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A VERY RIGID BOOLEAN ALGEBRA

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ABSTRACT

A Boolean algebra is constructed having only those endomorphisms corresponding to prime ideals, which are present in any BA. The BA constructed is of power c, has 2^c endomorphisms, and is not rigid in Bonnet's sense.

Rigid Boolean algebras — those without non-identity automorphisms — have been extensively studied (see van Douwen, Monk and Rubin [2] for a survey). In general algebra a stronger rigidity — no non-identity endomorphisms — has been studied (see, e.g., Hedrlin, Pultr and Vopenka [4]). This notion does not apply to Boolean algebras, since, e.g., non-identity endomorphisms exist corresponding to any maximal ideal (see below). But one can describe the endomorphisms inevitably present in any BA (we call them *simply definable endomorphisms*), and try to construct a BA in which these are the only endomorphisms. That is the main purpose of this article.

The paper is a sequel to Shelah [8], but is self-contained. In [8] the existence of a BA in which every endomorphism is simply definable (called henceforth a prime-rigid BA) was proved assuming $\diamondsuit_{\aleph_1}$. Later (unpublished) Shelah replaced $\diamondsuit_{\aleph_1}$ by CH, while the author established the existence of a BA with only definable endomorphisms (a weaker notion) in ZFC. This last construction was simplified by Shelah, and then the author saw how to modify and extend it to establish the existence of prime-rigid BA's in ZFC. Without these communications with Shelah the author would not have accomplished this work. In addition to Saharon Shelah the author is grateful for useful communications with Robert Bonnet, Neil Endsley, Sabine Koppelberg, Richard Laver, Matatyahu Rubin, and the referee.

To state the main result precisely we need several definitions. Two ideals I, J

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in A are *complementary* if $I \cap J = \{0\}$ while $I \cup J$ generates a maximal ideal of A. We call A *indecomposable* if A has no pair of non-principal complementary ideals. If f is an endomorphism of A we let $\ker f = \{a \in A : fa = 0\}$, $\operatorname{fix} f = \{a \in A : \text{for all } x \leqq a, fx = x\}$, and $\operatorname{exker} f = \{a + b : a \in \ker f, b \in \operatorname{fix} f\}$. We say that f is *definable* if $A/\operatorname{exker} f$ is finite. Finally, *A has only simply definable endomorphisms*, or is *prime-rigid*, if A is atomless, indecomposable, and has only definable endomorphisms. We give further information and background on these notions later. The main result is that there is a prime-rigid BA of power $2^{\aleph_0}$. We actually prove a stronger theorem, Theorem 12, which also gives some information about a weaker kind of rigidity in other powers. We begin with a discussion of indecomposability, and then give a more intuitive but more involved equivalent definition of prime-rigidity. Then we discuss the notion of a complicated BA, which plays an important technical role in the final construction. Most of the notions and results up to this point are in [8], perhaps in an implicit or less general form. Finally, we give the main construction, important corollaries, and mention some open problems. Some of the results stated are not needed for the main results, but are included for background.

Indecomposable BA's

It is easily checked that any complete BA is indecomposable. On the other hand, we have:

THEOREM 1. *If A is the denumerable atomless* BA, *then any maximal ideal in A is generated by the union of two non-principal complementary ideals. In particular, A is decomposable.*

PROOF. Let $\langle x_i : i < \omega \rangle$ be a free generating system for A. By the homogeneity of A it is enough to prove the theorem for the maximal ideal J generated by $\{x_i : i < \omega\}$. For each $i < \omega$ let

$$y_i = x_i \cdot \prod_{j<i} - x_j,$$

and let I_0 and I_1 be the ideals generated by $\{y_{2i} : i < \omega\}$ and $\{y_{2i+1} : i < \omega\}$ respectively. It is easily checked that I_0 and I_1 are complementary and that their union generates J.

We can extend this theorem as follows to higher cardinals.

THEOREM 2. *Assume Martin's axiom, and let B be a* BA *of power $< 2^{\aleph_0}$ which has a denumerable dense atomless subalgebra A. Then B is decomposable.*

PROOF. Let J be a maximal ideal in B which preserves all joins $b = \Sigma\{a \in A : a \leqq b\}$, for $b \in B$. Let $K = J \cap A$. Since K is a maximal ideal in A, by Theorem 1 let I_0 and I_1 be non-principal complementary ideals in A whose union generates K. Let I_0' and I_1' be the ideals in B generated by I_0 and I_1 respectively. Clearly I_0' and I_1' are non-principal, $I_0' \cap I_1' = \{0\}$, and $I_0' \cup I_1' \subseteq J$. Now let $b \in J$. Then $-b \notin J$ so, since $-b = \Sigma\{a \in A : a \leqq -b\}$ is preserved by J, we can choose $a \in A$ with $a \leqq -b$ and $a \notin J$. Say $-a = c + d$ with $c \in I_0$ and $d \in I_1$. Then $b = b \cdot c + b \cdot d$. Thus J is generated by $I_0' \cup I_1'$, as desired.

If A is a BA and X a set, we say that a BA $A(X)$ is a *free extension* of A by X if $A \cap X = 0$, $A(X)$ is generated by $A \cup X$, and for any BA B, any homomorphism f from A into B and any function g from X into B there is a homomorphism h of $A(X)$ into B which extends both f and g. The following theorem was independently found by P. Nyikos.

THEOREM 3. *Let $A(X)$ be a free extension of A, with X uncountable. Then $A(X)$ is indecomposable.*

PROOF. Suppose that I_0 and I_1 are non-principal complementary ideals in $A(X)$, and let J be the maximal ideal which their union generates. We may assume that $x \in J$ for all $x \in X$; say $x = y_x + z_x$ with $y_x \in I_0$ and $z_x \in I_1$. Since $y_x, z_x \leqq x$, we can write $y_x = x \cdot u_x$ and $z_x = x \cdot v_x$, where u_x and v_x are in the subalgebra generated by $A \cup (X - \{x\})$. Thus $x = x \cdot (u_x + v_x)$, so by the freeness of x, $u_x + v_x = 1$. But also $x \cdot u_x \cdot v_x = y_x \cdot z_x = 0$, so $u_x \cdot v_x = 0$. Thus $v_x = -u_x$. Now for each $x \in X$ let sx be a finite subset of $X - \{x\}$ such that u_x is in the subalgebra generated by $A \cup sx$. By Lázár [6], let Y be an infinite subset of X such that $x \notin sy$ for all $x, y \in Y$. Now for distinct $x, y \in Y$ we have $x \cdot u_x \cdot y \cdot -u_y = 0$, so, since x and y are free, $u_x \cdot -u_y = 0$. Hence $u_x = u_y$ for all $x, y \in Y$. Set $u_x = w$ for all $x \in Y$. Without loss of generality say $w \in J$. Hence write $w = s + t$, with $s \in I_0$ and $t \in I_1$. Choose s' with $s < s' \in I_0$. Choose $x \in Y$ so that s' is in the subalgebra generated by $A \cup (X - \{x\})$. Now $x \cdot s' \cdot -w \in I_0 \cap I_1$, so $x \cdot s' \cdot -w = 0$ and, by the freeness of x, $s' \cdot -w = 0$. But then $s' = s$, a contradiction.

To end our discussion of indecomposability let us show its relevance for the discussion of endomorphisms. Suppose that I and J are complementary ideals in A. Then, as is easily seen, $I \cup -I$ is a subalgebra of A isomorphic to A/J, where $-I = \{a \in A : -a \in I\}$. Hence the natural mappings $A \to A/J \to I \cup -I \subseteq A$ compose to give an endomorphism of A. Therefore, if we construct an indecomposable BA, it automatically fails to have this kind of endomorphism.

Prime-rigid Boolean algebras

The following theorem gives an indication of the strength of having only definable endomorphisms.

THEOREM 4. *If A is atomless and has only definable endomorphisms, then every one-one and every onto endomorphism of A is the identity.*

PROOF. First we take the case of a one-one endomorphism f. Suppose f is not the identity. Then there is a $y \neq 0$ with $y \cdot fy = 0$. For any distinct $u, v \leqq y$ we have $u/\text{exker} f \neq v/\text{exker} f$, so $A/\text{exker} f$ is infinite, contradiction.

Now suppose that f is an onto endomorphism different from the identity. Then f is not one-one, so $\ker f \neq \{0\}$; choose $0 \neq a \in \ker f$. For each $b \leqq a$ choose $c_b \in A$ with $fc_b = b$. Then for distinct $b, b' \leqq a$ we have $c_b/\text{exker} f \neq c_{b'}/\text{exker} f$. In fact, suppose $c_b/\text{exker} f = c_{b'}/\text{exker} f$ while $b \neq b'$, say $b \cdot - b' \neq 0$. Write $c_b \cdot - c_{b'} = d + e$ with $d \in \ker f$, $e \in \text{fix} f$. Applying f we get $a \geqq b \cdot - b' = e$, so $e \in \ker f \cap \text{fix} f$ and so $e = 0$. Thus $b \cdot - b' = 0$, contradiction. Thus again we have shown that $A/\text{exker} f$ is infinite, contradiction.

Now we want to give a more intuitive version of the notion of prime-rigid BA. Given BA's A, B and a maximal (prime) ideal I in A, the natural homomorphism $A \to A/I$ can be considered to be a homomorphism from A into B. For $A = B$, this gives an endomorphism of A (and shows, incidentally, that any BA has at least as many endomorphisms as maximal ideals). More complicated endomorphisms can be obtained, e.g., in the following way. For any $a \in A$ let $A \restriction a$ be the BA $\{x \in A : x \leqq a\}$. Let a, b, c, d be four pairwise disjoint non-zero elements of A with sum 1. Let I be a maximal ideal in $A \restriction d$. Then we can obtain an endomorphism of A as follows:

$$\begin{aligned} A &\simeq (A \restriction a) \times (A \restriction b) \times (A \restriction c) \times (A \restriction d) \\ &\to (A \restriction b) \times (A \restriction d) \\ &\to (A \restriction a) \times (A \restriction b) \times (A \restriction c) \times (A \restriction d) \\ &\cong A, \end{aligned}$$

where the first map is the natural isomorphism, the second one is projection ($\langle w, x, y, z \rangle$ goes to $\langle x, z \rangle$), the third one takes $\langle x, z \rangle$ to $\langle z/I, x, z/I, z \rangle$ (in a natural sense), and the fourth is again the natural isomorphism. If we analyze this situation carefully we can arrive at the following notion (the notion can be given various equivalent formulations; see, for example, Loats and Rubin [7] for another interesting one).

A *schema for a prime endomorphism* in A is a sequence

$$\langle a_0, a_1, b_0, \cdots, b_{m-1}, c_0, \cdots, c_{n-1}, b_0^*, \cdots, b_{m-1}^*, c_0^*, \cdots, c_{n-1}^*, I_0, \cdots, I_{m-1}, J_0, \cdots, J_{n-1}\rangle$$

such that the following conditions hold:

(1) $a_0, a_1, b_0, \cdots, b_{m-1}, c_0, \cdots, c_{n-1}$ are pairwise disjoint elements with join 1; $b_i \neq 0 \neq c_j$ for all $i < m$, $j < n$;

(2) $b_0^*, \cdots, b_{m-1}^*, c_0^*, \cdots, c_{n-1}^*$ are pairwise disjoint elements with join $a_0 + b_0 + \cdots + b_{m-1}$; $b_i^* \neq 0 \neq c_j^*$ for all $i < m$, $j < n$;

(3) for all $i < m$, I_i is a maximal ideal in $A \restriction b_i$;

(4) for all $j < n$, J_j is a maximal ideal in $A \restriction c_j$.

THEOREM 5. *Given a schema for a prime endomorphism as above, there is a unique endomorphism f of A with the following properties:*

(i) *for all $x \leqq a_0$, $fx = 0$; for all $i < m$ and for all $x \in I_i$, $fx = 0$;*

(ii) *for all $x \leqq a_1$, $fx = x$; for all $j < n$ and for all $x \in J_j$, $fx = x$;*

(iii) *for all $i < m$, $fb_i = b_i^*$, and for all $j < n$, $fc_j = c_j + c_j^*$.*

PROOF. The existence of f is seen from the following diagram:

$$\begin{aligned}
A &\cong (A \restriction a_0) \times (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i) \times \prod_{j<n} (A \restriction c_j) \\
&\to (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i) \times \prod_{j<n} (A \restriction c_j) \\
&\qquad \downarrow \text{identity} \qquad \downarrow \text{ideals } I_i \qquad \downarrow \text{ideals } J_i \\
&\qquad (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i^*) \times \prod_{j<n} (A \restriction c_j) \times (A \restriction c_j^*) \\
&\cong A.
\end{aligned}$$

($A \restriction b_i \to A \restriction b_i^*$ via $x \mapsto x/I_i$, while $A \restriction c_j \to (A \restriction c_j) \times (A \restriction c_j^*)$ via $x \mapsto \langle x, x/J_j\rangle$.) The uniqueness of f is clear, since the conditions (i)–(iii) uniquely determine f on each factor $A \restriction a_0, A \restriction a_1, A \restriction b_0, \cdots, A \restriction b_{m-1}, A \restriction c_0, \cdots, A \restriction c_{n-1}$.

In the situation of Theorem 5 we say that f is *simply defined* by the given schema.

LEMMA 6. *Let I and J be non-principal complementary ideals in an atomless* BA A, *and let f be the endomorphism of A obtained as described after the proof of Theorem* 3. *Then f is not simply defined by a schema.*

PROOF. Suppose on the contrary that f is simply defined by a schema, with

notation as above. Note that $fa = 0$ for all $a \in J$ and $fa = a$ for all $a \in I$. Let K be the (maximal) ideal generated by $I \cup J$. Then

(1) for all $j < n$, $c_j \notin K$.

For, otherwise write $c_j = d + e$ with $d \in I$, $e \in J$. Then $fc_j = d \leqq c_j$, so $c_j^* = 0$, contradiction. Next,

(2) for all $i < m$, $b_i \notin K$.

For, otherwise write $b_i = d + e$ with $d \in I$, $e \in J$. Then $b_i^* = fb_i = d \leqq b_i$, and $d \neq 0$ since $b_i^* \neq 0$, while $fx = x$ for all $x \leqq d$. But we can choose $0 \neq u \leqq d$ with $u \in I_i$, and then $fu = 0$, contradiction.

Suppose $m > 0$. Then by (1) and (2), $m = 1$ and $n = 0$. Also, $-b_0 \in K$, so we can write $-b_0 = d + e$, with $d \in I$ and $e \in J$. Hence $e = a_0$, $d = a_1$, and $b_0^* = a_0 + b_0$. Now choose $d' \in I$ with $d < d'$. Thus $d' \cdot - d \cdot b_0 \neq 0$, so we can choose $0 \neq u \leqq d' \cdot - d \cdot b_0$ with $u \in I_0$. Then $0 = fu = u$, contradiction.

Thus $m = 0$, and similarly $n = 0$. It is then easy to see that $a_0, a_1 \in K$, so $1 \in K$, contradiction.

The following theorem expresses our more intuitive version of prime rigidity.

THEOREM 7. *For any atomless* BA *the following conditions are equivalent*:

(i) *A is prime-rigid*;

(ii) *every endomorphism of A is simply defined by a schema for a prime endomorphism.*

PROOF. (ii) $\Rightarrow$ (i). Assume (ii). By Lemma 6, A is indecomposable. Assume the notation of the definition of a schema. Then $A/\operatorname{exker} f$, if it has more than one element, is a finite BA whose atoms are $b_0/\operatorname{exker} f, \cdots, b_{m-1}/\operatorname{exker} f$, $c_0/\operatorname{exker} f, \cdots, c_{n-1}/\operatorname{exker} f$.

(i) $\Rightarrow$ (ii). Assume (i). Thus $A/\operatorname{exker} f$ is finite. If $\operatorname{exker} f = A$, then $\ker f = \{0\}$ and $\operatorname{fix} f = A$, so f is the identity, defined by the schema $\langle 0, 1\rangle$. Now assume $\operatorname{exker} f \neq A$. Then there are pairwise disjoint elements $x_0, \cdots, x_{\kappa-1}$ of A with join 1 such that $x_0/\operatorname{exker} f, \cdots, x_{\kappa-1}/\operatorname{exker} f$ are the distinct atoms of $A/\operatorname{exker} f$. Each x_κ will give rise to from one to three terms of the desired schema, as follows (fix k). Let $I_0 = (\ker f) \cap (A \restriction x_\kappa)$, $I_1 = (\operatorname{fix} f) \cap (A \restriction x_\kappa)$, $J = (\operatorname{exker} f) \cap (A \restriction x_\kappa)$. Then I_0 and I_1 are complementary ideals in $A \restriction x$, their union generates J, and J is a maximal ideal in $A \restriction x_\kappa$. If I_0 and I_1 are non-principal, let $I_1' = \{a \in A : a \cdot x_\kappa \in I_1\}$. Then I_0 and I_1' are non-principal, complementary ideals in A, contradiction. So, I_0 and I_1 are not both non-principal. Also they are not both principal, since otherwise J would also be principal, which would imply the existence of an atom in $A \leqq x_\kappa$. Thus we have two cases:

Case 1. I_1 is principal, say generated by e. Note that $x_\kappa \cdot - e \neq 0$. We let e be a part of a_1, while $x_\kappa \cdot - e$ is to be a term b_l. Further, $b_l^* = fb_l$ and $I_l = (\operatorname{exker} f) \cap (A \restriction x_\kappa \cdot - e)$.

Case 2. I_0 is principal, say generated by e. Clearly

(1) for all $y \leqq x_\kappa \cdot - e$, if $y \in \operatorname{exker} f$ then $y \in I_1$. Also we claim

(2) $x_\kappa \cdot - e \leqq f(x_\kappa \cdot - e)$.

For, otherwise let $y = x_\kappa \cdot - e \cdot - f(x_\kappa \cdot - e)$. Then $y \cdot fy = 0$, so $y \notin I_1$ and hence $y \notin \operatorname{exker} f$ by (1). Hence $x_\kappa \cdot - y \in \operatorname{exker} f$, so by (1) $x_\kappa \cdot - y \cdot - e \in I_1$. Choose z such that $x_\kappa \cdot - y \cdot - e < z \in I_1$. Hence $z \leqq x_\kappa \cdot - e$ and $fz = z$, so

$$z \leqq x_\kappa \cdot - e \cdot f(x_\kappa \cdot - e) = x_\kappa \cdot - e \cdot - y,$$

contradiction. Thus (2) holds. Further, clearly $x_\kappa \cdot - e \neq 0$. Furthermore,

(3) $f(x_\kappa \cdot - e) \cdot - (x_\kappa \cdot - e) \neq 0$.

For, otherwise

(4) for all $y \leqq x_\kappa \cdot - e$, $fy = y$.

In fact, let $y \leqq x_\kappa \cdot - e$. If $y \in \operatorname{exker} f$, then $fy = y$ by (1). If $y \notin \operatorname{exker} f$, then $x_\kappa \cdot - e \cdot - y \in \operatorname{exker} f$, so by (1) $x_\kappa \cdot - e \cdot - y \in I_1$ and hence

$$\begin{aligned} fy &= f(x_\kappa \cdot - e \cdot - (x_\kappa \cdot - e \cdot - y)) \\ &= x_\kappa \cdot - e \cdot - (x_\kappa \cdot - e \cdot - y) \\ &= y. \end{aligned}$$

So (4). But by (4), $x_\kappa \cdot - e \in I_1$, so $x_\kappa \in \operatorname{exker} f$, contradiction. Hence (3) holds.

In this case we let e be a part of a_0, $x_\kappa \cdot - e$ a term c_l, and by (2) and (3) we write $fc_l = c_l + c_l^*$. Finally, we let $J_l = (\operatorname{exker} f) \cap A \restriction c_l$.

The various desired conditions in a schema and in Theorem 5 are now easily verified.

The following lemma is needed in the next section.

LEMMA 8. *If f is an endomorphism of a* BA A, $a \in A$, *and for all $x \leqq a$ we have $fx \leqq x$, then $a \in \operatorname{exker} f$.*

PROOF. Let $x \leqq a$. Then $f(x \cdot - fx) \leqq x \cdot fx$, and obviously $f(x \cdot - fx) \leqq fx$. Hence $f(x \cdot - fx) = 0$. Therefore,

(1) for all $x \leqq a$ we have $x \cdot - fx \in \ker f$.

Again, $x \leqq a$ implies $fx \leqq ffx$ by (1), and it implies $fx \leqq a$ and hence $ffx \leqq fx$, by hypothesis. Hence

(2) for all $x \leqq a$, $fx = ffx$.

Now we claim

(3) $fa \in \operatorname{fix} f$.

For, suppose $x \leqq fa$; we want to show $fx = x$. We have, using (2) and the fact that $x \cdot - fx \leqq fa$,

$$\begin{aligned} fa = ffa &= f(fa \cdot - (x \cdot - fx) + x \cdot - fx) \\ &= f(fa \cdot - (x \cdot - fx)) + f(x \cdot - fx) \\ &= f(fa \cdot - (x \cdot - fx)) \qquad \text{by (1)} \\ &\leqq fa \cdot - (x \cdot - fx) \qquad \text{by hypothesis.} \end{aligned}$$

Hence $fa \leqq -(x \cdot - fx)$. Since $x \cdot - fx \leqq fa$, it follows that $x \cdot - fx = 0$. Thus $x \leqq fx$, and $fx \leqq x$ by hypothesis. Hence $x = fx$, and we have established (3). By (1), (3) we have $a = (a \cdot - fa) + fa \in \operatorname{exker} f$.

Complicated BA's

Let A be a BA, L_A the first-order language for BA's enriched with constants for members of A. We shall consider 1-types in L_A i.e., sets of formulas in L_A with one free variable v. In fact we shall work only with very simple types. If $\langle a_\alpha : \alpha < \kappa \rangle$ is a sequence of elements of A, τ is a term in L_A in which, aside from the names of elements of A, only the variable v appears, and $S \subseteq \kappa$, by $[a_\alpha, \tau]^{\text{if } \alpha \in S}$ we mean the formula $a_\alpha \leqq \tau$ if $\alpha \in S$ and $a_\alpha \cdot \tau = 0$ if $\alpha \notin S$. A *standard κ-type over A* is a type of the form

$$\{[a_\alpha, v]^{\text{if } \alpha \in S} : \alpha < \kappa\},$$

where $\langle a_\alpha : \alpha < \kappa \rangle$ is a system of pairwise disjoint non-zero elements of A and $S \subseteq \kappa$.

A *κ-candidate over A* is a system $\langle (a_\alpha, b_\alpha) : \alpha < \kappa \rangle$ such that the a_α's are pairwise disjoint non-zero elements of A, as are the b_α's, and for all $\alpha < \kappa$, $b_\alpha \not\leqq a_\alpha$. Finally, we call A *κ-complicated* if for any such κ-candidate over A there is an $S \subseteq \kappa$ such that $\{[a_\alpha, v]^{\text{if } \alpha \in S} : \alpha < \kappa\}$ is realized in A but $\{[b_\alpha, v]^{\text{if } \alpha \in S} : \alpha < \kappa\}$ is omitted in A. The connection with the above rigidity notions is given in the following theorem.

THEOREM 9. *If A is ω-complicated, then all endomorphisms of A are definable.*

PROOF. Let f be an endomorphism of A, and suppose that $A/\operatorname{exker} f$ is infinite. Then there is a system $\langle a'_n : n \in \omega \rangle$ of pairwise disjoint non-zero

elements of A such that $a'_n/\ker f \neq 0$ for all n. By Lemma 8, for each $n \in \omega$ there is an $a_n \leqq a'_n$ with $fa_n \not\leqq a_n$. Thus $\langle (a_n, fa_n): n \in \omega \rangle$ is an ω-candidate over A. Hence we can choose $S \subseteq \omega$ so that $\{[a_n, v]^{\text{if } n \in S}: n \in \omega\}$ is realized in A, say by c, but $\{[fa_n, v]^{\text{if } n \in S}: n \in \omega\}$ is omitted in A. But clearly fc realizes this last type, contradiction.

The following theorem is needed in order to partially extend our main result to higher cardinals.

THEOREM 10. *Let A be an atomless κ-complicated* BA *with the property that for any non-zero $a \in A$ there is a system $\langle b_\alpha: \alpha < \kappa \rangle$ of non-zero pairwise disjoint elements of $A \restriction a$. Then every one-one endomorphism of A is the identity.*

PROOF. Suppose f is a one-one endomorphism of A which is not the identity. Then there is a non-zero $a \in A$ such that $a \cdot fa = 0$. Let $\langle b_\alpha: \alpha < \kappa \rangle$ be a system of non-zero pairwise disjoint elements of $A \restriction a$. Then $\langle (b_\alpha, fb_\alpha): \alpha < \kappa \rangle$ is a κ-candidate over A, which leads to a contradiction as above.

In the main theorem, Theorem 12, we establish the existence of κ-complicated BA's.

The main theorem

The following lemma is probably well-known.

LEMMA 11. *Let A satisfy the $(<\kappa)$-chain condition, i.e., suppose that every set of pairwise disjoint elements of A has power $<\kappa$. Let $A(X)$ be a free extension by X. Then $A(X)$ satisfies the $(<\kappa)$-chain condition.*

PROOF. By Erdös and Tarski [3] we can assume that κ is regular. Suppose $\langle b_\alpha: \alpha < \kappa \rangle$ is a system of pairwise disjoint non-zero elements of $A(X)$. We may assume that each b_α has the form $a_\alpha \cdot c_\alpha$, where $a_\alpha \in A$ and for some finite $Y_\alpha \subseteq X_\alpha$ and some $\varepsilon_\alpha \in {}^{Y_\alpha}2$, we have

$$c_\alpha = \prod_{x \in Y_\alpha} x^{\varepsilon_\alpha x}$$

(with $x^0 = -x$, $x^1 = x$). We may assume that $\langle Y_\alpha: \alpha < \kappa \rangle$ forms a Δ-system with kernel Z, and that $\varepsilon_\alpha \restriction Z = \varepsilon_\beta \restriction Z$ for all $\alpha, \beta < \kappa$. It then follows that $a_\alpha \cdot a_\beta = 0$ for $\alpha \neq \beta$, contradiction.

We also note that for any infinite cardinals κ and λ the following two conditions are equivalent:

(a) $\lambda = 2^\kappa = \kappa^{\aleph_0}$;

(b) $\lambda^\kappa = \lambda$ and there is a family F of λ denumerable subsets of κ with pairwise finite intersections.

In fact, (a) $\Rightarrow$ (b) is a well-known result of Tarski, and (b) $\Rightarrow$ (a) by the following computation:

$$2^\kappa \leqq \lambda^\kappa = \lambda = |F| \leqq \kappa^{\aleph_0} \leqq 2^\kappa.$$

Now we are ready for the main theorem.

THEOREM 12. *Assume that κ and λ are infinite cardinals such that $\lambda = 2^\kappa = \kappa^{\aleph_0}$. Then there is an atomless κ-complicated indecomposable* BA *C of power λ such that for any non-zero $a \in C$ there is a system $\langle b_\alpha : \alpha < \kappa \rangle$ of non-zero pairwise disjoint elements of $C \restriction a$.*

PROOF. Let $\langle\langle (a^\alpha_\beta, b^\alpha_\beta) : \beta < \kappa \rangle : \alpha < \lambda \rangle$ be a list all members of ${}^\kappa(\lambda \times \lambda)$, each member repeated λ times. One can easily construct a BA A of power κ such that for each non-zero $a \in A$ there is a system $\langle b_\alpha : \alpha < \kappa \rangle$ of non-zero pairwise disjoint elements of $A \restriction a$. Let $A(X)$ be a free extension of A with $|X| = \lambda$, and let $A(X)^*$ be the completion of $A(X)$. By Lemma 11, $A(X)^*$ satisfies the $(<\kappa^+)$-chain condition. We assume that $A(X)^* \subseteq \lambda$ as a set. Now we construct two sequences $\langle B_\alpha : \alpha \leqq \lambda \rangle$ and $\langle Q_\alpha : \alpha \leqq \lambda \rangle$ such that for all $\alpha, \beta < \lambda$,

(1) B_α is a subalgebra of $A(X)^*$, $|B_\alpha| \leqq |\alpha| + \kappa$, and $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta$;

(2) Q_α is a collection of standard κ-types over B_α omitted in B_α, and $|Q_\alpha| \leqq |\alpha| + \aleph_0$.

We let $B_0 = A$ and $Q_0 = 0$. For μ a limit ordinal $\leqq \lambda$ we let $B_\mu = \bigcup_{\alpha<\mu} B_\alpha$ and $Q_\mu = \bigcup_{\alpha<\mu} Q_\alpha$. The essential step is the successor step. So assume $\alpha < \lambda$, B_α and Q_α have been defined satisfying (1) and (2). The construction of $B_{\alpha+1}$ and $Q_{\alpha+1}$ takes two steps.

First we take care of a candidate, forming B'_α and Q'_α. If $\langle (a^\alpha_\beta, b^\alpha_\beta) : \beta < \kappa \rangle$ is not a κ-candidate over B_α, we let $B'_\alpha = B_\alpha$, $Q'_\alpha = Q_\alpha$. So assume it is. We drop the superscript α. Extend $\langle a_\beta : \beta < \kappa \rangle$ to a maximal pairwise disjoint system $\langle a_\beta : \beta < \gamma \rangle$; thus $\kappa \leqq \gamma < \lambda^+$. For each $S \subseteq \kappa$ let $cS = \Sigma_{\beta \in S} a_\beta$ (sum in $A(X)^*$), and set $B''_\alpha S = B_\alpha(cS)$ (simple extension within $A(X)^*$). We want to choose S so that

(3) $B''_\alpha S$ omits $p = \{[b_\beta, v]^{\text{if } \beta \in S} : \beta < \kappa\}$;

(4) $B''_\alpha S$ omits each member of Q_α.

Since cS realizes $\{[a_\beta, v]^{\text{if } \beta \in S} : \beta < \kappa\}$, this will take care of the current candidate. For each $\beta < \kappa$ choose $f\beta < \gamma$ so that $f\beta \neq \beta$ and $b_\beta \cdot a_{f\beta} \neq 0$; this is possible since $b_\beta \not\leqq a_\beta$. Now we claim

(5) there is an $S^* \subseteq \kappa$ with $|S^*| = \kappa$ such that for all $\beta \in S^*$, $f\beta \notin S^*$.
For, if for some $\Gamma \subseteq \kappa$ with $|\Gamma| < \kappa$ we have $f\beta \in \Gamma$ for all $\beta \in \kappa - \Gamma$, we can take $S^* = \kappa - \Gamma$. In the opposite case an easy inductive construction yields the desired set S^*.

The following fact will enable us to take care of (3).

(6) If d, e and g are pairwise disjoint elements of B_α such that $d + e \cdot cS + g \cdot - cS$ realizes p in $B''_\alpha S$, where $S \subseteq S^*$, then $S = \{\beta \in S^*: b_\beta \cdot a_{f\beta} \leqq d + g\}$.
Assume the hypothesis of (6). For any $\beta \in S^*$ we have $f\beta \notin S^*$ hence $f\beta \notin S$ and $a_{f\beta} \cdot cS = 0$. Thus $\beta \in S$ implies $b\beta \leqq d + e \cdot cS + g \cdot - cS$ and hence $b\beta \cdot a_{f\beta} \leqq d + g$. And $\beta \in S^* - S$ implies $b\beta \cdot (d + e \cdot cS + g \cdot - cS) = 0$ and so

$$b_\beta \cdot a_{f\beta} \cdot (d + g) = b_\beta \cdot a_{f\beta} \cdot (d + g \cdot - cS) = 0.$$

Thus (6) holds.

Now let K be a family of λ denumerable subsets of κ with pairwise finite intersections. The following will enable us to take care of (4).

(7) If d, e, g are pairwise disjoint elements of B_α, $q \in Q_\alpha$, $q = \{[h_\beta, v]^{\text{if } \beta \in T}: \beta < \kappa\}$, then there is at most one $S \in K$ such that $y_s = d + e \cdot cS + g \cdot - cS$ realizes q in $B''_\alpha S$.
Assume the hypotheses of (7). Let

$$k = - d \cdot - e \cdot - g,$$

$$l = \sum \{h_\beta \cdot (k + e): \beta \in T\} + \sum \{h_\beta \cdot (d + g): \beta \in \kappa \sim T\},$$

$$m = \sum \{h_\beta \cdot (k + g): \beta \in T\} + \sum \{h_\beta \cdot (d + e): \beta \in \kappa \sim T\}.$$

The following is easily verified:

(8) if $B_\alpha \subseteq C \subseteq A(X)^*$ and $u \in C$, then $q' = \{[h_\beta, d + e \cdot v + g \cdot - v]^{\text{if } \beta \in T}: \beta < \kappa\}$ is realized by u in C iff $l \leqq u$ and $m \cdot u = 0$.
As a consequence of (8) we have

(9) there is no $u \in B_\alpha$ such that $l \leqq u$ and $m \cdot u = 0$.
Now let L be the ideal in $A(X)^*$ generated by $\{a_\beta: \beta < \gamma\}$.

(10) If $l \in L$, then for any $S \in K$, $d + e \cdot cS + g \cdot - cS$ does not realize q in $B''_\alpha S$.
For, otherwise by (8) $l \leqq cS$ and $m \cdot cS = 0$. Hence for some finite join u of a_β's, $l \leqq u \cdot cS$ and $m \cdot u \cdot cS = 0$. But $u \cdot cS$ is clearly in B_α, so this contradicts (9). By (10), to establish (7) it suffices to assume that $l \notin L$. Now suppose that S_1 and S_2 are distinct elements of K such that y_{S_1} and y_{S_2} realize q in $B''_\alpha S_1$, $B''_\alpha S_2$ respectively. Then by (8), $l \leqq cS_i$ for $i = 1, 2$, so

$$l \leqq cS_1 \cdot cS_2 = \sum_{\beta \in S_1 \cap S_2} a_\beta,$$

hence $l \in L$, contradiction. Thus (7) holds.

By (1), (2), (6), (7), we can choose $S \subseteq S^*$ so that (3) and (4) hold. We let $B'_\alpha = B''_\alpha S$ and $Q'_\alpha = Q_\alpha \cup \{p\}$.

The second step in the construction of $B_{\alpha+1}$ and $Q_{\alpha+1}$ is to take care of indecomposability.

(11) There is an $x \in X$ free over B'_α.

In fact, for each $c \in B'_\alpha$ we can write $c = \Sigma D_c$, where D_c is a subset of $A(X)$ of power $\leqq \kappa$. For each $y \in A(X)$ there is a finite $E_y \subseteq X$ such that $y \in A(E_y)$. Let

$$F = \bigcup\{E_y : c \in B'_\alpha,\ y \in D_c\}.$$

By (1) we have $|F| < \lambda$, so choose $x \in X - F$. Clearly x is free over B'_α.

By (11) we choose $x \in X$ free over B'_α, and we set $B_{\alpha+1} = B'_\alpha(x)$. We let $Q_{\alpha+1}$ be Q'_α together with those of the two types

$$\{[a^\alpha_\beta \cdot x, v]^{\text{if } \beta \text{ even}} : \beta < \kappa\},$$

$$\{[a^\alpha_\beta \cdot -x, v]^{\text{if } \beta \text{ even}} : \beta < \kappa\}$$

which are standard κ-types omitted in $B_{\alpha+1}$.

This completes the construction. We claim that B_λ is the desired algebra C. Clearly $|B_\lambda| = \lambda$. Since at each stage $\alpha \to \alpha + 1$ the second step is to take a free extension, B_λ is clearly atomless. The following statement is easily proved by induction on α:

(12) for each $\alpha \leqq \lambda$ and each $0 \neq a \in B_\alpha$ there is a system $\langle b_\beta : \beta < \kappa \rangle$ of non-zero pairwise disjoint elements of $B_\alpha \restriction a$.

Hence the last statement of the theorem clearly holds. Since $\kappa < \operatorname{cf} \lambda$, it is clear that B_λ is κ-complicated. It remains only to show that it is indecomposable. Suppose I_0 and I_1 are non-principal complementary ideals in B_λ. Let $\langle a_\beta : \beta < \kappa, \beta \text{ even} \rangle$ be a maximal system of non-zero pairwise disjoint elements of I_0, and let $\langle a_\beta : \beta < \kappa,\ \beta \text{ odd} \rangle$ be a similar system for I_1. Let J be the (maximal) ideal generated by $I_0 \cup I_1$. Choose $\alpha < \lambda$ such that $\langle (a^\alpha_\beta, b^\alpha_\beta) : \beta < \kappa \rangle = \langle (a_\beta, a_\beta) : \beta < \kappa \rangle$ and $\{a_\beta : \beta < \kappa\} \subseteq B_\alpha$. Then at the αth step of the construction we let $B_{\alpha+1} = B'_\alpha(x)$, a free extension. We claim

(13) $\{[a_\beta, v]^{\text{if } \beta \text{ even}} : \beta < \kappa\}$ is omitted in B'_α.

For, suppose $y \in B'_\alpha$ realizes it. Without loss of generality say $y \in J$, and write $y = d + e$ with $d \in I_0$, $e \in I_1$. Then for β even we have $a_\beta \leqq y = d + e$, and $a_\beta \cdot e = 0$, so $a_\beta \leqq d$. This contradicts I_0 being non-principal, because of the maximality of $\{a_\beta : \beta \text{ even}\}$. So (13) holds.

(14) $\{[a_\beta \cdot x, v]^{\text{if } \beta \text{ even}}: \beta < \kappa\}$ and $\{[a_\beta \cdot -x, v]^{\text{if } \beta \text{ even}}: \beta < \kappa\}$ are omitted in $B_{\alpha+1}$.

For by symmetry we consider the first type only, and suppose $b + c \cdot x + d \cdot -x$ realizes it, where $b, c, d \in B'_\alpha$. Then for β even, $a_\beta \cdot x \leqq b + c \cdot x + d \cdot -x$, so $a_\beta \leqq b + c$. For β odd, $a_\beta \cdot x \cdot (b + c \cdot x + d \cdot -x) = 0$, so $a_\beta \cdot (b + c) = 0$. This contradicts (13), and establishes (14). By (14), the types there are omitted in B_λ as well (see the construction).

Now say without loss of generality $x \in J$, and write $x = d + e$ with $d \in I_0$, $e \in I_1$. Clearly then d realizes $\{[a_\beta \cdot x, v]^{\text{if } \beta \text{ even}}: \beta < \kappa\}$, contradiction. The proof is complete.

Taking $\kappa = \aleph_0$, $\lambda = 2^{\aleph_0}$ in Theorem 12 and using the previous theorems, we obtain

COROLLARY 13. *There is a prime-rigid* BA *of power* $2^{\aleph_0}$.

COROLLARY 14. *If* $\kappa^{\aleph_0} = 2^\kappa = \lambda$, *then there is a* BA *of power* λ *with no non-identity one-one endomorphisms.*

The hypothesis of Corollary 14 holds, e.g., if $\kappa = \beth_\alpha$ with $\operatorname{cf} \alpha = \aleph_0$ and $\lambda = \beth_{\alpha+1}$; thus it holds for arbitrarily large cardinals.

Concerning possible improvements of these results, we mention first a spectrum problem:

PROBLEM 1. In what cardinalities do there exist prime-rigid BA's?

We want to prove a theorem, implicit in S. Koppelberg [5], relevant to this problem. We need two lemmas.

LEMMA 15. *If A is atomless and has only definable endomorphisms, then the* BA *B of finite and cofinite subsets of ω is not a homomorphic image of A.*

PROOF. Assume otherwise, and let f be a homomorphism from A onto B. Since A has a subalgebra isomorphic to B, we may assume that B is a subalgebra of A. For every $n \in \omega$ choose $a_n \in A$ such that $fa_n = \{n\}$. Then for $m \neq n$ we have $a_m/\operatorname{exker} f \neq a_n/\operatorname{exker} f$. In fact, otherwise $a_m \Delta a_n = b + c$ for some $b \in \ker f$, $c \in \operatorname{fix} f$. Applying f, $\{m, n\} = c$. Say $0 < d < \{m\}$ (in A). But $d \leqq c$, so $fd = d \in B$, contradiction. Thus $A/\operatorname{exker} f$ is infinite.

LEMMA 16. *Let $\langle a_n : n \in \omega\rangle$ be a system of pairwise disjoint elements of A, and let $\langle D_n : n \in \omega\rangle$ be a system of ultrafilters of A with $a_n \in D_n$ for all $n \in \omega$. Assume that $\{n \in \omega : a \in D_n\}$ is finite or cofinite for every $a \in A$. Then the* BA *of finite and cofinite subsets of ω is a homomorphic image of A.*

PROOF. Let $fa = \{n \in \omega : a \in D_n\}$ for all $a \in A$.

THEOREM 17 (S. Koppelberg [5]).† *Assume Martin's axiom, and let A be a* BA *with $\aleph_0 \leqq |A| < 2^{\aleph_0}$. Then the* BA *of finite and cofinite subsets of ω is a homomorphic image of A. If in addition A is atomless, then A has undefinable endomorphisms.*

PROOF. By the above lemmas it suffices to construct $\langle a_n : n \in \omega\rangle$ and $\langle D_n : n \in \omega\rangle$ to satisfy the hypotheses of Lemma 16. Let $\langle a_n : n \in \omega\rangle$ be any system of non-zero pairwise disjoint elements, and let $a_n \in D_n$, D_n an ultrafilter on A, for each $n \in \omega$. For each $a \in A$ let $N_a = \{n \in \omega : a \in D_n\}$. Thus $\{a \in A : N_a \text{ is finite}\}$ is a proper ideal of A, and we extend it to a maximal ideal F of A. Let $P = \{(Q, R) : Q$ is a finite subset of ω and R is a finite subset of $F\}$. For $(Q, R), (Q', R') \in P$ we define $(Q, R) \leqq (Q', R')$ iff $Q \subseteq Q'$, $R \subseteq R'$, and $Q' \cap \bigcup_{a \in R} N_a \subseteq Q$. Clearly under this partial ordering P satisfies ccc. Let G be generic over P with respect to the dense sets

$$\{\{(Q, R) : |Q| \geqq m\} : m \in \omega\} \cup \{\{(Q, R) : a \in R\} : a \in F\}.$$

Let $x = \bigcup_{(Q,R) \in G} Q$. Then X is infinite, and for all $a \in F$, $X \cap N_a$ is finite (if $a \in R$ with $(Q, R) \in G$, then $X \cap N_a \subseteq Q$). Hence by relabeling, the desired conclusion follows.

There are two ways in which one might try to improve Corollary 13 to obtain even more rigid BA's:

Number of endomorphisms. It is easy to see that the BA C of Corollary 13 has a free subalgebra D with $|D| = 2^{\aleph_0}$. Hence C has $2^{2^{\aleph_0}}$ maximal ideals and also that many endomorphisms. It would be desirable to obtain a prime-rigid BA with no more endomorphisms than elements. The following theorem of Shelah, which we include with his kind permission, indicates some limits on constructing such an algebra — it must be of power at least $2^{\aleph_1}$.

THEOREM 18 (Shelah).† *If B is atomless and has only definable endomorphisms, then B has at least $2^{\aleph_1}$ ultrafilters.*

PROOF. Let $\langle a_n : n \in \omega\rangle$ be a system of non-zero pairwise disjoint elements of B, and let $\langle D_n : n \in \omega\rangle$ be a system of ultrafilters of B such that $a_n \in D_n$ for all $n \in \omega$. Now we define for all $\alpha < \omega_1$ and all $\eta \in {}^{\alpha}2$ a set $A_\eta \subseteq \omega$ and an element $b_\eta \in B$ by induction on α. Suppose defined for all $\beta < \alpha$, so that

†See note *Added in proof* at the end of the paper.

(1) $\beta < \alpha$ and $\eta \in {}^{\beta}2$ imply $|A_\eta| = \aleph_0$,

(2) $\beta < \gamma < \alpha$, $\eta \in {}^{\beta}2$, $\eta' \in {}^{\gamma}2$, $\eta \subseteq \eta'$ imply $A_{\eta'} \setminus A_\eta$ finite,

(3) $\beta + 1 < \alpha$, $\eta \in {}^{\beta}2$ imply $\{n \in A_\eta : -b_\eta \in D_n\} = A_{\eta 0}$ and $\{n \in A_\eta : b_\eta \in D_n\} = A_{\eta 1}$.

Now suppose $\eta \in {}^{\alpha}2$. If $\alpha = 0$, let $A_\eta = \omega$. If α is a non-zero limit ordinal, choose A_η so that $A_\eta \setminus A_{\eta \restriction \beta}$ is finite for all $\beta < \alpha$. Finally, for $\alpha = \beta + 1$ let $A_\eta = \{n \in A_{\eta \restriction b} : b_{\eta \restriction \beta}^{\eta\beta} \in D_n\}$, where $c^0 = -c$, $c^1 = c$ for all $c \in B$. By Lemmas 15 and 16, choose $b_\eta \in B$ such that $\{n \in A_\eta : b_\eta^\varepsilon \in D_n\}$ is finite for $\varepsilon = 0, 1$. This completes the construction.

Now let $\eta \in {}^{\omega_1}2$. Then by (2), $\{A_{\eta \restriction \alpha} : \alpha < \omega_1\}$ is contained in an ultrafilter E_η^* on ω. Set $E_\eta = \{a \in B : \{n : a \in D_n\} \in E_\eta^*\}$. Clearly E_η is an ultrafilter on B. Now suppose $\eta, \eta' \in {}^{\omega_1}2$ and $\eta \neq \eta'$. Choose α minimum such that $\eta\alpha \neq \eta'\alpha$. Then by (3),

$$\{n : b_{\eta \restriction \alpha}^{\eta\alpha} \in D_n\} = A_{\eta \restriction (\alpha+1)} \in E_\eta^*,$$

so $b_{\eta \restriction \alpha}^{\eta\alpha} \in E_\eta$. Similarly $b_{\eta' \restriction \alpha}^{\eta'\alpha} \in E_{\eta'}$, so $E_\eta \neq E_{\eta'}$. This completes the proof.

Bonnet-rigid BA's. Another strong version of rigidity was given by Bonnet [1]. A is *Bonnet-rigid* if for every BA B, every one-one homomorphism $f: A \to B$ and every onto homomorphism $g: A \to B$ we have $f = g$. He showed that there is such an algebra of power $2^{\aleph_0}$; his algebra is not prime-rigid. The algebra C of Corollary 13 is not Bonnet-rigid. For, let D be as above, and let h be a homomorphism from D onto C. Let I be the ideal in C generated by the kernel of h. Then there is a one-one homomorphism $f: C \to C/I$ with $fhd = d/I$ for every $d \in D$. Let $g: C \to C/I$ be the natural onto homomorphism. Clearly $f \neq g$. It would be nice to have a prime-rigid BA which is also Bonnet-rigid; such an algebra has been recently constructed by Shelah, assuming $\diamondsuit_{\aleph_1}$.

An interesting form of these problems is

PROBLEM 2. Is there a prime-rigid BA A which is also Bonnet-rigid and has only $|A|$ endomorphisms?

Concerning Corollary 14, we mention that Shelah has recently shown in ZFC that BA's with no non-identity one-to-one endomorphisms exist in each regular cardinality and many singular ones.

Added in proof (October 1979). S. Todorčević has brought to the author's attention that Theorem 18 follows easily from Lemma 15 and a theorem of S. P. Franklin (Proc. Amer. Math. Soc. **21**(1969), 597–599), while Malyhin and

Šapirovskiĭ in Sov. Math. Dokl. **14**(1973), 1746–1751 prove a theorem from which Theorem 17 follows under the weaker assumption that $|S| < \exp(\exp \aleph_0)$, S the Stone space of A.

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