Completions of BOOLEAN Algebras with operators

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The notion of a BOOLEan algebra with operators was introduced by JÓNSSON and TARSKI [5]. It encompasses as special cases relation algebras (TARSKI [9]), closure algebras (MCKINSEY-TARSKI [6]), cylindric algebras (HENKIN-TARSKI [4]), polyadic algebras (HALMOS [2]), and other algebras which have been studied in recent years. One of the basic results of [5] is that any BOOLEan algebra with operators can be extended to one that is complete and atomic. The extension does not preserve any BOOLEan sums (joins) which are essentially infinite, however. It is the main purpose of this paper to describe a completion that, while not atomic in general, does preserve all sums (and products).

In section 1 the theory of such completions is extensively developed, patterning the development after section 2 of [5]. It turns out that the proofs are much simpler than in [5], so they are given only briefly. The second short section of the paper deals briefly with completions of some of the special kinds of algebras mentioned in the preceding paragraph.

We adopt the notation of [5], with the following exceptions and additions. A BOOLEan algebra is treated as a structure $\mathfrak{A} = \langle A, +, \cdot, -\rangle$. ${}^{Y}X$ is the set of all functions mapping Y into X. ${}^{m}X$ is the set of all m-termed sequences of members of X. Id is the identity $\{(x, x)\colon x \text{ a set}\}$.

$f \restriction X$ is the restriction of f to X. Other set-theoretical conventions not mentioned in [5] are the usual ones. For the theory of BOOLEan algebras we refer to SIKORSKI [8]. Particular use will be made of the theory of completions (§ 35 of [8]; see in particular Theorem 35.2). A BOOLEan algebra $\mathfrak{A}$ is called *injective* if whenever $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{C}$ then there is a homomorphism f of $\mathfrak{C}$ into $\mathfrak{A}$ such that $Id \restriction B \subseteq f$. In [8] it is shown that $\mathfrak{A}$ is injective iff $\mathfrak{A}$ is complete. BOOLEan algebras with operators will be treated as algebras $\mathfrak{A} = \langle A, +, \cdot, -, f_i\rangle_{i\in I}$. We then let $\mathfrak{Bl}\,\mathfrak{A} = \langle A, +, \cdot, -\rangle$.

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1. Completions in general

Throughout this section, unless mentioned to the contrary, we assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\mathfrak{A}$ is a completion of $\mathfrak{B}$ (in the sense mentioned above). If f is an m-ary operation on $\mathfrak{B}$, we define f^+, an m-ary operation on A, by

$$f^+ x = \sum_{x \geqq y \in {}^m B} f y$$

for any $x \in {}^m A$. Thus f^+ is a monotonic operation on A.

Theorem 1.1. *If f is a monotonic operation on $\mathfrak{B}$, then $f \subseteq f^+$. Clearly if is not monotonic then $f \not\subseteq f^+$.*

Theorem 1.2. *If f is a completely additive operation on $\mathfrak{B}$, then f^+ is a completely additive operation on A.*

Proof. Say f is m-ary. Let $x^i \in {}^m A$ for each $i \in I$, where $I \neq 0$; suppose $j < m$ and $x^i_k = x^{i'}_k$ whenever $i, i \in I$ and $k \in m \sim \{j\}$. Let

$$y = \langle x^i_0, \ \ldots, x^i_{j-1}, \sum_{i \in I} x^i_j, \ \ x^i_{j+1}, \ldots, x^i_{m-1}\rangle$$

(for any $i \in I$). Then

(1) $$f^+ y = \sum_{y \geqq z \in {}^m B} f z;$$

(2) $$\sum_{i \in I} f^+ x^i = \sum_{i \in I} \sum_{x^i \geqq w \in {}^m B} f w.$$

Note that $y \geqq x^i$ for each $i \in I$. Hence, since f^+ is monotonic,

(3) $$\sum_{i \in I} f^+ x^i \leqq f^+ y.$$

Conversely, suppose $y \geqq z \in {}^m B$. Thus

(4) $$z_j \leqq y_j = \sum_{i \in I} x^i_j = \sum_{i \in I} \sum_{x^i_j \geqq u \in B} u.$$

Now for $i \in I$, $x^i_j \geqq u \in B$, and $k \in m \sim \{j\}$ let $v^{iu}_k = z_k$, and let $v^{iu}_j = z_j \cdot u$. Then, by the complete additivity of f and using (4)

$$f z = \sum_{i \in I} \sum_{x^i_j \geqq u \in B} f v^{iu}.$$

But if $i \in I$ and $x^i_j \geqq u \in B$, then $x^i \geqq v^{iu}$. Hence by (1), (2),

$$f^+ y \leqq \sum_{i \in I} f^+ x^i;$$

together with (3), this completes the proof.

Theorem 1.2 cannot be improved to assume, as in 2.4 of [5], simply that f is additive. For, since sums are preserved in going from $\mathfrak{B}$ to $\mathfrak{A}$ f^+ completely additive implies that f is completely additive. In fact, there is a one-one correspondence between completely additive operations on B and those on A which extend ones on B:

Theorem 1.3. *If f is an m-ary completely additive operation on A and $f \restriction^m B$ is an operation on B, then $f = (f \restriction^m B)^+$.*

Proof. For any $x \in {}^m A$ we have, by a simple inductive argument,

$$f\,x = \sum_{x_0 \geqq y_0 \in B} \sum_{x_1 \geqq y_1 \in B} \cdots \sum_{x_{m-1} \geqq y_{m-1} \in B} f\,y = \sum_{x \geqq y \in {}^m B} f\,y,$$

as desired.

Theorem 1.3. cannot be strengthened by assuming that f is merely additive. In fact, let $\mathfrak{B}$ be the Boolean algebra of finite and cofinite subsets of ω, $\mathfrak{A}$ the Boolean algebra of all subsets of ω. Thus $\mathfrak{A}$ is a completion of $\mathfrak{B}$. Let f be a homomorphism of $\mathfrak{A}$ onto its two-element subalgebra such that $f\,a = 0$ for each finite subset a of ω, and let $g = f \restriction B$. Then $g^+\,a = 1$ if a is a cofinite subset of ω, and $g^+\,a = 0$ otherwise. Hence $f \neq g^+$.

By 1.3, each completely additive operation on B has exactly one extension which is a completely additive operation on A. This is in contrast with the situation for perfect extensions (see [5], p. 913).

Theorem 1.4. *If g is an m-ary operation on B and f is a completely additive m-ary operation on A such that $f \restriction^m B \leqq g$, then $f \leqq g^+$.*

Proof. For any $x \in {}^m A$ we have

$$\begin{aligned} f\,x &= \sum_{x \geqq y \in {}^m B} f\,y \qquad \text{by } 1.3 \\ &\leqq \sum_{x \geqq y \in {}^m B} g\,y = g^+\,x. \end{aligned}$$

The example following 1.3 shows that one cannot merely take f additive in 1.4. In fact, in 1.4 one cannot even assume that g is completely additive when f is merely additive. For, let $\mathfrak{A}$ and $\mathfrak{B}$ be as before, let f be as in the former example, and for any $x \in B$ let $g\,x = \{j \colon \exists\, i\ (j \leqq 2\,i \in x)\}$. We may choose f so that $f\,\{1, 3, 5, \ldots\} = 1$; but $g^+\,\{1, 3, 5, \ldots\} = 0$.

Theorem 1.5.

(i) $+^+ = +$.
(ii) $\cdot^+ = \cdot$.
(iii) *If $f = B \times \{b\}$, then $f^+ = A\,x\,\{b\}$.*
(iv) *If $f\,x = x_i$ for all $x \in {}^m B$ (where $i < m$), then $f^+\,x = x_i$ for all $x \in^m A$.*

Theorem 1.6. *If f and g are respectively m- and $(m+n)$-ary operations on B with $g(x^\frown y) = fx$ for all $x \in {}^mB$ and $y \in {}^nB$, then $g^+(x^\frown y) = f^+ x$ for all $x \in {}^mA$ and $y \in {}^nA$.*

Theorem 1.7. *If f is a completely additive m-ary operation on B and $g_0, \ldots, g_{m-1}$ are n-ary operations on B, then*

$$(f\langle g_0, \ldots, g_{m-1}\rangle)^+ = f^+\langle g_0^+, g_0^+, \ldots, g_{m-1}\rangle.$$

Proof. Assume that $x^i \in {}^nA$ for each $i < n$, and $y = x^{0\frown} \cdots {}^\frown x^{m-l}$. Then

$$\begin{aligned}(f\langle g_0, \ldots, g_{n-1}\rangle)^+ y &= \sum_{y \geqq z \in {}^{mn}B} (f\langle g_0, \ldots, g_{m-1}\rangle) z \\ &= \sum_{x^0 \geqq z^0 \in {}^nB} \cdots \sum_{x^{m-1} \geqq z^{m-1} \in {}^nB} (f\langle g_0, \ldots, g_{m-1}\rangle)(z^{0\frown} \cdots {}^\frown z^{m-1}) \\ &= \sum_{x^0 \geqq z^0 \in {}^nB} \cdots \sum_{x^{m-1} \geqq x^{m-1} \in {}^nB} f(g_0 u^0, \ldots, g_{m-1} z^{m-1});\end{aligned}$$

an easy inductive argument using the complete additivity of f^+, which follows from 1.2, then gives

$$\begin{aligned}(f\langle g_0, \ldots, g_{m-1}\rangle)^+ y &= f^+\Big(\sum_{x^0 \geqq z \in {}^nB} g_0 z, \ldots, \sum_{x^{m-1} \geqq z \in {}^nB} g_{n-1} z\Big) \\ &= f^+(g_0^+ x^0, \ldots, g_{m-1}^+ x^{m-1}) \\ &= (f^+\langle g_0^+, \ldots, g_{m-1}{}^+\rangle) y,\end{aligned}$$

as desired.

Again, in 1.7 f must be assumed to be completely additive. In the example following 1.4, let $f' = f \upharpoonright B$. Then f' is additive, g is completely additive, while $(f' \cdot g)^+ \neq f'^+ \circ g^+$ since $(f' \circ g)^+ \{0, 2, \ldots\} = 0$ while $f'^+ g^+ \{0, 2, \ldots\} = 1$. This observation is relevant to the following theorem also.

Theorem 1.8. *If f is a completely additive m-ary operation on B and $g_0, \ldots, g_{m-1}$ are monotone n-ary operations on B, then*

$$(f[g_0, \ldots, g_{m-1}])^+ = f^+[g_0^+, \ldots, g_{m-1}].$$

Proof. Let $x \in {}^nA$. Then

$$\begin{aligned}(f[g_0, \ldots, g_{m-1}])^+ x &= \sum_{x \geqq y \in {}^nB} f(g_0 y, \ldots, g_{m-1} y) \\ &= \sum_{x \geqq y \in {}^nB} f^+(g_0^+ y, \ldots, g_{m-1}^+ y) \\ &\leqq f^+(g_0^+ x, \ldots, g_{m-1}^+ x) \\ &= (f^+[g_0^+, \ldots, g_{m-1}^+]) x.\end{aligned}$$

To establish the other direction, an easy induction using the complete additivity of f^+ gives

$$(f^+[g_0^+, \ldots, g_{m-1}^+])\, x = f^+\Big(\sum_{x \geqq y_0 \in {}^nB} g_0\, y_0, \ldots, \sum_{x \geqq y_{m-1} \in {}^nB} g_{m-1}\, y_{m-1}\Big)$$
$$= \sum_{x \geqq y_0 \in {}^nB} \cdots \sum_{x \geqq y_{m-1} \in {}^nB} f(g_0 y_0, \ldots, g_{m-1} y_{m-1}).$$

Now if $x \geqq y_0, \ldots, y_{m-1}$, let $z = y_0 + \cdots + y_{m-1}$; since g_i is monotone, $g_i\, y_i \leqq g_i z$. Thus

$$(f^+[g_0^+, \ldots, g_{m-1}^+])\, x \leqq \sum_{x \geqq z \in {}^nB} f(g_0\, z, \ldots, g_{m-1}\, z)$$
$$= (f[g_0, \ldots, g_{m-1}])^+\, x,$$

which completes the proof.

Let $\mathfrak{B} = \langle B, +, \cdot, -, f_i\rangle_{i \in I}$ be a Boolean algebra with operators. A *completion* of $\mathfrak{B}$ is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, f_i^+\rangle_{i \in I}$, where $\mathfrak{Bl}\,\mathfrak{A}$ is a completion of $\mathfrak{Bl}\,\mathfrak{B}$. If each operation f_i is completely additive, we call $\mathfrak{B}$ *completely additive.* Thus if $\mathfrak{B}$ is completely additive, then $\mathfrak{A}$ is complete. With a given Boolean algebra with operators $\mathfrak{A}$ we suppose associated a first-order logic $\mathscr{L}_{\mathfrak{A}}$. A term σ of $\mathscr{L}_{\mathfrak{A}}$ is *positive* if $-$ does not occur in it; an equation $\sigma = \tau$ is positive if both σ and τ are positive. If σ is a term with variables among $v_0, \ldots, v_{n-1}$, then $\overset{n\sim\mathfrak{A}}{\sigma}$ is the naturally associated n-ary operation on A.

Theorem 1.9. *If $\mathfrak{B}$ is a completion of a completely additive* Boolean *algebra with operators $\mathfrak{A}$, then a positive equation $\sigma = \tau$ holds in $\mathfrak{A}$ iff it holds in $\mathfrak{B}$.*

Proof. By induction on terms one easily shows that $\overset{n\sim\mathfrak{B}+}{\sigma} = \overset{n\sim\mathfrak{A}}{\sigma}$ for any positive term σ with variables among $v_0, \ldots, v_{n-1}$. Hence 1.9 follows (cf. Ribeiro [7]).

Theorem 1.10. *With $\mathfrak{A}$ and $\mathfrak{B}$ as in 1.9, if φ is a conjunction or disjunction of formulas of the form $\sigma = 0$ or $\sigma \neq 0$, σ positive, and if τ and ϱ are positive, then $\varphi \rightarrow \tau = \varrho$ holds in $\mathfrak{A}$ iff it holds in $\mathfrak{B}$.*

Corollary 1.11. If $f, g \in {}^BB$ and f and g are conjugate, then f^+ and g^+ are conjugate.

As is shown in [5], p. 921, Theorem 1.9 does not extend to inequalities $\sigma \neq \tau$, even if σ and τ are positive; it does not extend to equations $\sigma = \tau$ with σ and τ arbitrary, as is shown in Henkin, Monk, Tarski [3], Remark 2.7.19.

One of the basic properties of completions of Boolean algebras is their minimality (35.2 (iv) of [8]): if $\mathfrak{A}$ is a completion of $\mathfrak{B}$ and $\mathfrak{B}$ is a subalgebra of a complete algebra $\mathfrak{C}$, then there is an isomorphism f of $\mathfrak{C}$ into $\mathfrak{C}$ such that $Id \restriction B \subseteq f$. This extends in the following modified form to Boolean algebras with operators.

Theorem 1.12. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean algebras with operators, $\mathfrak{A}$ a completion of $\mathfrak{B}$, $\mathfrak{B}$ completely additive, $\mathfrak{B}$ a subalgebra of $\mathfrak{C}$, $\mathfrak{C}$ complete, $\mathfrak{B}\mathfrak{l}\,\mathfrak{B}$ a regular subalgebra of $\mathfrak{B}\mathfrak{l}\,\mathfrak{C}$. Then there is an isomorphism f of $\mathfrak{A}$ into $\mathfrak{C}$ such that $Id \restriction B \subseteq f$.*

Proof. For any $a \in A$, let $f a = \sum\limits_{a \leq b \in B}^{\mathfrak{C}} b$. Although it is well-known that f is a complete Boolean isomorphism into, we present a proof for the sake of completeness. We have $f a \cdot f(-a) = 0$. Further,

$$1 = a + (-a) = \sum_{a \geqq b \in B}^{\mathfrak{A}} b + \sum_{-a \geqq c \in B}^{\mathfrak{A}} c = \sum_{b \in T}^{\mathfrak{A}} b,$$

where $T = \{b \in B\colon\ a \geqq b \text{ or } -a \geqq b\}$. Obviously, then, $1 = \sum\limits_{b \in T}^{\mathfrak{B}} b$, hence $1 = \sum\limits_{b \in T}^{\mathfrak{C}} b$. But this means that $f a + f(-a) = 1$. Thus

$$f(-a) = -f a.$$

Next, obviously $\sum\limits_{i \in I}^{\mathfrak{C}} f a_i \leqq f \sum\limits_{i \in I}^{\mathfrak{A}} a_i$. If $\sum\limits_{i \in I}^{\mathfrak{A}} a_i \geqq b \in B$, then

$$\begin{aligned} b &= \sum_{i \in I}^{\mathfrak{A}} \sum_{b \cdot a_i \geqq c \in B}^{\mathfrak{A}} c = \sum^{\mathfrak{A}} \{c \in B\colon \exists\, i \in I\, (c \leqq b \cdot a_i)\} \\ &= \sum^{\mathfrak{B}} \{c \in B\colon \exists\, i \in I\, (c \leqq b \cdot a_i)\} = \sum^{\mathfrak{C}} \{c \in B\colon \exists\, i \in I\, (c \leqq b \cdot a_i)\} \\ &= \sum_{i \in I}^{\mathfrak{C}} \sum_{b \cdot a_i \geqq c \in B}^{\mathfrak{C}} c \leqq \sum_{i \in I}^{\mathfrak{C}} \sum_{a_i \geqq c \in B}^{\mathfrak{C}} c = \sum_{i \in I}^{\mathfrak{C}} f a_i . \end{aligned}$$

Hence $f \sum\limits_{i \in I}^{\mathfrak{A}} a_i \leqq \sum\limits_{i \in I}^{\mathfrak{C}} f a_i$, as desired.

Finally, f preserves non-Boolean operations: let $\mathfrak{A} = \langle A, +, \cdot, -, g_i^{\mathfrak{A}} \rangle_{i \in I}$, $i \in I$, g_i m-ary. Then for any $x \in {}^m A$,

$$\begin{aligned} f g_i^{\mathfrak{A}} x &= f \sum_{x \geqq y \in {}^m B} g_i^{\mathfrak{B}} y = \sum_{x \geqq y \in {}^m B}^{\mathfrak{C}} g_i^{\mathfrak{B}} y \\ &= g_i^{\mathfrak{C}} \sum_{x \geqq y \in {}^m B} y = g_i^{\mathfrak{C}} (f \circ x), \end{aligned}$$

and the proof is complete.

The restriction in 1.12 that $\mathfrak{B}\mathfrak{l}\,\mathfrak{B}$ is a regular subalgebra of $\mathfrak{B}\mathfrak{l}\,\mathfrak{C}$ is essential. In fact, let $\mathfrak{A}$ and $\mathfrak{B}$ be as in the example following 1.3, with the following two non-BOOLEan operations: $f_0^{\mathfrak{B}}\, b = \{j\colon \exists\, i\, (j \leqq 2\, i \in b)\}$, $f_1^{\mathfrak{B}}\, b = \{j\colon \exists\, i\, (j \leqq 2\, i + 1 \in b)\}$, $f_0^{\mathfrak{A}} = f_0^{\mathfrak{B}}, f_1^{\mathfrak{A}} = f_1^{\mathfrak{B}+}$. Let $\mathfrak{C}$ be a perfect extension of $\mathfrak{B}$. It is easily verified that $\mathfrak{C}$ has exactly one more atom, say x, than $\mathfrak{B}$, and that $x \leqq b$ for each cofinite subset b of ω. Now suppose that g is an isomorphism from $\mathfrak{A}$ into $\mathfrak{C}$ with $Id \upharpoonright B \subseteq g$. Then $x \leqq g\,\{0, 2, 4, \ldots\}$ or $x \leqq g\,\{1, 3, 5, \ldots\}$; say $x \leqq g\,\{0, 2, 4, \ldots\}$. Then, as is easily checked, $g\,\{0, 2, 4, \ldots\} = \sum\limits_{i \in \omega}^{\mathfrak{C}} \{2\, i\} + x$. Then

$$g f_1^{\mathfrak{A}}\,\{0, 2, ,4 \ldots\} = 0;$$

$$f_1^{\mathfrak{C}}\, g\,\{0, 2, 4, \ldots\} \geqq f_1^{\mathfrak{C}}\, x = \prod_{x \leqq b \in B}^{\mathfrak{C}} f_1^{\mathfrak{B}}\, b \geqq x,$$

a contradiction.

We may mention that another important property of BOOLEan algebras, that any algebra can be imbedded in an injective — because complete algebras are injective (see [8], section 33) —, fails for BOOLEan algebras with operators, as we will see in our discussion of special algebras below.

2. Completions of special algebras

The discussion of the preceding section is fully applicable only to completely additive BOOLEan algebras with operators. For this reason we shall concern ourselves only with relation and cylindric algebras. It is well-known that the operations in closure and polyadic algebras are not, in general, completely additive.

From 1.10 and the axioms given for relation algebras in [5], it follows that a completion of a relation algebra is again a relation algebra; by 4.10 of [5], simplicity is preserved. An easy argument shows that a completion of an integral relation algebra is again integral. We do not know whether a completion of a representable relation algebra is again representable. Since by TARSKI [10] the class $\mathcal{RRA}$ of representable relation algebras is equational, this problem is related to the problem whether $\mathcal{RRA}$ can be characterized relative to the BOOLEan algebra axioms by positive equations. The following are partial results relevant to these problems.

Theorem 2.1. *If $\mathfrak{A}$ satisfies the hypothesis of* 4.32 *or of* 4.33 *of* [5] *and* $\mathfrak{B}$ *is a completion of* $\mathfrak{A}$, *then* $\mathfrak{B}$ *is representable.*

Theorem 2.2. *If* $\mathfrak{A}$ *is a relation algebra of class* i ($i = 1, 2, 3$), *then so is any completion of* $\mathfrak{A}$.

Both theorems can be shown by adjoining O' as a new O-ary operation.

Turning to cylindric algebras, we see that a completion of a (simple) $\mathscr{CA}_\alpha$ is again a (simple) $\mathscr{CA}_\alpha$, by [3], remark 1.3.6 and 2.3.14 [2]). It can be shown that a completion of a hereditarily non-discrete $\mathscr{CA}_\alpha$ is never locally finite. Again it is an open problem whether the completion of a representable $\mathscr{CA}_\alpha$ is representable. In this connection the following two partial results may be of interest. We use the notation of [3].

Theorem 2.3. *Any completion of a dimension complemented $\mathscr{CA}_\alpha$ of infinite dimension is representable.*

Proof. Say $\mathfrak{A}$ is a completion of $\mathfrak{B}$. For $\varkappa < \alpha$ and $a \in A$ we then have

$$\begin{aligned} c_\varkappa a &= \sum_{a \geqq b \in B} c_\varkappa b = \sum_{a \geqq b \in B} \sum_{\lambda < \alpha} S^\varkappa_\lambda b \qquad \text{by 1.11.6 of [3]} \\ &= \sum_{\lambda < \alpha} \sum_{a \geqq b \in B} S^\varkappa_\lambda b \\ &= \sum_{\lambda < \alpha} S^\varkappa_\lambda a; \end{aligned}$$

hence $\mathfrak{A}$ is representable by 2.6.54 of [3].

Theorem 2.4. *If $\mathfrak{A}$ is a completion of $\mathfrak{B}$ and $\mathfrak{B} = \mathfrak{Nr}_\alpha \mathfrak{B}'$, $\mathfrak{B}'$ a complete $\mathscr{CA}_{\alpha+\omega}$, then $\mathfrak{A}$ is representable.*

Proof. If $\langle b_i : i \in I \rangle$ is a system of elements of B such that $\sum_{i \in I} b_i$ exists, then $\sum_{i \in I}^{\mathfrak{B}} b_i = \sum_{i \in I}^{\mathfrak{B}} b_i$. In fact, suppose $x \in B'$ is an upper bound for each b_i. Then for each $i \in I$ and each finite $\Gamma \subseteqq (\alpha + \omega) \sim \alpha$, $c^\partial_{(\Gamma)} x \geqq b_i$. Let $y = \prod_{\Gamma \in T} \mathfrak{B}' c^\partial_{(\Gamma)} x$, $T = \{\Gamma : \Gamma \text{ finite}, \Gamma \subseteqq (\alpha + \omega) \sim \alpha\}$. Then $\Delta y = \alpha$, y is an upper bound for each b_i, so $\sum_{i \in I}^{\mathfrak{B}} b_i \leqq y \leqq x$, as desired. Thus $\mathfrak{B}\mathfrak{l}\,\mathfrak{B}$ is a regular subalgebra of $B\mathfrak{l}\,\mathfrak{B}'$. The desired conclusion now follows by 1.12 and 2.6.36 of [3].

We understand the notion of injective $\mathscr{CA}_\alpha$ in the natural way.

Theorem 2.3. *The only injective $\mathscr{CA}_1$'s are the one-element $\mathscr{CA}_1$s.*

Proof. Suppose $\mathfrak{A}$ is an injective $\mathscr{CA}_1$ with more than one element. Then there is a simple $\mathscr{CA}_1$ $\mathfrak{B}$ such that $|\mathfrak{B}| > |A|$ and $\mathfrak{C} \subseteqq \mathfrak{B}$, where $\mathfrak{C}$ is the two-element subalgebra of $\mathfrak{A}$. But clearly then there is no homomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, contradiction.

For more information on injective $\mathscr{CA}_\alpha$s see Comer [1].

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2) Except for simplicity this was shown in Mangani [11].

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