Solutions to exercises in Chapter 3

E3.1 *Do the case* $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ *for some m-ary relation symbol and terms* $\sigma_0, \ldots, \sigma_{m-1}$ *in the proof of Theorem 3.1, (L3).*

We are assuming that v_i does not occur in $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$; hence it does not occur in any term σ_i.

$$\begin{aligned} \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{R}}(a)\rangle \in \mathbf{R}^{\overline{A}} \\ &\text{iff} \quad \langle \sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{R}}(b)\rangle \in \mathbf{R}^{\overline{A}} \\ &\qquad \text{(by Proposition 2.4)} \\ &\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[b]. \end{aligned}$$

E3.2 *Prove that (L6) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\rho = \sigma)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\rho^{\overline{A}}(a) = \sigma^{\overline{A}}(a)$, so $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, hence $\overline{A} \models (\rho = \tau)[a]$.

E3.3 *Prove that (L8) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Assume that

$$\begin{aligned} &\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})[a]; \text{ hence} \\ &\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}; \text{ hence} \\ &\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}; \text{ hence} \\ &\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a]; \end{aligned}$$

hence (L8) is universally valid.

E3.4 *Finish the proof of Proposition 3.11.*

We are assuming inductively that φ is $\forall v_s \psi$ with ψ a formula and $s \in \omega$. Thus φ is $\langle 4, 5(s+1)\rangle^\frown\psi$. If $i = 0$, then φ itself is the desired segment, unique by Proposition 2.6(iii). Suppose that $i > 0$. Then by the hypothesis of the proposition, actually $i > 1$, since φ_1 is $5(s+1)$. So φ_i is an entry in ψ and hence by the inductive assumption, there is a segment $\langle \varphi_i, \varphi_{i+1}, \ldots \varphi_m\rangle$ which is a formula; this is also a segment of φ, and it is unique by Proposition 2.6(iii).

E3.5 *Indicate which occurrences of the variables are bound and which ones free for the following formulas.*

$\exists v_0(v_0 < v_1) \wedge \forall v_1(v_0 = v_1)$.
$v_4 + v_2 = v_0 \wedge \forall v_3(v_0 = v_1)$.
$\exists v_2(v_4 + v_2 = v_0)$.

First formula: the first and second occurrences of v_0 are bound, and the third one is free. The first occurrence of v_1 is free, and the other two are bound.

Second formula: the occurrence of v_3 is bound. All other occurrences of variables are free.

Third formula: the two occurrences of v_2 are bound. The other occurrences of variables are free.

E3.6 *Finish the proof of Proposition 3.13.*

Suppose that φ is an atomic non-equality formula; so there is a relation symbol **R** and terms $\sigma_0, \ldots \sigma_{n-1}$ such that φ is $\langle \mathbf{R} \rangle^\frown \sigma_0^\frown \cdots ^\frown \sigma_{n-1}$. Hence $i > 0$, and it is inside some term σ_j. By Proposition 3.12 there is a term which is a segment of σ_j beginning at i; it is also a segment of φ, and it is unique by Proposition 2.2(iii).

Suppose inductively that φ is $\neg\psi$, i.e., it is $\langle 0 \rangle^\frown \psi$. Then $i > 0$, so that it is inside ψ. Hence the inductive hypothesis gives the desired result.

Suppose inductively that φ is $\psi \to \chi$, i.e., it is $\langle 1 \rangle^\frown \psi^\frown \chi$. Then $i > 0$ and i is inside ψ or χ; the inductive hypothesis gives the desired result.

Suppose inductively that φ is $\forall v_k \psi$, i.e., it is $\langle 4, 5(k+1) \rangle^\frown \psi$. So $i > 0$. If $i = 1$, then φ_i is $5(k+1)$, so that $\langle 5(k+1) \rangle$ is a term which is a segment of φ, unique by Proposition 2.2(iii). If $i > 1$, then it is inside ψ, and the inductive hypothesis gives the desired result.

E3.7 *Indicate all free and bound occurrences of terms in the formula* $v_0 = v_1 + v_1 \to \exists v_2(v_0 + v_2 = v_1)$.

v_0 is free in both of its occcurrences.
v_1 is free in all three of its occurrences.
v_2 is bound in both of its occurrences.
$v_1 + v_1$ is free in its occurrence.
$v_0 + v_2$ is bound in its occurrence.

E3.8 *Prove Proposition 3.16*

Induction on φ. Suppose that φ is $\rho = \xi$. Then by Proposition 3.13, σ occurs in ρ or ξ. Suppose that it occurs in ρ. Let ρ' be obtained from ρ by replacing that occurrence of σ by τ. Then ρ' is a term by Proposition 3.14. Since ψ is $\rho' = \xi$, ψ is a formula. The case in which σ occurs in ξ is similar. Now suppose that φ is $\mathbf{R}\eta_0 \ldots \eta_{m-1}$ with **R** an m-ary relation symbol and $\eta_0, \ldots, \eta_{m-1}$ are terms. Then the occurrence of σ is within some η_i. Let η_i' be obtained from η_i by replacing that occurrence by τ. Now ψ is $\mathbf{R}\eta_0 \ldots \eta_{i-1}\eta_i' \ldots \eta_{m-1}$, so ψ is a formula.

Now suppose that the result holds for φ', and φ is $\neg\varphi'$. Then σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\neg\psi'$ also ψ is a formula.

Next, suppose that the result holds for φ' and φ'', and φ is $\varphi' \to \varphi''$. Then the occurrence of σ is within φ' or is within φ''. If it is within φ', let ψ' be obtained from φ' by replacing that occurrence of σ by τ. Then ψ' is a formula by the inductive hypothesis. Since ψ is $\psi' \to \varphi''$, also ψ is a formula. If the occurrence is within φ'', let ψ'' be obtained from φ'' by replacing that occurrence of σ by τ. Then ψ'' is a formula by the inductive hypothesis. Since ψ is $\varphi' \to \psi''$, also ψ is a formula.

Finally, suppose that the result holds for φ', and φ is $\forall v_k \varphi'$. If $i = 1$, then σ is v_k, and by hypothesis τ is some variable v_l. Then ψ is $\forall v_l \varphi'$, which is a formula. If $i > 1$, then

σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\forall v_k\psi'$ also ψ is a formula.

E3.9 *Show that the condition in Proposition 3.17 that the resulting occurrence of τ is free is necessary. Hint: use Theorem 3.2; describe a specific formula of the type in Proposition 3.17, but with τ not free, such that the formula is not universally valid.*

Consider the language for (ω, S), and the formula

$$v_0 = v_1 \to (\exists v_1(\mathbf{S}v_0 = v_1) \leftrightarrow \exists v_1(\mathbf{S}v_1 = v_1)).$$

Taking an assignment $a : \omega \to \omega$ with $a_0 = a_1$ makes this sentence false; hence it is not provable, by Theorem 3.2.

E3.10 *Prove Proposition 3.19*

Induction on φ. If φ is atomic, then ψ is equal to φ, and θ is equal to χ and hence is a formula. Suppose the result is true for φ' and φ is $\neg\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\neg\theta'$, also θ is a formula.

Now suppose the result is true for φ' and φ'', and φ is $\varphi' \to \varphi''$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ' or is within the subformula φ''. If it is within φ' and θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\theta' \to \varphi''$, also θ is a formula. If it is within φ'' and θ'' is obtained from φ'' by replacing that occurrence by χ, then θ'' is a formula by the inductive hypothesis. Since θ is $\varphi' \to \theta''$, also θ is a formula.

Finally, suppose the result is true for φ' and φ is $\forall v_i\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\forall v_i\theta'$, also θ is a formula.

E3.11 *Prove that the hypothesis of Theorem 3.27 is necessary.*

Consider the formula

$$\forall v_0\exists v_1(v_0 < v_1) \to \exists v_1(v_1 < v_1).$$

This formula is not universally valid; it fails to hold in $(\omega, <)$, for example. In the notation of Theorem 3.27 we have $i = 0$, φ is the formula $\exists v_1(v_0 < v_1)$, σ is v_1, and $\mathrm{Subf}^{v_i}_\sigma$ is $\exists v_1(v_1 < v_1)$. Note that the free occurrence of v_0 in $\exists v_1(v_0 < v_1)$ is within a subformula of $\exists v_1(v_0 < v_1)$ of the form $\forall v_1\psi$ with v_1 occurring in σ. Namely, $\exists v_1(v_0 < v_1)$ is by definition $\neg\forall v_1\neg(v_0 < v_1)$, and the subformula is $\forall v_1\neg(v_0 < v_1)$.

E3.12 *Prove Proposition 3.31.*

Proof. By definition, $\exists v_i\neg\varphi$ is $\neg\forall v_i\neg\neg\varphi$. Now $\vdash \varphi \leftrightarrow \neg\neg\varphi$ by a tautology. Hence using generalization and (L2) we get $\vdash \forall v_i\varphi \leftrightarrow \forall v_i\neg\neg\varphi$. Hence another tautology yields $\vdash \neg\forall v_i\varphi \leftrightarrow \neg\forall v_i\neg\neg\varphi$, i.e., $\vdash \neg\forall v_i\varphi \leftrightarrow \exists v_i\neg\varphi$. □

E3.13 *Prove Proposition 3.32.*

Proof. $\neg\exists v_i\varphi$ is the formula $\neg\neg\forall v_i\neg\varphi$, so a simple tautology gives the result. □

E3.14 *Prove Proposition 3.33.*

Proof. By Theorem 3.27 we have $\vdash \forall v_i\neg\varphi \to \mathrm{Subf}^{v_i}_\sigma(\neg\varphi)$. Since clearly $\mathrm{Subf}^{v_i}_\sigma(\neg\varphi)$ is the same as $\neg\mathrm{Subf}^{v_i}_\sigma\varphi$, a tautology gives $\vdash \mathrm{Subf}^{v_i}_\sigma\varphi \to \exists v_i\varphi$. □

E3.15 *Prove Proposition 3.35.*

Proof. By Corollary 3.28, Corollary 3.34, and a tautology. □

E3.16 *Prove Proposition 3.36.*

Proof. $\vdash \varphi \to \exists v_i\varphi$ by Corollary 3.34. $\vdash \neg\varphi \to \forall v_i\neg\varphi$ by Proposition 3.29. Hence the result follows by a tautology. □

E3.17 *Prove Proposition 3.43.*

Proof. Assume that $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \varphi \to \psi$. Generalization and (L2) then give $\vdash \forall v_i\varphi \to \forall v_i\psi$. Similarly, $\vdash \forall v_i\psi \to \forall v_i\varphi$. Now a tautology finishes the proof. □

E3.18 *Prove Proposition 3.44.*

Proof. Assume that $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \neg\varphi \leftrightarrow \neg\psi$. Then by Proposition 3.43, $\vdash \forall v_i\neg\varphi \leftrightarrow \forall v_i\neg\psi$. Now a tautology finishes the proof. □

E3.19 *Find a formula in prenex normal form equivalent to the following formula:*

$$\forall v_0\exists v_1(v_0 < v_1) \wedge \exists v_1\forall v_0(v_0 < v_1).$$

First solution. By Theorem 3.37 we have

$$\vdash \exists v_1\forall v_0(v_0 < v_1) \to \forall v_0\exists v_1(v_0 < v_1). \quad (1)$$

Now (1) $\to [\forall v_0\exists v_1(v_0 < v_1) \wedge \exists v_1\forall v_0(v_0 < v_1) \leftrightarrow \exists v_1\forall v_0(v_0 < v_1)]$ is a tautology. It follows that $\exists v_1\forall v_0(v_0 < v_1)$ is a formula in prenex normal form equivalent to the given formula.

Second solution. (This solution indicates a pattern which can be followed in many other cases.)

By the change of bound variable theorem 3.25,

$$\vdash \exists v_1\forall v_0(v_0 < v_1) \leftrightarrow \exists v_2\forall v_0(v_0 < v_2) \quad (1)$$

Again by 3.25,

$$\vdash \exists v_2\forall v_0(v_0 < v_2) \leftrightarrow \exists v_2\forall v_3(v_3 < v_2) \quad (2)$$

By (1), (2), and a tautology,

$$\vdash \exists v_1\forall v_0(v_0 < v_1) \leftrightarrow \exists v_2\forall v_3(v_3 < v_2);$$

then another tautology gives

(3) $\vdash \forall v_0 \exists v_1 (v_0 < v_1) \wedge \exists v_1 \forall v_0 (v_0 < v_1) \leftrightarrow \forall v_0 \exists v_1 (v_0 < v_1) \wedge \exists v_2 \forall v_3 (v_3 < v_2).$

Now by Theorem 3.48 we have

$$\vdash v_0 < v_1 \wedge \exists v_2 \forall v_3 (v_3 < v_2) \leftrightarrow \exists v_2 \forall v_3 (v_0 < v_1 \wedge v_3 < v_2)$$

Applying Propositions 3.43 and 3.44 to this we get

(4) $\vdash \forall v_0 \exists v_1 (v_0 < v_1 \wedge \exists v_2 \forall v_3 (v_3 < v_2) \leftrightarrow \forall v_0 \exists v_1 \exists v_2 \forall v_3 (v_0 < v_1 \wedge v_3 < v_2)$

Now by Theorem 3.47 we have

(5) $\vdash \forall v_0 \exists v_1 (v_0 < v_1) \wedge \exists v_2 \forall v_3 (v_3 < v_2) \leftrightarrow \forall v_0 \exists v_1 (v_0 < v_1 \wedge \exists v_2 \forall v_3 (v_3 < v_2)$

Now (3), (4), (5) and a tautology give the result of the exercise.

E3.21 *Prove that*

$$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow \forall v_0 (v_0 = v_1 \vee v_0 = v_2).$$

$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_1;$ Cor. 3.28 twice, taut. (1)
$\vdash \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_2;$ Thm. 3.27 (2)
$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_2;$ (2), Cor. 3.28, taut. (3)
$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_1 \vee v_0 = v_2;$ (1), (3), taut. (4)
$\vdash \forall v_0 \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow \forall v_0 (v_0 = v_1 \vee v_0 = v_2);$ (4), (L2), taut. (5)
$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \rightarrow \forall v_0 (v_0 = v_1 \vee v_0 = v_2).$ (5), Prop. 3.29, taut.

E3.22 *Prove that*

$$\vdash \exists v_0 (\neg v_0 = v_1 \wedge \neg v_0 = v_2) \rightarrow \exists v_0 \exists v_1 (\neg v_0 = v_1).$$

$\vdash \neg \forall v_0 (v_0 = v_1 \vee v_0 = v_2) \rightarrow \neg \forall v_0 \forall v_1 (v_0 = v_1);$ E3.21, taut. (1)
$\vdash \neg \forall v_0 (v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0 \neg (v_0 = v_1 \vee v_0 = v_2);$ Prop. 3.31 (2)
$\vdash \neg (v_0 = v_1 \vee v_0 = v_2) \leftrightarrow (\neg (v_0 = v_1) \wedge \neg (v_0 = v_2));$ taut. (3)
$\vdash \exists v_0 \neg (v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0 (\neg (v_0 = v_1) \wedge \neg (v_0 = v_2));$ (3), Prop. 3.44 (4)
$\vdash \neg \forall v_0 (v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0 (\neg (v_0 = v_1) \wedge \neg (v_0 = v_2));$ (2), (4), taut. (5)
$\vdash \neg \forall v_1 (v_0 = v_1) \leftrightarrow \exists v_1 \neg (v_0 = v_1);$ Prop. 3.31 (6)
$\vdash \exists v_0 \neg \forall v_1 (v_0 = v_1) \leftrightarrow \exists v_0 \exists v_1 \neg (v_0 = v_1);$ (6), Prop. 3.44 (7)
$\vdash \neg \forall v_0 \forall v_1 (v_0 = v_1) \leftrightarrow \exists v_0 \neg \forall v_1 (v_0 = v_1);$ Prop. 3.31 (8)
$\vdash \neg \forall v_0 \forall v_1 (v_0 = v_1) \leftrightarrow \exists v_0 \exists v_1 \neg (v_0 = v_1)$ (7), (8), taut. (9)
$\vdash \exists v_0 (\neg v_0 = v_1 \wedge \neg v_0 = v_2) \rightarrow \exists v_0 \exists v_1 (\neg v_0 = v_1).$ (1), (5), (9), taut. □