Solutions to exercises in Chapter 5

E5.1 *Give an estimate for the size of $gn(\varphi)$, where φ is the Peano Postulate (P1).*

(P1) is the formula $\forall v_0 \forall v_1[\mathbf{S}v_0 = \mathbf{S}v_1 \rightarrow v_0 = v_1]$, or as a sequence

$$\langle 4, 5, 4, 10, 2, 3, 6, 5, 6, 10, 3, 5, 10\rangle.$$

Hence

$$\begin{aligned} gn(\varphi) &= p_0^4 \cdot p_1^5 \cdot p_2^4 \cdot p_3^{10} \cdot p_4^2 \cdot p_5^3 \cdot p_6^6 \cdot p_7^5 \cdot p_8^6 \cdot p_9^{10} \cdot p_{10}^3 \cdot p_{11}^5 \cdot p_{12}^{10} \\ &= 2^4 \cdot 3^5 \cdot 5^4 \cdot 7^{10} \cdot 11^2 \cdot 13^3 \cdot 17^6 \cdot 19^5 \cdot 23^6 \cdot 29^{10} \cdot 31^3 \cdot 37^5 \cdot 41^{10} \\ &\approx 1.9 \cdot 10^{85}. \end{aligned}$$

E5.2 *Describe $G(gn(v_0 = v_0))$ and express it as a product of primes.*

For brevity let $m = gn(v_0 = v_0)$; recall from the notes that $m = 6,075,000$. Now by definition, $G(gn(v_0 = v_0)) = gn(\mathrm{Subff}^{v_0}_{\overline{m}}(v_0 = v_0))$; thus $G(gn(v_0 = v_0))$ is the Gödel number of the formula

$$\forall v_m[v_m = \overline{m} \rightarrow \forall v_0[v_0 = v_m \rightarrow v_0 = v_0]].$$

As an actual sequence of integers, this formula is

$$\langle 4, 5(m+1), 2, 3, 5(m+1), 6(m \text{ times}), 8, 4, 5, 2, 3, 5, 5(m+1), 3, 5, 5\rangle.$$

Hence $G(gn(v_0 = v_0))$ is the number

$$p_0^4 \cdot p_1^{5(m+1)} \cdot p_2^2 \cdot p_3^3 \cdot p_4^{5(m+1)} \cdot \prod_{i<m} p_{5+i}^6 \cdot p_{5+m}^8 \cdot p_{5+m+1}^4$$
$$\cdot p_{5+m+2}^5 \cdot p_{5+m+3}^2 \cdot p_{5+m+4}^3 \cdot p_{5+m+5}^5 \cdot p_{5+m+6}^{5(m+1)} \cdot p_{5+m+7}^3 \cdot p_{5+m+8}^5 \cdot p_{5+m+9}^5$$

Suppose that Γ is a set of sentences containing $\mathbf{P}'$. A formula ρ with at most v_0 free is a Γ-*provability condition* iff for any sentence φ, $\Gamma \vdash \varphi$ iff $\Gamma \vdash \rho(\overline{gn(\varphi)})$.

E5.3 *Suppose that Γ is a set of sentences containing $\mathbf{P}'$, and $\overline{M} \stackrel{\mathrm{def}}{=} (\omega, S, 0, +, \cdot)$ is a model of Γ. Let χ be as in the proof of Gödel's incompleteness theorem, and let π be the formula $\exists v_1\chi$. Prove that π is a Γ-provability condition.*

Let φ be a sentence. First suppose that $\Gamma \vdash \varphi$. Let Φ be a Γ-proof with last entry φ. Thus $(gn(\varphi), gn_1(\Phi)) \in \mathrm{Prf}_\Gamma$, so $\mathbf{P} \vdash \chi(\overline{g(\varphi)}, \overline{g_2(\Phi)})$. Hence $\mathbf{P} \vdash \exists v_1\chi(\overline{gn(\varphi)})$, so also $\Gamma \vdash \exists v_1\chi(\overline{gn(\varphi)})$.

Second suppose that $\Gamma \vdash \exists v_1\chi(\overline{gn(\varphi)})$. Then $\overline{M} \models \exists v_1\chi(\overline{gn(\varphi)})$, so we can choose $m \in \omega$ such that $\overline{M} \models \chi(\overline{gn(\varphi)})[m, m]$. We claim that $(gn(\varphi), m) \in \mathrm{Prf}_\Gamma$. Otherwise we get $\mathbf{P} \models \neg\chi(\overline{gn(\varphi)}, \overline{m})$, hence $\overline{M} \models \neg\chi(\overline{gn(\varphi)})[m, m]$, contradiction. This proves the claim. Let Φ be a Γ-proof with last entry φ such that $m = gn_1(\Phi)$. Hence $\Gamma \vdash \varphi$.

$\boxed{\text{E5.4}}$ *Suppose that* Γ *is a set of sentences containing* $\mathbf{P}'$, *and* $\overline{M} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$ *is a model of* Γ. *Suppose that* ρ *is a* Γ*-provability condition. Apply the fixed point theorem to get a sentence* ψ *such that* $\mathbf{P} \vdash \psi \leftrightarrow \neg\rho(\overline{gn(\psi)})$, *as in the proof of Gödel's incompleteness theorem. Prove that not(*$\Gamma \vdash \psi$*) and not(*$\Gamma \vdash \neg\psi$*).*

Suppose that $\Gamma \vdash \psi$. Then $\Gamma \vdash \rho(\overline{gn(\psi)})$ since ρ is a provability condition, but also $\Gamma \vdash \neg\rho(\overline{gn(\psi)})$ by the choice of ψ. So Γ is inconsistent. This contradicts the assumption that $\overline{M}$ is a model of Γ. Hence not($\Gamma \vdash \psi$). Suppose that $\Gamma \vdash \neg\psi$. Then by the choice of ψ, $\Gamma \vdash \rho(\overline{gn(\psi)})$. It follows that $\Gamma \vdash \psi$ since ρ is a provability condition, contradiction.

$\boxed{\text{E5.5}}$ *Let* χ *be as in the proof of Gödel's incompleteness theorem, and let* π *be the formula* $\exists v_1\chi$. *The following can be shown for* χ.

(i) For any sentences φ *and* ψ,

$$\Gamma \vdash \pi(\overline{gn(\varphi \to \psi)}) \to (\pi(\overline{gn(\varphi)}) \to \pi(\overline{gn(\psi)})).$$

("Γ *proves that if* $\varphi \to \psi$ *is provable, then from the provability of* φ *it follows that* ψ *is provable")*

(ii) For any sentence φ,

$$\Gamma \vdash \pi(\overline{gn(\varphi)}) \to \pi(\overline{gn(\pi(\overline{g_s(\varphi)}))}).$$

("Γ *proves that if* φ *is provable, then it is provable that* φ *is provable."*

By the fixed point theorem, let ψ *be a sentence such that* $\Gamma \vdash \psi \leftrightarrow \pi(\overline{gn(\psi)})$. *Note that* ψ *says "I am provable". By the fixed point theorem again, let* θ *be a sentence such that* $\Gamma \vdash \theta \leftrightarrow (\pi(\overline{gn(\theta)}) \to \psi)$. *Thus* θ *says "If I am provable, then* ψ *holds."*

Show that if $\Gamma \vdash \theta$, *then* $\Gamma \vdash \psi$.

Let Φ be a Γ-proof with last entry θ. Thus $(gn(\theta), gn_1(\Phi)) \in \text{Prf}_\Gamma$, and it follows that $\mathbf{P} \vdash \chi(\overline{gn(\theta)}, \overline{gn_1(\Phi)})$, hence $\Gamma \vdash \pi(\overline{gn(\theta)})$. Since also $\Gamma \vdash \theta$, it follows that $\Gamma \vdash \psi$.

$\boxed{\text{E5.6}}$ *(Continuing E5.5.) Show that* $\Gamma \vdash \pi(\overline{gn(\theta)}) \to \pi(\overline{gn(\psi)})$.

By the choice of θ we have $\Gamma \vdash \theta \to (\pi(\overline{gn(\theta)}) \to \psi)$, and then by a tautology we have $\Gamma \vdash \pi(\overline{gn(\theta)}) \to (\theta \to \psi)$. By exercise E5.3 we then get

$$\Gamma \vdash \pi(\overline{gn(\pi(\overline{gn(\theta)}) \to (\theta \to \psi))}),$$

and E5.5(i) gives

$$\Gamma \vdash \pi(\overline{gn(\pi(\overline{gn(\pi(\overline{gn(\theta)}))}))}) \to \pi(\overline{gn(\theta \to \psi)}). \tag{1}$$

Another instance of E5.5(i) is

$$\Gamma \vdash \pi(\overline{gn(\theta \to \psi)}) \to (\pi(\overline{gn(\theta)}) \to \pi(\overline{gn(\psi)})). \tag{2}$$

Two instances of (ii) are

(3) $$\Gamma\vdash\pi(\overline{gn(\theta)})\to\pi(\overline{gn(\pi(\overline{gn(\theta)}))})\quad\text{and}$$

(4) $$\Gamma\vdash\pi(\overline{gn(\pi(\overline{gn(\theta)}))})\to\pi(\overline{gn(\pi(\overline{gn(\pi(\overline{gn(\theta)}))}))}).$$

Now a simple tautology gives $\Gamma\vdash\varphi(\overline{gn(\theta)})\to\pi(\overline{gn(\theta\to\psi)})$, using (3), (4), (1). The more complicated tautology

$$[S_0\to(S_1\to S_2)]\to[(S_1\to S_0)\to(S_1\to S_2)]$$

gives the desired conclusion. [Substitute $\pi(\overline{g(\theta\to\psi)})$ for S_0, $\pi(\overline{gn(\theta)})$ for S_1, and $\pi(\overline{gn(\psi)})$ for S_2.]

E5.7 *(Continuing E5.5.) Show that $\Gamma\vdash\psi$.*

By the choice of θ we have $\Gamma\vdash(\pi(\overline{g(\theta)})\to\psi)\to\theta$. Using the definition of ψ and a tautology, we get $\Gamma\vdash(\pi(\overline{gn(\theta)})\to\pi(\overline{gn(\psi)}))\to\theta$. Then by exercise 5.6 it follows that $\Gamma\vdash\theta$. Hence by exercise 5.5, $\Gamma\vdash\psi$.

E5.8 *Assume that $\Gamma\vdash\neg(\overline{m}=\overline{n})$ for all distinct $m,n\in\omega$. Let χ be as in the proof of Göde's incompleteness theorem, and let θ be the sentence $\forall v_0(v_0=v_0)$. Prove that the following formula $\rho(v_0)$ is a provabilty condition:*

$$v_0=\overline{gn(\theta)}\vee\exists v_1\chi(v_0,v_1)$$

Let φ be any sentence. If $\Gamma\vdash\varphi$, then $\Gamma\vdash\exists v_1\chi(\overline{gn(\varphi)},v_1)$ by exercise E5.3, and so clearly $\Gamma\vdash\rho(\overline{gn(\varphi)})$. Suppose that $\Gamma\vdash\rho(\overline{gn(\varphi)})$. If $\varphi=\theta$, obviously $\Gamma\vdash\varphi$. If $\varphi\neq\theta$, then $gn(\varphi)\neq gn(\theta)$ and hence by assumption $\Gamma\vdash\neg(\overline{gn(\varphi)}=\overline{gn(\theta)})$, and a tautology gives $\Gamma\vdash\exists v_1\chi(\overline{gn(\varphi)},v_1)$. Hence by exercise E5.3, $\Gamma\vdash\varphi$.

E5.9 *(Continuing E5.8) Prove that $\Gamma\vdash\theta\leftrightarrow\rho(\overline{gn(\theta)})$ (so that θ asserts its own provability with respect to this condition).*

In fact, clearly $\Gamma\vdash\theta$, and also $\Gamma\vdash\rho(\overline{g(\theta)})$, so the conclusion follows.

E5.10 *(Continuing E5.8) Prove that $\Gamma\vdash\theta$.*

This is clear.

E5.11 *Assume that $\Gamma\vdash\neg(\overline{m}=\overline{n})$ for all distinct $m,n\in\omega$. Let χ be an in the proof of Gödel's incompleteness theorem, and let θ be the sentence $\neg\forall v_0(v_0=v_0)$. Prove that the following formula $\rho(v_0)$ is a provabilty condition:*

$$\neg(v_0=\overline{gn(\theta)})\wedge\exists v_1\chi(v_0,v_1)$$

Let φ be a sentence. Suppose that $\Gamma\vdash\varphi$. Then by exercise E5.3, $\Gamma\vdash\exists v_1\chi(\overline{g(\varphi)},v_1)$. Now $\varphi\neq\theta$, since $\Gamma\vdash\neg\theta$, and Γ is consistent since $\overline{M}$ is a model of it. Hence $gn(\varphi)\neq gn(\theta)$. So by assumption $\Gamma\vdash\neg(\overline{gn(\varphi)}=\overline{gn(\theta)})$. Hence $\Gamma\vdash\rho(\overline{gn(\varphi)})$.

Suppose that $\Gamma \vdash \rho(\overline{gn(\varphi)})$. Then also $\Gamma \vdash \exists v_i \chi(\overline{gn(\varphi)}, v_1)$, so by exercise E5.3, $\Gamma \vdash \varphi$.

E5.12 *(Continuing E5.11) Show that $\Gamma \vdash \theta \leftrightarrow \rho(\overline{g(\theta)})$, so that θ asserts its own provability.*

In fact, clearly $\Gamma \vdash \neg\theta$ and also $\Gamma \vdash \neg\rho(\overline{g(\theta)})$, so the exercise follows.

E5.13 *(Continuing E5.11) Show that $\Gamma \vdash \neg\theta$.*

This is clear.