Solutions for exercises in Chapter 2

$\boxed{\text{E2.1}}$ *Give the exact definition of the language for the structure* $(\omega, <)$.

The quadruple $(\{11\}, \emptyset, \emptyset, rnk)$, where rnk is the function with domain $\{11\}$ such that $rnk(11) = 2$.

$\boxed{\text{E2.2}}$ *Give the exact definition of the language for the set A (no individual constants, function symbols, or relation symbols).*

The quadruple $(\emptyset, \emptyset, \emptyset, \emptyset)$. Note that the last $\emptyset$ is the empty function.

$\boxed{\text{E2.3}}$ *Describe a term construction sequence which shows that* $+\bullet v_0 v_0 v_1$ *is a term in the language for* $(\mathbb{R}, +, \cdot, 0, 1, <)$.

$\langle v_0, \bullet v_0 v_0, v_1, +\bullet v_0 v_0 v_1\rangle$.

$\boxed{\text{E2.4}}$ *In any first-order language, show that the sequence* $\langle v_0, v_0\rangle$ *is not a term. Hint: use Proposition 2.2.*

Suppose that $\langle v_0, v_0\rangle$ is a term. This contradicts Proposition 2.2(ii).

$\boxed{\text{E2.5}}$ *In the language for* $(\omega, S, 0, +, \cdot)$, *show that the sequence* $\langle +, v_0, v_1, v_2\rangle$ *is not a term. Hint: use Proposition 2.2.*

Suppose it is a term. By Proposition 2.2(ii)(c), there are terms σ, τ such that $\langle +, v_0, v_1, v_2\rangle$ is $\langle +\rangle^\frown\sigma^\frown\tau$. Thus $\langle v_0, v_1, v_2\rangle = \sigma^\frown\tau$. So the term v_0 is an initial segment of the term σ. By Proposition 2.2(iii) it follows that $v_0 = \sigma$. Hence $\langle v_1, v_2\rangle = \tau$. This contradicts Proposition 2.2(ii).

$\boxed{\text{E2.6}}$ *Show how the structure* (A, f) *introduced at the beginning of the chapter can be put in the general framework of structures.*

(A, f) can be considered to be the structure (A, Rel', Fcn', Cn') where $Rel' = Cn' = \emptyset$ and Fcn' is the function with domain $\{13\}$ such that $Fcn'(13) = f$.

$\boxed{\text{E2.7}}$ *Show how the structure* $(\omega, S, 0, +, \cdot)$ *introduced at the beginning ot the chapter can be put in the general framework of structures.*

$(\omega, S, 0, +, \cdot)$ can be considered to be the structure $(\omega, Rel', Fcn', Cn')$ where $Rel' = \emptyset$, Cn' is the function with domain $\{8\}$ such that $Cn'(8) = 0$, and Fcn' is the function with domain $\{6, 7, 9\}$ such that $Fcn'(6) = S$, $Fcn'(7) = +$, and $Fcn'(9) = \cdot$.

$\boxed{\text{E2.8}}$ *Prove that in the language for the structure* $(\omega, +)$, *a term has length* m *iff* m *is odd.*

First we show by induction on terms that every term has odd length. This is true for variables. Suppose that it is true for terms σ and τ. Then also $\sigma + \tau$ has odd length. Hence every term has odd length.

Second we prove by induction on m that for all m, there is a term of length $2m + 1$. A variable has length 1, so our assertion holds for $m = 0$. Assume that there is a term σ of length $2m + 1$. Then $\sigma + v_0$ has length $2m + 3$. This finishes the inductive proof.

$\boxed{\text{E2.9}}$ *Complete the proof of Proposition 2.5.*

We show by complete induction on i that $\varphi_i \in \Gamma$ for all $i < m$. So, suppose that $i < m$ and $\varphi_j \in \Gamma$ for all $j < i$. By the definition of formula construction sequence, we have the following cases.

Case 1. φ_i is an atomic formula. Then $\varphi_i \in \Gamma$ by (i).

Case 2. There is a $j < i$ such that φ_i is $\neg\varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (ii), $\varphi_i \in \Gamma$.

Case 2. There are $j, k < i$ such that φ_i is $\varphi_j \to \varphi_k$. By the inductive hypothesis, $\varphi_j \in \Gamma$ and $\varphi_k \in \Gamma$. Hence by (iii), $\varphi_i \in \Gamma$.

Case 4. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (iv), $\varphi_i \in \Gamma$.

This completes the inductive proof.

$\boxed{\text{E2.10}}$ *Give a formula φ in the language for $(\mathbf{Q}, +, \cdot)$ such that for any $a : \omega \to \mathbb{Q}$, $(\mathbb{Q}, +, \cdot) \models \varphi[a]$ iff $a_0 = 1$.*

Let φ be the formula $\forall v_1[v_0 \cdot v_1 = v_1]$.

$\boxed{\text{E2.11}}$ *Give a formula φ which holds in a structure, under any assignment, iff the structure has at least 3 elements.*

$$\exists v_0 \exists v_1 \exists v_2 (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_1 = v_2)).$$

$\boxed{\text{E2.12}}$ *Give a formula φ which holds in a structure, under any assignment, iff the structure has exactly 4 elements.*

$$\exists v_0 \exists v_1 \exists v_2 \exists v_3 (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_0 = v_3) \wedge \neg(v_1 = v_2) \wedge \neg(v_1 = v_3) \wedge \neg(v_2 = v_3)$$
$$\wedge \forall v_4 (v_0 = v_4 \vee v_1 = v_4 \vee v_2 = v_4 \vee v_3 = v_4)).$$

$\boxed{\text{E2.13}}$ *Write the formula given in (1) at the end of this chapter as a sequence of integers.*

$$\langle 4, 5, 4, 10, 2, 3, 6, 5, 6, 10, 3, 5, 10 \rangle.$$

$\boxed{\text{E2.14}}$ *Write a formula φ in the language for $(\omega, <)$ such that for any assignment a, $(\omega, <) \models \varphi[a]$ iff $a_0 < a_1$ and there are exactly two integers between a_0 and a_1.*

$$\begin{aligned} &v_0 < v_1 \wedge \exists v_2 \exists v_3 [v_0 < v_2 \wedge v_2 < v_3 \wedge v_3 < v_1 \\ &\wedge \forall v_4 [v_0 < v_4 \wedge v_4 < v_1 \to v_4 = v_2 \vee v_4 = v_3]]. \end{aligned}$$

$\boxed{\text{E2.15}}$ *Prove that the formula*

$$v_0 = v_1 \to (\mathbf{R} v_0 v_2 \to \mathbf{R} v_1 v_2)$$

is universally valid, where $\mathbf{R}$ *is a binary relation symbol.*

Let $\overline{A}$ be a structure and $a : \omega \to A$ an assignment. Suppose that $\overline{A} \models (v_0 = v_1)[a]$. Then $a_0 = a_1$. Also suppose that $\overline{A} \models \mathbf{R} v_0 v_2[a]$. Then $(a_0, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $(a_1, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $\overline{A} \models \mathbf{R} v_1 v_2[a]$, as desired.

$\boxed{\text{E2.16}}$ *Give an example showing that the formula*

$$v_0 = v_1 \rightarrow \forall v_0(v_0 = v_1)$$

is not universally valid.

Consider the structure $\overline{A} \stackrel{\text{def}}{=} (\omega, <)$, and let $a : \omega \rightarrow \omega$ be defined by $a(i) = 0$ for all $i \in \omega$. Then $\overline{A} \models (v_0 = v_1)[a]$. Now $\overline{A} \not\models (v_0 = v_1)[a_1^0]$ since $1 \neq 0$, so $\overline{A} \not\models \forall v_0(v_0 = v_1)[a]$. Therefore $\overline{A} \not\models (v_0 = v_1 \rightarrow \forall v_0(v_0 = v_1))[a]$.

$\boxed{\text{E2.17}}$ *Prove that* $\exists v_0 \forall v_1 \varphi \rightarrow \forall v_1 \exists v_0 \varphi$ *is universally valid.*

Assume that $a : \omega \rightarrow A$ and $\overline{A} \models \exists v_0 \forall v_1 \varphi[a]$. Choose $u \in A$ so that $\overline{A} \models \forall v_1 \varphi[a_u^0]$. In order to show that $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, let $w \in A$ be given. Then $\overline{A} \models \varphi_{uw}^{01}$. It follows that $\overline{A} \models \exists v_0 \varphi[u_w^1]$. Hence $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, as desired.