in J_i$, then

$$f_i + f_j = S_{t_i} - \sum_{s \in J_i \cup J_j,\ s \neq t_j} S_s;$$

similarly, if f_i has the form (2). Thus we may assume that (i)-(v) hold.

The following supplement to the normal form lemma is sometimes useful.

LEMMA 1.5. (i) Suppose $b \leqq \sum_{t \in J} S_t$ for some $J \in [T]^{<\omega}$ and $b = \sum_{i \leqq m} f_i$ and $c = \sum_{i \leqq n} g_i$ are the normal forms of b and c. Then $b \leqq c$ iff for all $i \leqq m$, there is $j \leqq n$ such that $f_i \leqq g_j$.

(ii) Let $\{s,t\} \cup J \cup K \in [T]^{<\omega}$, with J and K pairwise incomparable sets. (a) If J is a set of successors to s and K a set of successors to t, then $S_s - \sum_{u \in J} S_u \leq S_t - \sum_{u \in K} S_u$ iff the following conditions hold: (I) $t \leq s$, (II) for any $u \in K$, $u \not\leq s$, and (III) for all $u \in K$, $s < u \rightarrow$ there is $v \in J$, $v \leq u$. (b) If J is a set of successors to s, then $S_s - \sum_{u \in J} S_u \leq - \sum_{u \in K} S_u$ iff (II) and (III) hold. (c) If K is a set of successors to t, then $- \sum_{u \in J} S_u \leq S_t - \sum_{u \in K} S_u$ iff either $\sum_{u \in J} S_u = 1$ or else the following conditions hold: (A) t is a root, (B) $\{r \in T : r \text{ is a root}, r \neq t\}$ is a subset of J, (C) for all $u \in K$, there is $v \in J$, $v \leq u$. (d) $- \sum_{u \in J} S_u \leq - \sum_{u \in K} S_u$ iff for all $u \in K$, there is $v \in J$, $v \leq u$.

(iii) If $b \leq \sum_{u \in J} S_u$ for some $J \in [T]^{<\omega}$, and $b = \sum_{i \leq m} f_i = \sum_{j \leq n} g_j$ are two normal forms for b, then $n = m$, $g_m = f_m = 0$, and $\{g_i : i < m\} = \{f_i : i < m\}$.

PROOF. (i). We only need to show ($\rightarrow$). Say $f_i = S_{t_i} - \sum_{s \in J_i} S_s$ for all $i < m$, $f_m = 0$, $g_i = S_{u_i} - \sum_{s \in K_i} S_s$ for all $i < n$, $g_n = - \sum_{s \in K_n} S_s$ or $g_n = 0$, with all conditions of the normal form lemma satisfied. Suppose there is $i \leq m$, for all $j \leq n$, $f_i \not\leq g_j$. Thus $i < m$. Now $t_i \in f_i \leq b \leq c$, so $t_i \in g_j$ for some $j \leq n$. Since $f_i \not\leq g_j$, we can choose $w \in f_i - g_j$ of lowest level. Since $t_i \leq w$ and $t_i \in g_j$, it follows that $w \in S_s$ for some $s \in K_j$. Now

(1) $s = w$

For, assume $s < w$. Then $t_i \leq w$, $s < w$, so t_i and s are comparable. Since $t_i \in g_j$ and $s \in K_j$, we have $t_i < s$. Thus $s \in f_i - g_j$, contradicting the choice of w.

Now $w \in f_i \leq b \leq c$, so choose $k \leq n$ so that $w \in g_k$. Recall that $w \notin g_j$, so $k \neq j$. Also, $u_k \leq w$, and by 1.4(v), $u_k \neq w$.

CASE 1. $j,k < n$. So $u_k < w$ and $u_j < w$. It follows that $u_k \leq u_j$ - hence $u_j \in g_k$, $g_j \cdot g_k \neq 0$, a contradiction; or $u_j < u_k$ - hence $u_k \in g_j$, $g_j \cdot g_k \neq 0$, again a contradiction.

CASE 2. $j < n$, $k = n$. Since $t_i \leq w \in g_k$, we also have $t_i \in g_k$, so $t_i \in g_j \cdot g_k$, a contradiction.

CASE 3. $j = n$, $k < n$. Then $u_k \in g_k \cdot g_n$, a contradiction.

It is straightforward to verify (ii). Condition (iii) is immediate from (i).

We note that the hypothesis $b \leq \sum_{t \in J} S_t$ for some $J \in [T]^{<\omega}$ is necessary in 1.5(i) and 1.5(iii). For example, if T has infinitely many roots $r_0, r_1, \ldots$, then $b = c = T = S_{r_0} + -S_{r_0} = S_{r_1} + -S_{r_1}$ are two normal forms that violate 1.5(i) and 1.5(iii).

COROLLARY 1.6. (i) For any $b \in B_T$, b is an atom iff there is $t \in T$, A_t is finite and $b = S_t - \sum_{s \in A_t} S_s$.

(ii) $\{S_t : t \in T, A_t \text{ is infinite}\} \cup \{b \in B_T : b \text{ is an atom}\}$ is dense in B_T.

COROLLARY 1.7. B_T is atomless iff for all $t \in T$, A_t is infinite. B_T is atomic iff for all $t \in T$, there is $s \geq t$, A_s is finite.

It can be shown that B_T is hereditarily atomic iff T does not contain a subtree T' with exactly one root, height ω, such that $|A_t| = \omega$ for all $t \in T'$.

The following result is frequently useful.

THEOREM 1.8. For any tree T there is a tree T' with a single root such that $B_T \cong B_{T'}$.

PROOF. CASE 1. T has a finite set R of roots. Fix $r \in R$. Let $T' = T$, and

$$\leq' = \{(x,y) : x,y \in T' \text{ and either } x \leq y \text{ or } x = r\}$$

The identity on $\{S_t^{T'} : t \in T \setminus \{r\}\}$ extends to an isomorphism from $B_{T'}$ onto B_T.

CASE 2. T has an infinite set of roots. Fix $z \notin T$, and let $T' = T \cup \{z\}$,

$$\leq' = \{(x,y) : x,y \in T' \text{ and either } x \leq y \text{ or } x = z\}$$

The identity on $\{S_t^T : t \in T\}$ extends to an isomorphism from B_T onto $B_{T'}$.

THEOREM 1.9. $B_T \upharpoonright S_t = B_{S_t}$.

THEOREM 1.10. If T and T' are trees and $T' \subseteq T$, then $B_{T'}$ embeds in B_T.

PROOF. First suppose that T' contains all the roots of T, or T' has infinitely many roots. In this case the map $S_t^{T'} \mapsto S_t^T$, $t \in T'$, extends to an embedding.

Second, suppose T' has only finitely many roots $t_1, \ldots, t_m$. For all $i \leq m$, let $S_i = S_{t_i}^T$ and $R_i = S_{t_i}^{T'}$. Now by the first case, B_{R_i} embeds in B_{S_i} for all i, so

$$\begin{aligned} B_{T'} &\cong (B_{T'} \restriction R_1) \times \cdots \times (B_{T'} \restriction R_m) \\ &= B_{R_1} \times \cdots \times B_{R_m} \end{aligned}$$

embeds in

$$\begin{aligned} B_{S_1} \times \cdots \times B_{S_m} &= (B_T \restriction S_1) \times \cdots \times (B_T \restriction S_m) \\ &\cong B_T \restriction (S_1 + \cdots + S_m) \end{aligned}$$

which, as is well-known, embeds in B_T.

2. CHAINS CONTAINED IN TREE ALGEBRAS. Some of our methods here are adapted from McKenzie, Monk [MM].

THEOREM 2.1. Let κ be uncountable and regular, and suppose C is a chain of cardinality κ in B_T. Then there is a $W \in [C]^\kappa$ which is well-ordered or inversely well-ordered, and T has a chain of type κ.

PROOF. By 1.8 and its proof, we may assume that T has only one root. We say that $b \in B_T$ has <u>wedge - size</u> n iff in the normal form of 1.4 we have $b = \sum_{i \leq n} f_i$ (recall also that $f_n = 0$ in our case). It suffices to show by induction on n that the conclusion of the theorem holds when all members of C have wedge-size n.

First suppose that n = 1. Say $b = S_{t_b} - \sum_{s \in J_b} S_s$, for each $b \in C$, in normal form. By 1.5(ii), if $b, c \in C$ and $b < c$, then $t_c \leq t_b$.

CASE 1. There is $C' \in [C]^\kappa$, for all $b, c \in C'$, $t_b = t_c$. Say $t_b = t$, for all $b \in C'$. Then by 1.5(ii) again, we have

(1) if $b, c \in C'$ and $b < c$, then for all $s \in J_c$, there is $u \in J_b$, $u \leq s$.

Now we choose $C'' \in [C']^\kappa$ so that $(J_b : b \in C'')$ forms a Δ-system, say with kernel K, and $J_b \neq K$ for all $b \in C''$. Then (1) clearly extends to

(2) if $b,c \in C''$ and $b < c$, then for all $s \in J_c \setminus K$, there is $u \in J_b \setminus K$, $u < s$.

Now we claim

(3) There is $f \in \prod_{b \in C''} (J_b \setminus K)$, for all $b,c \in C''$, $b < c \to fb < fc$.

This follows from

(4) For all $n \in \omega$, for all $b \in {}^nC''$, ($b_0 < \ldots < b_{n-1} \to$ There is $c \in \prod_{i \in n} (J_{b_i} \setminus K)$ such that $c_0 < \ldots < c_{n-1}$).

Condition (4) is clear from (2). One can derive (3) from (4) by using the Tychonoff product theorem or the compactness theorem. By (3), C'' is a chain of type $\geqq \kappa$ in B_T and $\{fb : b \in C''\}$ is a chain of type $\geqq \kappa$ in T.

CASE 2. Otherwise, there is a $C' \in [C]^\kappa$ such that for all $b,c \in C'$, $b < c \to t_c < t_b$. Hence C' is inversely well-ordered, and $\{t_b : b \in C'\}$ is a chain of type $\geqq \kappa$ in T.

Now suppose inductively that $n > 1$. Say $b = \sum_{i < n} f_i^b$, $f_i^b = s_{t_{bi}} - \sum_{s \in J_{bi}} s_s$ in normal form, for each $b \in C$. By 1.5(i) we have:

If $b,c \in C$ and $b < c$, then for all $i < n$, there is $j < n$, $f_i^b \leqq f_j^c$.

Hence by the Tychonoff product theorem, there is an $x : C \to n$ such that for all $b,c \in C$, if $b < c$ then $f_{xb}^b \leqq f_{xc}^c$. Without loss of generality, $xb = 0$ for all $b \in C$. Thus $f_0^b \leqq f_0^c$ whenever $b,c \in C$ and $b < c$.

CASE 1. There is $C' \in [C]^\kappa$, for all $b,c \in C'$, $f_0^b = f_0^c$. For all $b \in C'$ let $b' = b - f_0^b$. Thus

(5) if $b,c \in C'$ then ($b < c$ iff $b' < c'$).

Now $\{b' : b \in C'\}$ is a chain of cardinality κ, all members of which have wedge-size $n-1$. By the induction hypothesis, there is a $C'' \in [C']^\kappa$ such that $\{b' : b \in C''\}$ is well-ordered or inversely well-ordered, and T has a chain of type κ. By (5), C'' itself is well-ordered or inversely well-ordered.

CASE 2. Otherwise, there is $C' \in [C]^\kappa$, for all $b,c \in C'$, $b < c \to f_0^b < f_0^c$. By the case $n = 1$, there is a $C'' \in [C']^{\kappa}$ such that $\{f_0^b : b \in C''\}$ is well-ordered or inversely well-ordered, and T has a chain of type κ. Clearly C'' is itself well-ordered or inversely well-ordered.

COROLLARY 2.2. Let κ be regular and uncountable. Then the following conditions are equivalent:

(i) T has no branch of type $\geqq\kappa$.

(ii) B_T has no chain of cardinality κ.

COROLLARY 2.3. If T is an infinite tree, then length B_T = depth B_T = $\sup\{|X| : X \text{ is a branch in } T\}$.

PROOF. Clearly $\sup\{|X| : X \text{ is a branch in } T\}$ = depth $B_T \leqq$ length B_T. If the corollary is false, then B_T has a chain of size $(\sup\{|X| : X \text{ is a branch in } T\})^+$, contradicting 2.1.

PROPOSITION 2.4. If κ is singular, then there is a tree T of cardinality κ and height κ which has no branch of size κ, but B_T has a chain C of order type κ and a chain D of size κ with no well-ordered or inversely well-ordered subset of D of size κ.

PROOF. Let $(\lambda_\alpha : \alpha < \text{cf } \kappa)$ be a strictly increasing sequence of infinite cardinals with supremum κ. Let $T = \{t_{\alpha\beta} : \alpha < \text{cf } \kappa,\ \beta < \lambda_\alpha\}$ and define

$$t_{\alpha\beta} \leqq t_{\alpha'\beta'} \text{ iff } (\alpha = \alpha' \text{ and } \beta \leqq \beta') \text{ or } (\alpha \leqq \alpha' \text{ and } \beta = 0).$$

Clearly T contains no branch of length κ, $|T| = \kappa$, and T has height κ. Let $C = \{S_{t_{\alpha\beta}} + S_{t_{\alpha+1,0}} : \alpha < \text{cf } \kappa,\ \beta < \lambda_\alpha\}$, $D = \{S_{t_{\alpha+1,0}} + (S_{t_{\alpha,0}} - S_{t_{\alpha,\beta}}) : \alpha < \text{cf } \kappa,\ 0 < \beta < \lambda_\alpha\}$. It is easily checked that C is inversely well-ordered in type κ, and D satisfies the desired conditions.

One might ask about the very existence of a non-well-ordered chain in B_T, since no such exists in T. If for all $t \in T$, $|A_t| \geqq \omega$, then B_T is atomless, and so η is embeddable in B_T. However, this is not a necessary condition. For example, let T be the full binary tree of height ω (T has one root, height ω, and $|A_t| = 2$ for every

$t \in T$). It is not hard to see that T', the full ω-tree of height ω is embeddable in T; see, e.g. Rabin [R] (T' has one root, height ω, and $|A_t| = \omega$ for all $t \in T'$). Hence $B_{T'}$, which is atomless, is embeddable in B_T by 1.10, so η is embeddable in B_T. See the comment following corollary 1.7. In this connection we have:

PROPOSITION 2.5. Let T be a splitting tree, and let $\omega \leq \kappa < \min\{\lambda : T$ has no branch of power $\lambda\}$. Then in B_T there is a dense chain of size κ.

PROOF. For all $t \in T$, let $R_t \subseteq B_T \upharpoonright S_t$ be a chain of type η. Let C be an initial chain of limit type in T of size κ. For every $t \in C$ let s_t be an immediate successor to t which is different from the immediate successor u_t to t in C. Set $D = \{S_{u_t} + r : t \in C,\ r \in R_{s_t}\}$. It is easily checked that D is as desired.

3. THE RELATIONSHIP OF TREE ALGEBRAS TO INTERVAL ALGEBRAS.

THEOREM 3.1. For any T, B_T embeds in an interval algebra.

PROOF. By theorem 1.8 we may assume that T has a single root, and by 1.10 that T is limit-normal and infinitely splitting.

For all $t \in T$, let $\leqq_t$ well-order A_t in a type with last element. Let $L = \{b : b$ is a branch in $T\}$. We define an ordering $\leqq_L$ on L as follows: for $b = c$, $b \leqq_L c$; for $b \neq c$, since T is limit-normal and has one root, the first ordinal ξ where $b(\xi) \neq c(\xi)$ is a successor, say $\xi = \zeta + 1$. We let $b \leqq_L c$ iff $b(\xi) \leqq_{b(\zeta)} c(\xi)$.

The map $S_t \mapsto [b_t^{min}, b_t^{max})$ for $t \in T$, where b_t^{min} and b_t^{max} are, respectively, the $\leqq_L$-minimal and $\leqq_L$-maximal elements of $\{b : b$ is a branch in T and $t \in b\}$, is easily seen to extend to an embedding of B_T into the interval algebra on L.

A BA B is <u>retractive</u> iff for all homomorphisms f mapping B onto A, there is a one-one homomorphism g mapping A into B such that $f \cdot g$ is the identity on A. Rubin [Ru] contains a proof that if A is a subalgebra of an interval algebra, then A is retractive. So 3.1 has the following immediate corollary.

COROLLARY 3.2. If A is a tree algebra, then A is retractive.

Now we show that not every interval algebra embeds in a tree algebra.

PROPOSITION 3.3. For X an uncountable subset of the real numbers, Int(X), the interval algebra on X, cannot be embedded in any tree algebra.

PROOF. Given X as above, clearly Int(X) contains a chain C of power ω_1. Suppose for contradiction that Int(X) embeds in B_T for some T. Let D be the image of C under the embedding. By theorem 2.1, there is $D' \subseteq D$ such that D' has order type ω_1 or $\omega_1{}^*$ (the order type of ω_1 under $\geq$). So the preimage of D' in Int(X) has order type ω_1 or $\omega_1{}^*$, which as is well-known cannot occur.

Next we exhibit tree algebras that are not isomorphic to any interval algebra. The finite-cofinite algebra on any cardinal κ (consisting of all finite or cofinite subsets of κ) is isomorphic to the tree algebra generated by a tree of κ roots. In [MM] it is shown that no finite-cofinite algebra contains a chain of type $\omega + \omega$. Thus for any uncountable κ, the finite-cofinite algebra is not isomorphic to any interval algebra. We use theorem 2.1 to obtain atomless examples.

PROPOSITION 3.4. For any regular uncountable κ, if T is a tree satisfying

(1) $|T| = \kappa$,

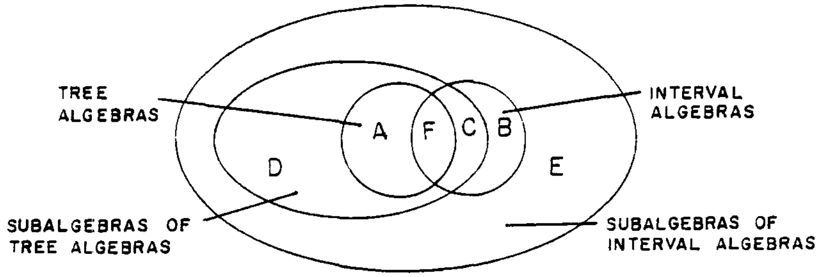
(2) height $T < \kappa$,

(3) for all $t \in T$, A_t is infinite,

then B_T is atomless and is not isomorphic to any interval algebra.

PROOF. Suppose $B_T \cong \text{Int}(L)$ for some L. By 2.1 and condition (2) on T, B_T does not contain a chain of cardinality κ. However, $\kappa = |T| = |B_T| = |\text{Int}(L)| = |L|$. So Int(L) contains a chain of cardinality κ.

Theorem 3.1, proposition 3.3 and proposition 3.4 yield the following diagram.



Proposition 3.4 gives examples that lie in region A. Proposition 3.3 gives examples that lie in region B. Examples that lie in region D are given in the first author's dissertation. These results will appear in a forthcoming paper.

The denumerable atomless BA is isomorphic to the tree algebra generated from any infinitely splitting countable tree. It is an example that lies in region F.

Finally, for any uncountable cardinal κ, if A is the algebra obtained from κ as in proposition 3.4 and B is the interval algebra on the reals, then $A \times B$ is atomless and lies in region E (If A is the finite-cofinite algebra on κ, then $A \times B$ is an atomic example.). Only region C may possibly be empty. Thus we ask the following question.

QUESTION. Is there a BA which is isomorphic to an interval algebra and to a subalgebra of a tree algebra but is not isomorphic to any tree algebra?

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